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Composite Concave Cupolae as Geometric and Architectural Forms

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Abstract. In this paper, the geometry of concave cupolae has been the starting point for the generation of composite polyhedral structures, usable as formative patterns for architectural purposes. Obtained by linking paper folding geometry with the geometry of polyhedra, concave cupolae are polyhedra that follow the method of generating cupolae (Johnson's solids: J3, J4 and J5); but we removed the convexity criterion and omitted squares in the lateral surface. Instead of alternating triangles and squares there are now two or more paired series of equilateral triangles. The criterion of face regularity is respected, as well as the criterion of multiple axial symmetry. The distribution of the triangles is based on strictly determined and mathematically defined parameters, which allows the creation of such structures in a way that qualifies them as an autonomous group of polyhedra — concave cupolae of sorts II, IV, VI (2N). If we want to see these structures as polyhedral surfaces (not as solids) connecting the concept of the cupola (dome) in the architectural sense with the geometrical meaning of (concave) cupola, we remove the faces of the base polygons. Thus we get a deltahedral structure — a shell made entirely from equilateral triangles, which is advantageous for the purpose of prefabrication. Due to the congruence of the major $2n$-sided bases of concave cupolae of sort II with the minor bases of the corresponding concave cupolae of sort IV, it is possible to combine these polyhedra in composite polyhedra. But also their elongation with concave antiprisms of sort II or the augmentation with concave pyramids of sort II could be performed. Based on the foregoing, we examine the possibilities of combining the considered polyhedra into unified composite structures.

Key Words: concave polyhedra, cupola, deltahedron, equilateral triangle
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1. Introduction

This paper discusses the formation of composite polyhedral structures based on the geometry of concave polyhedra of the second (and higher) sort, by joining the initial polyhedra, matching and combining their forms in order to obtain a new matrix, applicable in architectural design. Composite polyhedra are differently defined, depending on whether they are observed from the mathematical or engineering point of view, so TIMOFEENKO in [17] provided the following definition of composite polyhedra:

> *"If a convex polyhedron with regular faces can be divided by some plane into two polyhedra with regular faces, then it is said to be composite",*

which applies only to convex polyhedra. Since the forms in the present research are investigated for the purpose of engineering solutions, we accepted a broader definition of composite solids, which was adopted by P. HUYBERS [2] and D.G. EMMERICH [1], who granted composite polyhedra to be also polyhedra obtained by augmentations of the uniform polyhedra using JOHNSON's cupolae and rotundae [5] whereat the concave structures arise.

Composite polyhedra that are the subject of this study, occur by augmentation of concave cupolae of sort IV using concave cupolae of sort II and concave pyramids of sort II, and by (gyro)elongations, using antiprisms and concave antiprisms of sort II, i.e., using the polyhedra which belong to the family of concave polyhedra whose geometry is based on folding the deltahedral lateral surface.

2. Concave M-m polyhedra

The family of concave polyhedra obtained by folding a lateral surface net of double (or an even number of) series of equilateral triangles, consists of: concave cupolae of sort II [11], concave pyramids of sort II [11, 16], concave antiprisms of sort II [9, 15], as well as concave cupolae of IV and higher sorts [8]. The process of generating these polyhedra is similar to the formation of polygrammatic antiprisms [3] or pseudo-cylinders [7]. It is also akin to the origami technique, since the folding of a planar net along the assigned edges produces 3D structures. Their common features are that they are formed over a regular polygonal base, they are regularly faced, they have multiple axial symmetry, and they are obtained by a polar array of unit concave cells consisting of equilateral triangles (spatial hexahedral or pentahedral cells) arranged around the axis of the solid.

Over the same polygonal base, by a polar arrangement of a unit cell, two types of deltahedral belt can be formed: with greater height (M) or with the smaller height (m). For this reason, the considered concave polyhedra are denoted briefly as M-m polyhedra. Their parameters, i.e., lengths and angles, are strictly determined and geometrically defined, as shown in previous studies [9, 11]. In this paper we give just a brief overview of these solids.

2.1. Concave cupolae of sort II

Concave cupolae of sort II (CC II) [11] are polyhedra that follow the method of generating cupolae (JOHNSON's solids: J3, J4 and J5); but we exclude the convexity criterion and omit squares in the lateral surface. Instead of alternating triangles and squares, now appear two series of equilateral triangles. Such polyhedra are formed by connecting two regular polygons, a n-gon and a $2n$-gon (adopted bases of the solid) in parallel planes, by two rows of equilateral triangles which make lateral surface of a solid. The sort of cupola is determined by the number of rows of equilateral triangles in the lateral surface (Figure 1a).

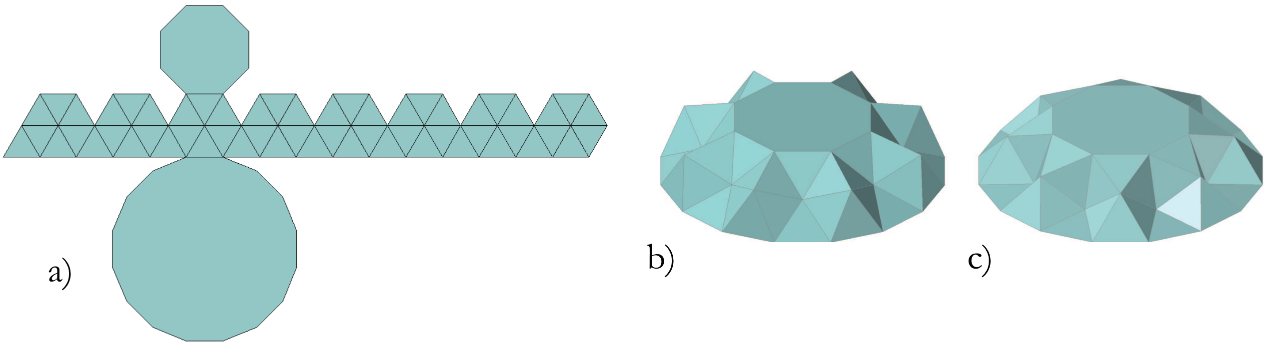

Figure 1: a) The lateral surface net of CC II-8, b) CC II-8M, c) CC II-8m

For each basic polygon, the lateral surface net can be folded in two ways: with central vertex of the unite hexahedral cell indented, giving the major height of the solid and the first type, CC II-M, or protruded, giving the minor height of the solid, and the second type, CC II m. There are fourteen possible representatives of CC II altogether [11], from $n = 4$ to $n = 10$, each of them with two types, M (Figure 1b) and m (Figure 1c).

2.2. Concave pyramids of sort II

Concave pyramids of the second sort (CP II) are polyhedra which follow the method of generating concave cupolae of the second sort (CC II) [16], using the same method of folding the double row of equilateral triangle's plane net, as shown in Figures 2a and 3a. These polyhedra are also organized over a regular n-sided polygonal base and regular faced, with a deltahedral lateral surface consisting of equilateral triangles, as for Johnson's pyramids (J1 and J2). Unlike CC, the unit cell that forms the solid by its radial array is now a spatial pentahedral cell, having a common vertex in the apex of the solid.

There are two types of concave cupolae of sort II. The first type, CP II-A, has the number of unit cells equal to the number of the base polygons' sides, since it covers all the bases from $n = 6$ to $n = 9$, both odd and even. The second type, CP II-B, formed with the halved number of sides is possible only for the even bases, $n = 6$ and $n = 8$.

The number of vertices (V), edges (E) and faces (F) for any CP II-A over a n-sided polygonal base can be calculated by the formulae:

$$V = 3n + 1, \quad E = 8n, \quad F = 5n + 1, \tag{1}$$

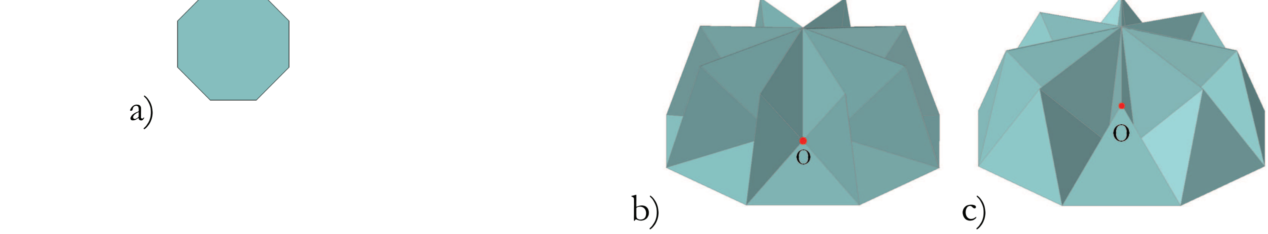

Figure 2: a) The lateral surface net of CP II-8A, b) CP II-8MA, c) CP II-8mA

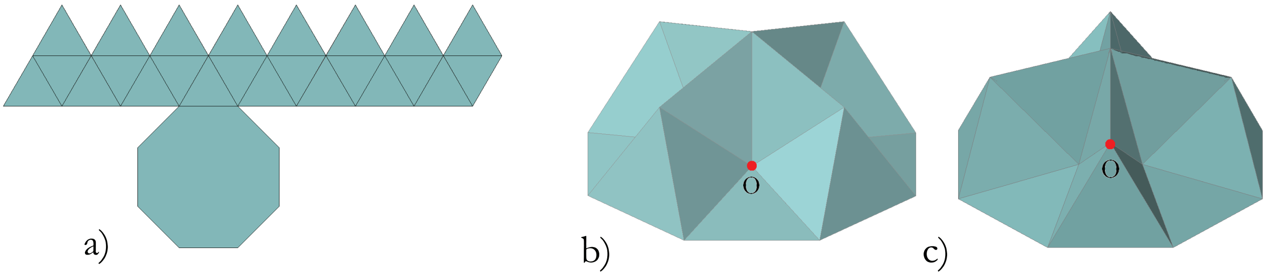

Figure 3: a) The lateral surface net of CP II-8B, b) CP II-8MB, congruent to c) CP II-8mB

and for any CP II-B:

$$V = 2n + 1, \quad E = 5n, \quad F = 3n + 1. \tag{2}$$

There are seven representatives of CP II-A: four CP II-MA with greater (major) height (indented central vertex O of the spatial hexahedral unit cell), Figure 2b (from $n = 6$ to $n = 9$), and three CP II-mA (from $n = 6$ to $n = 8$) [16] with smaller (minor) height (protruding central vertex O of the spatial hexahedral unit cell), Figure 2c, while there are two (and the conditional third) representatives of CP II-B for $n = 6$ and $n = 8$, whereat the cases CP II-MB and CP II-mB are congruent [11] (Figure 3).

2.3. Concave antiprisms of sort II

Concave antiprisms of the second sort (CA II) are polyhedra consisting of two congruent regular polygonal bases in parallel planes and deltahedral lateral surface [9, 15]. The lateral surface consists of two-row strip of equilateral triangles, arranged in such a way to form a spatial hexahedral elements by whose polar arrangement around the axis that connects bases' centroids the polyhedron is created (Figure 4).

Over the same polygonal base, the two different types of concave antiprisms of sort II can be formed: CA II-M, with the greater height (indented central vertex O of the spatial hexahedral unit cell), and CA II-m, with smaller height (protruding central vertex O of the spatial hexahedral unit cell), Figure 4.

The constructive procedure for generating CA II and determining their metric relations was elaborated in detail [9, 15]. The number of vertices (V), edges (E) and faces (F) for any CA II over n-sided polygonal base is calculated by formulae:

$$V = 4n, \quad E = 10n, \quad F = 6n + 2. \tag{3}$$

Concave antiprisms of sort II can be formed over any regular polygonal base, so they represent an infinite family of concave regular-faced polyhedra.

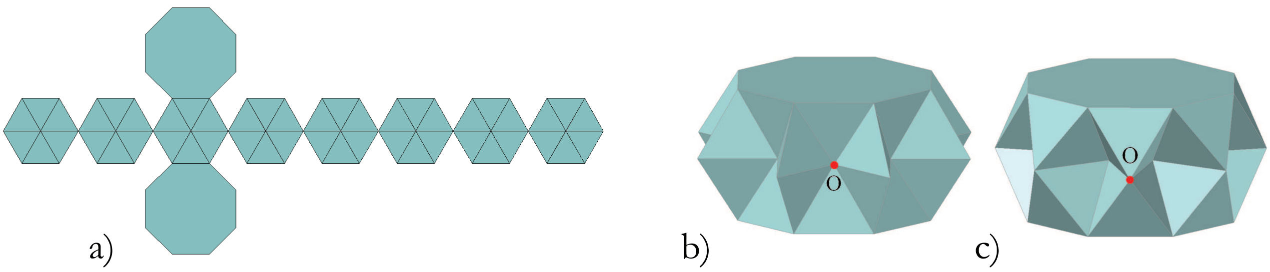

Figure 4: a) The lateral surface net of CA II-8, b) CA II-8M, c) CA II-8m

2.4. Concave cupolae of sort IV

Concave cupolae of sort IV (CC IV) are polyhedra which follow the method of generating CC II, only with a four-fold strip of equilateral triangles in the planar net and with the additional quadrilateral cells inserted between the hexahedral cells in the middle zone of the lateral surface, in order to enclose the inner space and to form a deltahedral shell modeled by strictly geometrically and mathematically defined principles [8]. CC IV is formed, like CC II, over the regular polygonal bases, n-sided (Ω_1) and $2n$-sided (Ω_2), which lie in parallel planes (Figure 5).

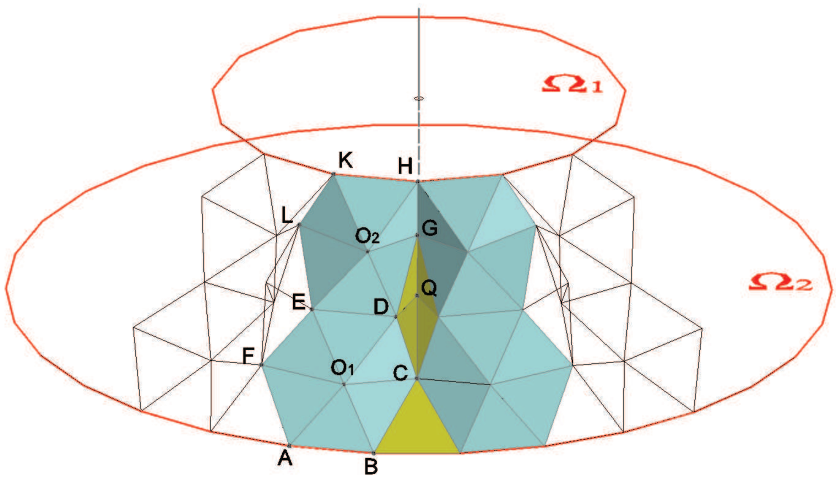

Figure 5: The generation of CC IV by the polar array of the spatial hexahedral unit cells

The positions and the heights of the vertices of CC IV are constructed using the intersections of the vertical plane β (Figure 6), determined by the axis of the solid and the vertex of the base polygon, with the spheres of radii $r = a$ (equal to the edge length of the solid). The sphere centers are set in the neighboring vertices of the spatial hexahedral cell (Figure 6). For a more detailed description of the constructive procedure see [9].

Examining the four different types of CC IV with the same polygonal base, shown in Figure 7, we can notice that their fundamental difference lies in the positions of the central vertices (O_1 and O_2) of the spatial hexahedal unit cell, which may be indented or protruding, as seen from outside.

2.5. Concave cupolae of higher sorts

The sort of the cupola is determined by the number of rows of equilateral triangles in the lateral surface needed to form the spatial unit cells. The arrangement of the triangles around a common vertex requires two linked series of triangles; consequently, the concave cupolae are always of an even sort [9].

For each sort of concave cupolae we can form the lateral shell (with the even number of equilateral triangles' rows) over eleven additional polygonal bases, as compared to each preceding lower sort of CC [9]. The only exception are CC II, because they can be generated over seven different polygonal bases ($4 \leq n \leq 10$), as described in [11]. Also, CC II are an exception as they are the only concave cupolae that do not have bonding spatial quadrihedral cells in their lateral surface net.

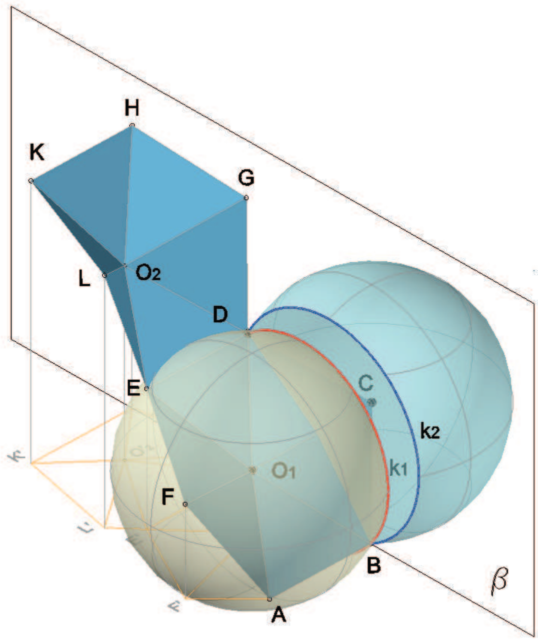

Figure 6: 3D model of the CC IV unit cell, the construction of the vertices B and D positions for the spatial hexahedral cell $ABCDEFO_1$, [9]

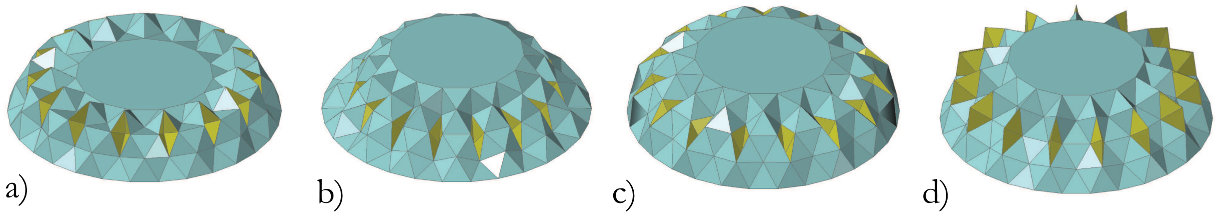

Figure 7: a) CC IV-15Mm, b) CC IV-15mm, c) CC IV-15mM, d) CC IV-15MM, [9]

The number (ν) of the possible different concave cupolae of the sort x over the same polygonal base is calculated according to the formula:

$$\nu = 2^{\frac{x}{2}} \tag{4}$$

The number of vertices (V), edges (E) and faces (F) for any CC of the sort x over a n-sided polygonal base is calculated by formulae [9]:

$$V = \frac{5x}{2}n, \quad E = \left(\frac{15x}{2} - 3\right)n, \quad F = (5x-3)n + 2. \tag{5}$$

3. Composite polyhedral structures based on geometry of concave M-m polyhedra

3.1. Augmentations

An *augmented* polyhedron is a polyhedron with one or more other solids adjoined [4]. At the regular faces of polyhedron, we can subjoin other figures with bases identical to the respective face of the polyhedron. Most often the augmentation involves adding pyramids (regular-faced,

Table 1: Possible mutual combinations of CC II

CC		II-4 m	II-4 M	II-5 m	II-5 M
II-8	m	•	•		
II-8	M	•	•		
II-10	m			•	•
II-10	M			•	•

though not necessarily) (P1, J1, J2), cupolae (J3, J4, J5) or pentagonal rotunda (J6) [5]. In this paper, we used CC IV as the initial figure which can be augmented by CC II and by CP II.

The research results on mutual combinations of CC II are shown in Table 1, while the possible combinations of CC II and CC IV are shown in Table 2, comprising the 32 possible cases of matching pairs of concave cupolae. Joining the congruent polygonal faces (bases) of CC IV, CC II and CP II, we obtain polyhedral structures with compositive deltahedral shell, which completely enclose the space confined by the polygonal basis.

The composite polyhedra formed in this manner, which could be (according to Gestalt principles) related to the cupola as a form, i.e., to a dome in the architectural sense are marked

Table 2: Possible combinations of CC II and CC IV

CC		II-6 m	II-6 M	II-7 m	II-7 M	II-8 m	II-8 M	II-9 m	II-9 M	II-10 m	II-10 M
IV-12	Mm	×	×								
IV-12	mm	•	•								
IV-12	mM										
IV-12	MM	•	•								
IV-14	Mm			•	•						
IV-14	mm			•	•						
IV-14	mM			•	•						
IV-14	MM			•	•						
IV-16	Mm					•	•				
IV-16	mm					×	×				
IV-16	mM					•	•				
IV-16	MM					•	•				
IV-18	Mm							×	×		
IV-18	mm							×	×		
IV-18	mM							•	•		
IV-18	MM							•	•		
IV-20	Mm									×	×
IV-20	mm										
IV-20	mM									•	•
IV-20	MM										

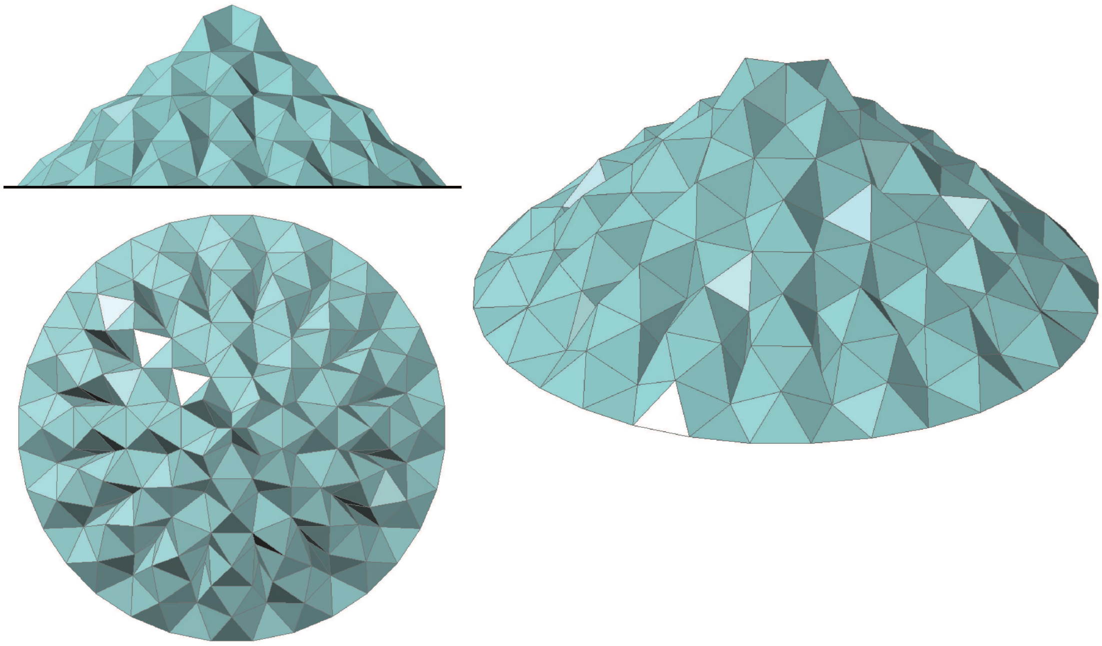

Figure 8: Front view, top view and 3D model of the composite polyhedron obtained by joining CC IV-16mm, CC II-8m and CP II-8B

with • in the Tables 1 and 2. The cases where only a deltahedral lateral surface (not the whole solid) is obtained are marked with ×. Thus formed structure can be further augmented by CP II-A. In the case that the CC II basic polygon is a hexagon, an octagon or a decagon, we can also augment the cupola observed by CP II-B. The representatives of CC IV cannot be cross-combined because in the span of $11 \leq n \leq 21$ there are no representatives A and B such that $2n_A \equiv n_B$.

The composite polyhedral structure consisting of CC IV-16mm, CC II-8m and CP II-8B, is presented in Figure 8.

This paper illustrates just a few examples of composite structures based on the geometry of concave M-m polyhedra, whose form is similar to domes (cupolas) in the architectural sense (Figures 9–11). A complete gallery would require far more space, since all variations, including augmentations and (gyro)-elongations, far exceed the 32 cases given in Tables 1 and 2.

3.2. Elongations

In Johnson's classification of convex polyhedra [5] there are 35 solids obtained by *elongations* or *gyro-elongations* of (bi)pyramids J1 and J2 or (bi)cupolae J3, J4 and J5.

> *Elongation is a method of joining prisms of the congruent bases to the base of the initial solid, while gyro-elongation is the method of joining antiprisms.*

In this paper, we expanded the meaning of elongation by including CA II, with bases congruent to the bases of the initial solids. Since the sides of CA II bases are identical and mutually parallel, we have the cases of (conca-) elongations [13], and not gyro-elongations.

Using the method of elongation, the solids can be additionally extended, resulting in the increase of the surface area / volume of the new structures, which provides larger interior space, or contributes to aesthetically more harmonious proportions. In order to achieve the desired goal: all the lateral surfaces being transformed into deltahedral ones, in this paper we

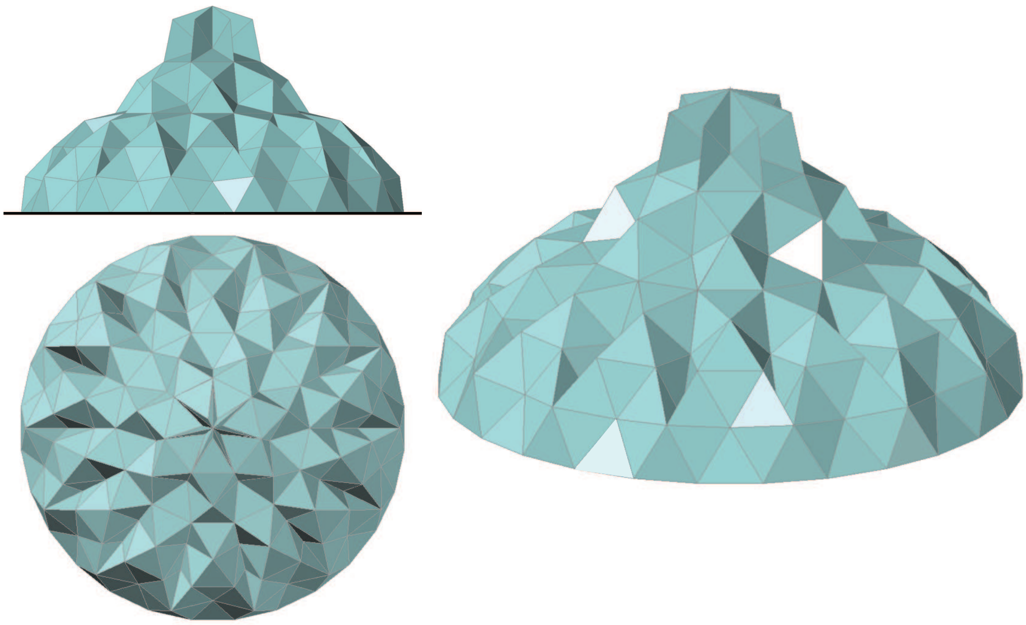

Figure 9: Front view, top view and 3D model of the composite polyhedron obtained by joining CC IV-14mM, CC II-7m and CP II-7MA

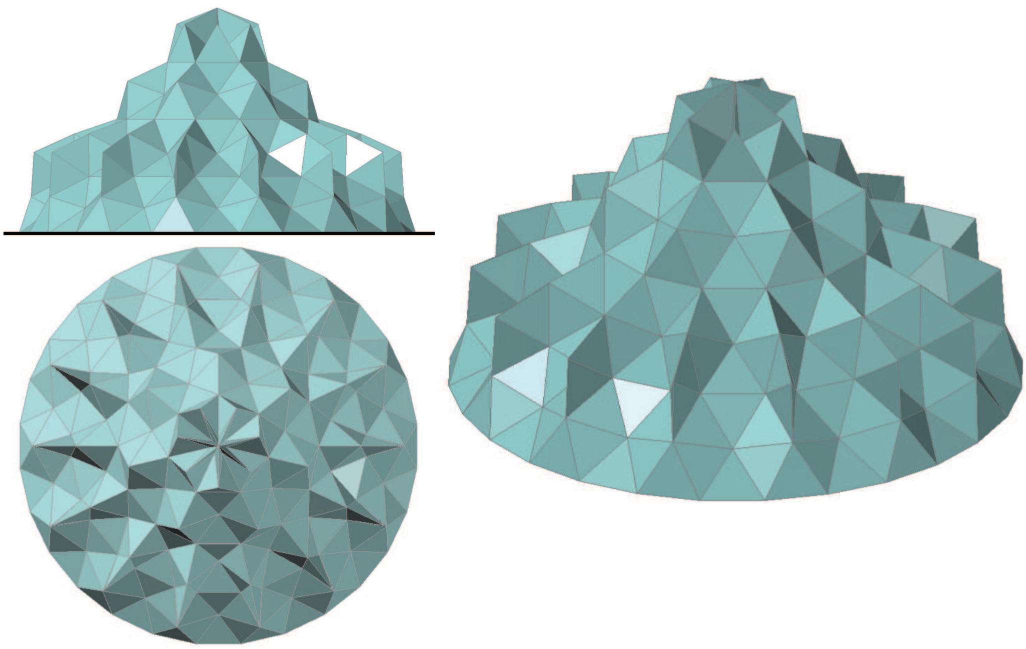

Figure 10: Front view, top view and 3D model of composite polyhedron obtained by joining CC IV-14MM, CC II-7M and CP II-7mA

consider only the cases of elongation by surfaces composed strictly of equilateral triangles:

1. antiprisms (Figure 12),
2. CA II (Figure 13).

The group of polyhedra obtained by elongations and gyro-elongations of CC II is described in [9, 13]. The group of polyhedra obtained by elongations and gyro-elongations of CP II is

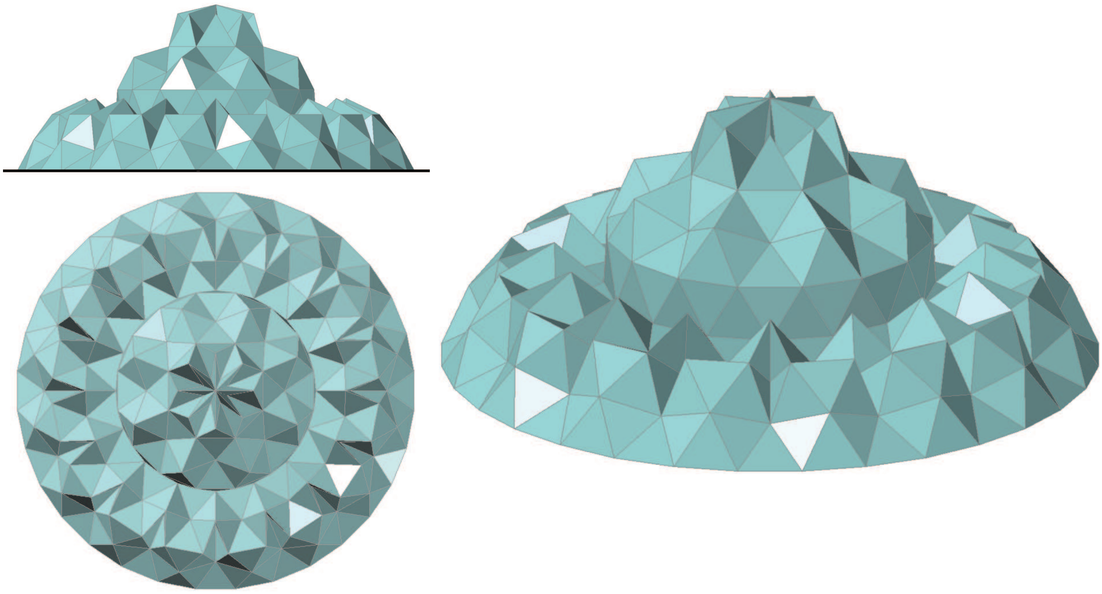

Figure 11: Front view, top view and 3D model of composite polyhedron obtained by joining CC IV-16Mm, AP-16, CC II-8M and CP II-8mA

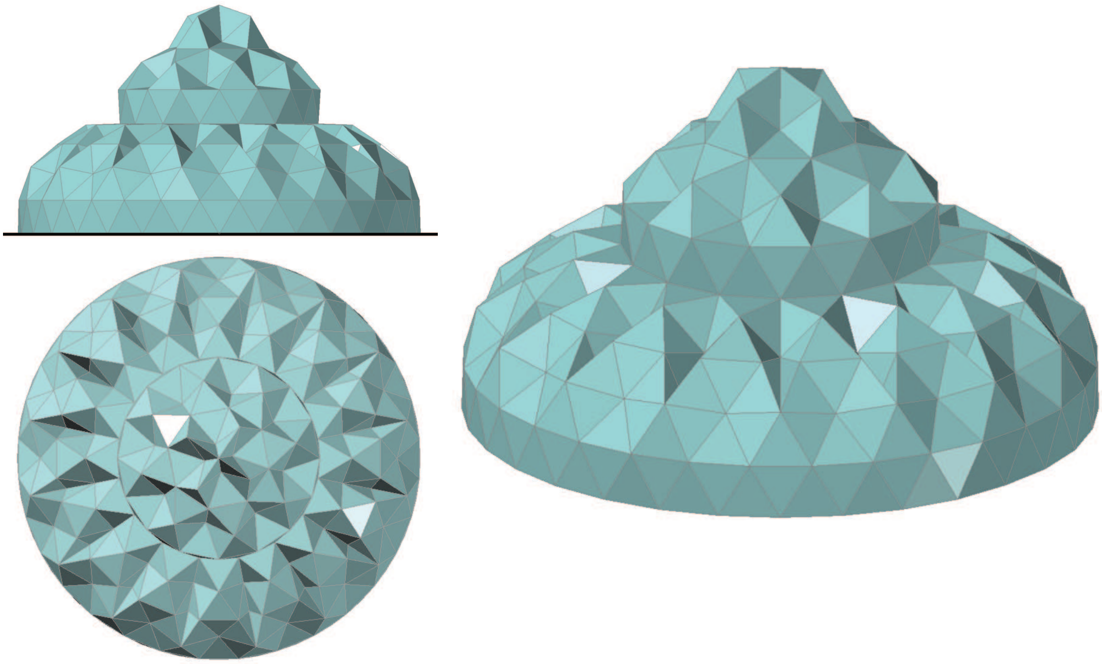

Figure 12: Front view, top view and 3D model of the composite polyhedron obtained by joining AP-32, CC IV-16mM, AP-16, CC II-8m and CP-8B

described in [16].

For elongations of CC IV, CC II and CP II we use deltahedral belts formed over regular polygons congruent to their bases. The composite polyhedron obtained by joining AP-32, CC IV-16mM, AP-16, CC II-8m and CP-8B is presented in Figure 12. The composite polyhedron obtained by joining CA II-32m, CC IV-16mM, CA-16m, CC II-8m, CP-8mA is presented in Figure 13.

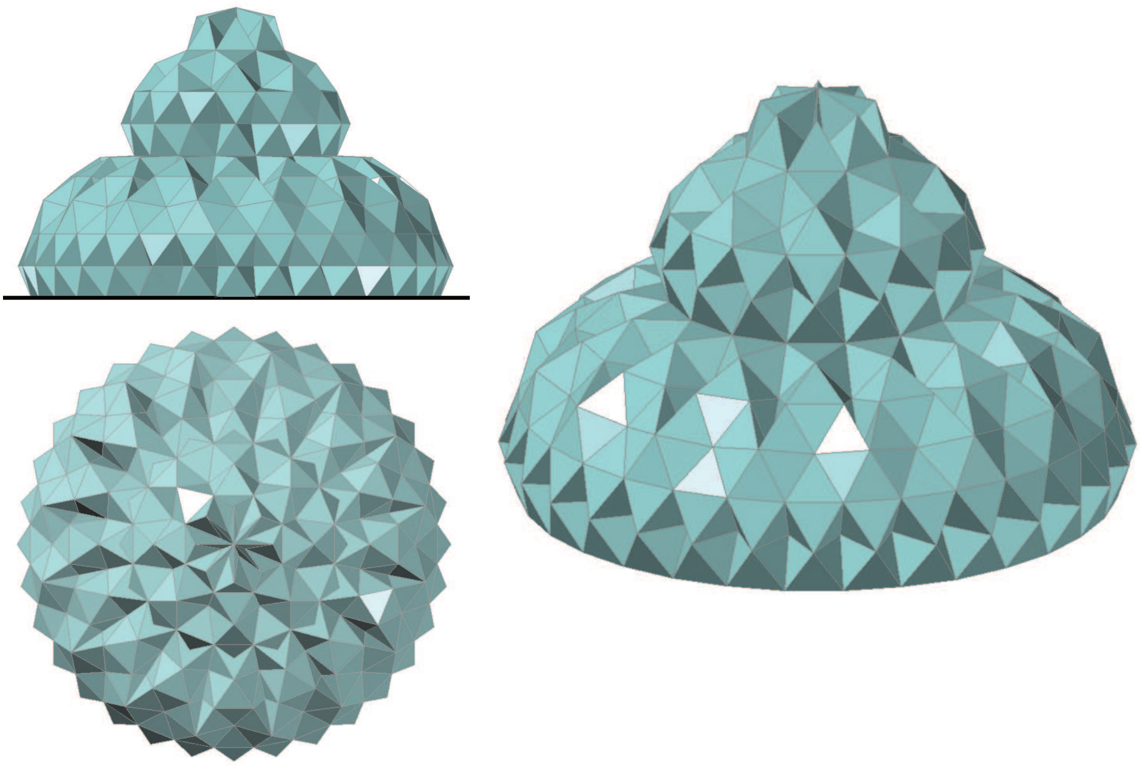

Figure 13: Front view, top view and 3D model of the composite polyhedron obtained by joining CA II-32m, CC IV-16mM, CA-16m, CC II-8m, and CP-8mA

As noted above, over the same polygonal base it is always possible to construct two different types of concave antiprisms of the second sort: CA II-M and CA-II m. Therefore, the observed composite cupola can be elongated in three different ways including antiprisms. Thus the number of so formed composite polyhedral structures increases by factor 3 for each polyhedron participating in the composition.

4. Applicability of composite M-m structures in architectural practice

The research on possibilities of concave cupola's application in architectural practice [9, 11, 14], has revealed a number of advantages. Primarily, there is a deltahedral surface, wherein the presence of equilateral triangles gives a construction the necessary unification, which behoves prefabrication. Due to the congruence, i.e., commonality of the lateral faces in the surface, it is possible to produce serially prefabricated elements which could easily form a structure that follows the geometry of these polyhedra.

The deltahedral shell itself can be formed by mounting triangular panels, frames, reinforced slabs, precast, y-profile rails, or even spatial tetrahedral grids, which allow spanning over a much larger range. Moreover, the triangle as a stable geometric figure enables organization of the structure into geometric form with the exact position of vertices, which further ensures immobility of the construction. The analysis in [14] showed that the structure based on the geometry of the concave cupolae has good dynamic and static properties. Besides, the possible use of a diagrid structural system allows a free organization of the interior space.

One of the deficiencies of concave cupolae as engineering forms is the existence of large flat surfaces of polygonal bases. This was the primary motivation of our research of new composite polyhedral structures completely converted into deltahedral lateral surfaces, whose geometry would come closer to the form of architectural dome. Herewith, the architectural potential of polyhedral structures [6] is once again confirmed.

5. Conclusions

The concave M-m polyhedra, including CC II, CP II, CA II and CC IV with common polygonal bases can be joined, merged, combined and integrated into a whole, which makes them suitable for the formation of composite polyhedral structures. In this paper, we have explored possible solutions whose shape would resemble the dome in the architectural sense. For this purpose, we started from CC IV which we transformed by the procedures of augmentation and (gyro)elongations into complex polyhedral forms. By the augmentations of CC IV and CC II we obtained deltahedral surface which completely encloses interior space. A possible need for greater height or volume can be successfully met by elongation of the resulting structure either by antiprisms or by CA II. In addition, all the desirable geometrical, static and aesthetic traits of concave cupola (as an architectural form) are transferred to the newly created composite polyhedral structure.

Acknowledgments

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References

[1] D.G. Emmerich: *Composite Polyhedra.* Int. J. Space Struct. **5**, 281–296 (1990).

[2] P. Huybers: *Form Generation of Polyhedric Building Shapes.* Int. J. Space Struct. Special issue on Morphology and Architecture, **11**/1, 173–181 (1996).

[3] P. Huybers: *Prism Based Structural Forms.* Eng. Struct. **23**, 12–21 (2001).

[4] P. Huybers: *Polyhedroids.* In R. Motro (ed.): *An Anthology of Structural Morphology*, World Scientific Publishing Co. Pte. Ltd., Singapore 2009, pp. 49–62.

[5] N.W. Johnson: *Convex Solids with Regular Faces.* Canad. J. Math. **18**, 169–200 (1966).

[6] I.J. Markov, J.F. Gabriel: *Spatial and structural aspects of polyhedra.* Eng. Struct. **23**, 4–11 (2001).

[7] K. Miura: *Proposition of Pseudo-Cylindrical Concave Polyhedral Shells.* Institute of Space and Aeronautical Science, University of Tokyo, Report No. 442, Tokyo 1969.

[8] S. Mišić, M. Obradović: *Forming the cupolae with concave polihedral surfaces by corrugating a fourfold strip of equilateral triangles.* Proc. 2nd Internat. Scientific Conference moNGeometrija, Belgrade 2010, pp. 363–374.

[9] S. Mišić: *Constructive geometric generating of cupolae with concave polyhedral surfaces.* PhD Thesis, University of Belgrade, Faculty of Architecture, 2012.

[10] S. Mišić, M. Obradović, B. Popkonstantinović: *The structural transformation of concave cupolae of fourth sort using different variants of constructive procedure.* Proc. 4th Internat. Scientific Conference moNGeometrija, Vlasina/Serbia 2014, pp. 148–158.

[11] M. Obradović: *Constructive-geometrial elaboration on toroidal deltahedra with regular polygonal bases*. PhD Thesis, University of Belgrade, Faculty of Architecture, 2006.

[12] M. Obradović, S. Mišić: *Concave Regular Faced Cupolae of Second Sort*. Proc. 13th ICGG – International Conference on Geometry and Graphics, Dresden/Germany 2008.

[13] M. Obradović: *A group of polyhedra arised as variations of concave bicupolae of second sort*. Proc. 3rd Internat. Scientific Conference moNGeometrija, Novi Sad/Serbia 2012, pp. 95–108.

[14] M. Obradović, S. Mišić, B. Popkonstantinović, M. Petrović, B. Malešević, R. Obradović: *Investigation of Concave Cupolae Based Polyhedral Structures and Their Potential Application in Architecture*. Technics Technologies Education Management **8**, no. 3, 1198–1214 (2013).

[15] M. Obradović, B. Popkonstantinović, S. Mišić: *On the Properties of the Concave Antiprisms of Second Sort*. FME Transactions **41**/3, 256–263 (2013).

[16] M. Obradović, S. Mišić, B. Popkonstantinović: *Concave pyramids of second sort – the occurrence, types, variations*. Proc. 4th Internat. Scientific Conference moNGeometrija, Vlasina / Serbia 2014, pp. 159–170.

[17] A.V. Timofeenko: *The non-Platonic and non-Archimedian noncomposite polyhedral*. J. Math. Sci. (N. Y.) **162**, no. 5, 710–729 (2009).

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