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The Extended Oloid and Its Contacting Quadrics

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Abstract. The oloid is the convex hull of two circles with equal radius in perpendicular planes so that the center of each circle lies on the other circle. It is part of a developable surface which we call *extended oloid*. We determine the tangential system of all contacting quadrics $\mathcal{Q}_\lambda$ of the extended oloid $\mathcal{O}$ where λ is the system parameter. From this result we conclude parameter equations of the touching curve $\mathcal{C}_\lambda$ between $\mathcal{O}$ and $\mathcal{Q}_\lambda$, and of the edge of regression of $\mathcal{O}$. Properties of the curves $\mathcal{C}_\lambda$ are investigated, including the case that $\lambda \to \infty$. The self-polar tetrahedron of the tangential system $\mathcal{Q}_\lambda$ is obtained. The common generating lines of $\mathcal{O}$ and any ruled surface $\mathcal{Q}_\lambda$ are determined. Furthermore, we derive the curves which are the images of $\mathcal{C}_\lambda$ when $\mathcal{O}$ is developed onto the plane.

Key Words: Oloid, developable surface, torse, tangential system of quadrics, touching curve, edge of regression, self-polar tetrahedron, ruled surface

MSC 2010: 53A05, 51N20, 51N15

1. Introduction

The *oloid* was discovered by Paul Schatz in 1929. It is the convex hull of two circles with equal radius r in perpendicular planes so that the center of each circle lies on the other circle. The oloid has the remarkable properties that its surface area is equal to $4\pi r^2$ and the two circles terminate on each generator a segment of the same length $r\sqrt{3}$. The boundary of the oloid is part of a developable surface [3, 9].

In the following this developable surface is called *extended oloid*. According to [3, pp. 105-106], in an appropriate cartesian coordinate frame the circles with $r = 1$ can be defined by

$$\begin{aligned} k_A &:= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + (y+1/2)^2 = 1 \wedge z = 0\right\}, \\ k_B &:= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; (y-1/2)^2 + z^2 = 1 \wedge x = 0\right\}. \end{aligned}$$

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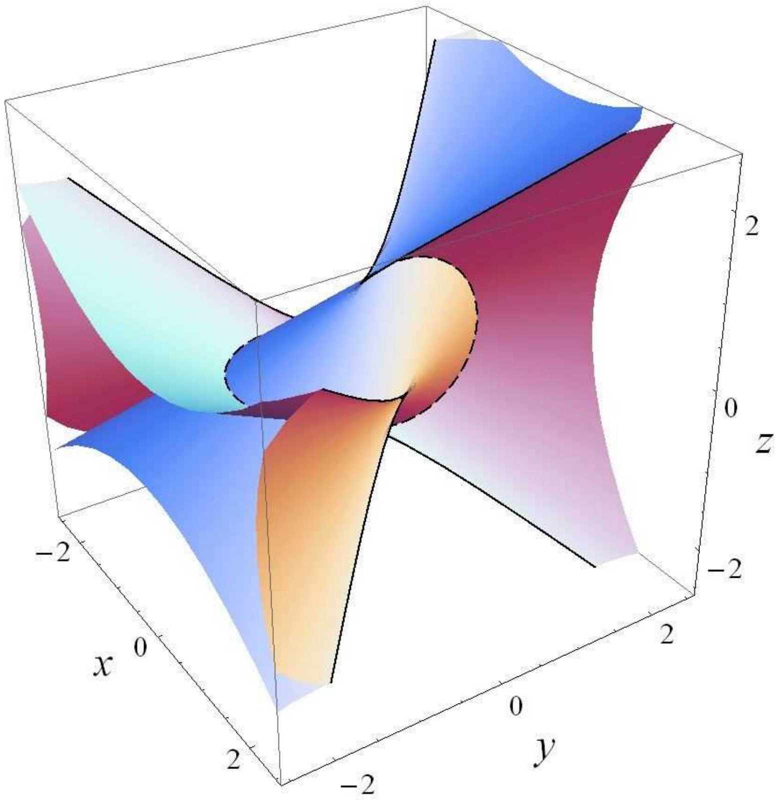

Figure 1: The extended oloid $\mathcal{O}$, the circles k_A, k_B (dashed lines), and the edge of regression $\mathcal{R}$ (solid lines) in the box $-2.5 \le x, y, z \le 2.5$

In this case we denote the extended oloid by $\mathcal{O}$ (see Figure 1).

Now we introduce homogeneous coordinates x_0, x_1, x_2, x_3 with $x = x_1/x_0$, $y = x_2/x_0$, $z = x_3/x_0$. Then the real projective space — as a set of points $P = [x_0, x_1, x_2, x_3]$ — is given by

$$\mathbb{P}_3(\mathbb{R}) = \left\{[x_0, x_1, x_2, x_3] \mid (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \setminus \{0\}\right\},$$

where $[x_0, x_1, x_2, x_3] = [y_0, y_1, y_2, y_3]$ if there exist a $\mu \in \mathbb{R} \setminus \{0\}$ such that $x_j = \mu y_j$ for $j = 0, 1, 2, 3$. If necessary, complex coordinates will be used instead of the real ones. For the description of the corresponding projective circles $\mathcal{K}_A$ and $\mathcal{K}_B$ to k_A and k_B, respectively, we write

$$\mathcal{K}_A = \left\{[x_0, x_1, x_2, x_3] \in \mathbb{P}_3(\mathbb{R}) \;\middle|\; 3x_0^2 - 4x_0x_2 - 4x_1^2 - 4x_2^2 = 0 \wedge x_3 = 0\right\},$$
$$\mathcal{K}_B = \left\{[x_0, x_1, x_2, x_3] \in \mathbb{P}_3(\mathbb{R}) \;\middle|\; 3x_0^2 + 4x_0x_2 - 4x_2^2 - 4x_3^2 = 0 \wedge x_1 = 0\right\}.$$

2. Contacting quadrics

Lemma 1. *The sets of the tangent planes $\bar{u} = [u_0, \ldots, u_3]$ of the circles $\mathcal{K}_A$ and $\mathcal{K}_B$ satisfy the respective equations*

$$F_0(\bar{u}) = 4u_0^2 - 4u_0u_2 - 4u_1^2 - 3u_2^2 = 0 \quad \textit{and} \quad F_1(\bar{u}) = 4u_0^2 + 4u_0u_2 - 3u_2^2 - 4u_3^2 = 0\,.$$

Proof. Instead of following [4, pp. 41–42] or [5, pp. 160, 164–165], we conclude as follows: The plane $[u_0, u_1, u_2, u_3]$ is tangent to the circle $\mathcal{K}_A$ if and only if its intersection with the plane

$x_3 = 0$ satisfies the tangential equation of $\mathcal{K}_A$. Therefore, the coefficient matrix of $F_0(\bar{u})$ is obtained — up to a real factor $\neq 0$ — by inverting the symmetric 3×3 coefficient matrix of the equation of $\mathcal{K}_A$,

$$\begin{pmatrix} 3 & 0 & -2 \\ 0 & -4 & 0 \\ -2 & 0 & -4 \end{pmatrix}^{-1} = \frac{1}{16} \begin{pmatrix} 4 & 0 & -2 \\ 0 & -4 & 0 \\ -2 & 0 & -3 \end{pmatrix},$$

hence $F_0(\bar{u}) = 4u_0^2 - 4u_0u_2 - 4u_1^2 - 3u_2^2$. Analogously, one finds $F_1(\bar{u})$ from the coefficient matrix of $\mathcal{K}_B$. □

Remark 1. If the coordinates $u_0, \ldots, u_3$ are considered as homogenized cartesian coordinates $x_0, \ldots, x_3$ (cf. [1, p. 139]), then the equations in Lemma 1 define elliptic cylinders. Their inhomogeneous equations are

$$\frac{x^2}{\left(2/\sqrt{3}\right)^2} + \frac{(y+2/3)^2}{(4/3)^2} = 1 \qquad \text{and} \qquad \frac{(y-2/3)^2}{(4/3)^2} + \frac{z^2}{\left(2/\sqrt{3}\right)^2} = 1\,.$$

This means that the extended oloid, which is the connecting torse of the two circles k_A and k_B, is dual to the curve of intersection of two elliptic cylinders. The equations in Lemma 1 were already given in [3, p. 115].

Theorem 2. *All regular quadrics which contact the extended oloid $\mathcal{O}$ along a curve are given by*

$$\mathcal{Q}_\lambda = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; f_\lambda(x,y,z) = 0\right\},$$

where

$$f_\lambda(x,y,z) = \begin{cases} \dfrac{x^2}{1-\lambda} + \dfrac{(y-\lambda+1/2)^2}{1-\lambda+\lambda^2} + \dfrac{z^2}{\lambda} - 1 & \text{if } \lambda \in \mathbb{R} \setminus \{0,1\}\,, \\ x^2 - z^2 + 2y & \text{if } \lambda = \infty\,. \end{cases}$$

Proof. Because of the homogeneity of the coordinates we may write the tangential equations of all contacting quadrics in question as linear combinations of $F_0(\bar{u})$ and $F_1(\bar{u})$ (see Lemma 1),

$$F_\lambda(\bar{u}) := (1-\lambda)\,F_0(\bar{u}) + \lambda\,F_1(\bar{u}) \quad \text{or} \quad F_\infty(\bar{u}) := -F_0(\bar{u}) + F_1(\bar{u}).$$

The formula for $F_\infty(\bar{u})$ is the limit of $F_\lambda(\bar{u})/\lambda$ for $\lambda \to \infty$. The homogeneous point equation $\tilde{f}_\lambda(x_0,x_1,x_2,x_3) = 0$ of any regular $\mathcal{Q}_\lambda$ is obtained — up to a real factor $\neq 0$ — by inverting the symmetric coefficent matrix of $F_\lambda(\bar{u})$,

$$\begin{pmatrix} 4 & 0 & 2(2\lambda-1) & 0 \\ 0 & 4(\lambda-1) & 0 & 0 \\ 2(2\lambda-1) & 0 & -3 & 0 \\ 0 & 0 & 0 & -4\lambda \end{pmatrix}^{-1} = -\frac{1}{4} \begin{pmatrix} -\dfrac{3}{4(1-\lambda+\lambda^2)} & 0 & \dfrac{1-2\lambda}{2(1-\lambda+\lambda^2)} & 0 \\ 0 & \dfrac{1}{1-\lambda} & 0 & 0 \\ \dfrac{1-2\lambda}{2(1-\lambda+\lambda^2)} & 0 & \dfrac{1}{1-\lambda+\lambda^2} & 0 \\ 0 & 0 & 0 & \dfrac{1}{\lambda} \end{pmatrix},$$

hence

$$\begin{aligned} \tilde{f}_\lambda(x_0,x_1,x_2,x_3) &= -\frac{3x_0^2}{4(1-\lambda+\lambda^2)} + \frac{x_1^2}{1-\lambda} + \frac{x_2^2}{1-\lambda+\lambda^2} + \frac{x_3^2}{\lambda} + \frac{(1-2\lambda)x_0x_2}{1-\lambda+\lambda^2} \\ &= \frac{x_1^2}{1-\lambda} + \frac{\left(x_2 + \left(\frac{1}{2}-\lambda\right)x_0\right)^2}{1-\lambda+\lambda^2} + \frac{x_3^2}{\lambda} - x_0^2\,, \end{aligned} \tag{1}$$

and

$$\tilde{f}_\infty(x_0,x_1,x_2,x_3) = \lim_{\lambda\to\infty} \lambda \tilde{f}_\lambda(x_0,x_1,x_2,x_3) = -(x_1^2 + 2x_0x_2 - x_3^2)\,.$$

We write the equations in inhomogeneous coordinates as

$$\begin{aligned} f_\lambda(x,y,z) &= \tilde{f}_\lambda(1,x,y,z) = \frac{x^2}{1-\lambda} + \frac{(y-\lambda+1/2)^2}{1-\lambda+\lambda^2} + \frac{z^2}{\lambda} - 1\,,\\ f_\infty(x,y,z) &= -\tilde{f}_\infty(1,x,y,z) = x^2 - z^2 + 2y\,. \end{aligned}$$

□

Corollary 3. *a) Every quadric $\mathcal{Q}_\lambda$ and the extended oloid $\mathcal{O}$ are symmetric with respect to the planes $x = 0$ and $z = 0$.*
b) The axial symmetries $(x,y,z) \mapsto (z,-y,x)$ and $(x,y,z) \mapsto (-z,-y,-x)$ exchange the quadrics $\mathcal{Q}_\lambda$ and $\mathcal{Q}_{1-\lambda}$.

Proof. a) Since $f_\lambda(-x,y,z) = f_\lambda(x,y,z)$ and $f_\lambda(x,y,-z) = f_\lambda(x,y,z)$, every quadric is symmetric with respect to $x = 0$ and $z = 0$. The symmetry of $\mathcal{O}$ with respect to these planes follows immediately.

b) From

$$\begin{aligned} f_{1-\lambda}(x,y,z) &= \frac{x^2}{1-(1-\lambda)} + \frac{(y-(1-\lambda)+1/2)^2}{1-(1-\lambda)+(1-\lambda)^2} + \frac{z^2}{1-\lambda} - 1\\ &= \frac{z^2}{1-\lambda} + \frac{(-y-\lambda+1/2)^2}{1-\lambda+\lambda^2} + \frac{x^2}{\lambda} - 1 = f_\lambda(z,-y,x) \end{aligned}$$

and $f_\lambda(x,y,z) = f_{1-\lambda}(z,-y,x)$ follows that the isometry $(x,y,z) \mapsto (z,-y,x)$ exchanges the quadrics $\mathcal{Q}_\lambda$ and $\mathcal{Q}_{1-\lambda}$, while all points $(t,0,t)$ with $t \in \mathbb{R}$ remain fixed. The same holds for $(x,y,z) \mapsto (-z,-y,-x)$, which fixes all points $(t,0,-t)$. □

Table 1: Types of contacting quadrics $\mathcal{Q}_\lambda$

Parameter	*Equation of* $\mathcal{Q}_\lambda$	*Type of* $\mathcal{Q}_\lambda$
$\lambda < 0$	$\frac{x^2}{a^2} + \frac{(y-\lambda+1/2)^2}{b^2} - \frac{z^2}{c^2} = 1$	Hyperboloid of one sheet
$\lambda = 0$	$x^2 + (y+1/2)^2 = 1 \wedge z = 0$	Circle k_A
$0 < \lambda < 1$	$\frac{x^2}{a^2} + \frac{(y-\lambda+1/2)^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipsoid
$\lambda = 1$	$(y-1/2)^2 + z^2 = 1 \wedge x = 0$	Circle k_B
$\lambda > 1$	$\frac{(y-\lambda+1/2)^2}{b^2} + \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$	Hyperboloid of one sheet
$\lambda = \infty$	$x^2 - z^2 + 2y = 0$	Hyperbolic paraboloid

Table 1 shows the classification of the quadrics $\mathcal{Q}_\lambda$, $\lambda \in \mathbb{R} \cup \{\infty\}$, in the Euclidean space. In this table the abbreviations $a^2 := |1-\lambda|$, $b^2 := 1-\lambda+\lambda^2 > 0$ and $c^2 := |\lambda|$ are used.

A point of the circle k_A is given by

$$A = \big(\alpha_1(t), \alpha_2(t), \alpha_3(t)\big) \quad \text{with} \quad \alpha_1(t) = \sin t\,,\ \ \alpha_2(t) = -\frac{1}{2} - \cos t\,,\ \ \alpha_3(t) = 0\,.$$

There are two points B_1, B_2 of the circle k_B such that the connecting lines AB_1 and AB_2 are generators of $\mathcal{O}$:

$$B_1 = \big(\beta_1(t), \beta_2(t), \beta_3(t)\big)\,, \quad B_2 = \big(\beta_1(t), \beta_2(t), -\beta_3(t)\big)\,,$$

where

$$\beta_1(t) = 0\,, \quad \beta_2(t) = \frac{1}{2} - \frac{\cos t}{1+\cos t}\,, \quad \beta_3(t) = \frac{\sqrt{1+2\cos t}}{1+\cos t}$$

(see [3, pp. 106–107]). Hence, for fixed $t \in [-2\pi/3, 2\pi/3]$, a parametrization of the line AB_1 is

$$\omega_i(m,t) := (1-m)\alpha_i(t) + m\beta_i(t),\ i = 1,2,3,\ \text{with } m \in \mathbb{R}\,.$$

We obtain

$$\left.\begin{aligned} \omega_1(m,t) &= (1-m)\sin t, \\ \omega_2(m,t) &= \frac{2(m-1)\cos^2 t + (2m-3)\cos t + 2m - 1}{2(1+\cos t)}\,, \\ \omega_3(m,t) &= \frac{m\sqrt{1+2\cos t}}{1+\cos t}\,. \end{aligned}\right\} \tag{2}$$

Consequently,

$$\left.\begin{aligned} &1)\ x = \omega_1(m,t), \quad y = \omega_2(m,t), \quad z = \omega_3(m,t)\,, \\ &2)\ x = \omega_1(m,t), \quad y = \omega_2(m,t), \quad z = -\omega_3(m,t)\,, \end{aligned}\right\}\ t \in [-2\pi/3, 2\pi/3],\ m \in \mathbb{R}\,,$$

is a parametrization of $\mathcal{O}$ with the generators as m-lines. The restriction of m to the interval $[0,1]$ yields the oloid in the narrow sense, which is the convex hull of k_A and k_B.

Corollary 4. *For a fixed value of $\lambda \in \mathbb{R}$, we can parametrize the touching curve $\mathcal{C}_\lambda$ between $\mathcal{O}$ and $\mathcal{Q}_\lambda$ as*

$$\gamma(\lambda,\cdot):\ [-2\pi/3,\ 2\pi] \to \mathbb{R}^3\,, \quad t \mapsto \gamma(\lambda,t) = \begin{cases} \gamma_1(\lambda,t) & \text{if } t \in [-2\pi/3,\ 2\pi/3]\,, \\ \gamma_2(\lambda,t) & \text{if } t \in (2\pi/3,\ 2\pi]\,, \end{cases}$$

where

$$\begin{aligned} \gamma_1(\lambda,t) &= \big(\kappa_1(\lambda,t),\ \kappa_2(\lambda,t),\ \kappa_3(\lambda,t)\big)\,, \\ \gamma_2(\lambda,t) &= \big(\kappa_1(\lambda,\ 4\pi/3 - t),\ \kappa_2(\lambda,\ 4\pi/3 - t),\ -\kappa_3(\lambda,\ 4\pi/3 - t)\big)\,, \end{aligned}$$

and

$$\kappa_1(\lambda,t) = \frac{(1-\lambda)\sin t}{1+\lambda\cos t}\,, \quad \kappa_2(\lambda,t) = \frac{2\lambda - 1 + (\lambda-2)\cos t}{2(1+\lambda\cos t)}\,, \quad \kappa_3(\lambda,t) = \frac{\lambda\sqrt{1+2\cos t}}{1+\lambda\cos t}\,.$$

Proof. For a fixed t the generator

$$\mathcal{L}_t := \big\{(x,y,z) \in \mathbb{R}^3 \;\big|\; x = \omega_1(m,t),\ y = \omega_2(m,t),\ z = \omega_3(m,t);\ m \in \mathbb{R}\big\}$$

of $\mathcal{O}$ is tangent to $\mathcal{Q}_\lambda$ for one value $\widetilde{m}$ of m. As a double root of the equation

$$f_\lambda(\omega_1(m,t), \omega_2(m,t), \omega_3(m,t)) = 0$$

we find

$$\widetilde{m} = \psi(\lambda,t) := \frac{\lambda(1+\cos t)}{1+\lambda\cos t}\,,$$

and consequently

$$\omega_1(\psi(\lambda,t),t) = \frac{(1-\lambda)\sin t}{1+\lambda\cos t}\,,\quad \omega_2(\psi(\lambda,t),t) = \frac{2\lambda-1+(\lambda-2)\cos t}{2(1+\lambda\cos t)}\,,$$
$$\omega_3(\psi(\lambda,t),t) = \frac{\lambda\sqrt{1+2\cos t}}{1+\lambda\cos t}\,.$$

We put $\kappa_j(\lambda,t) := \omega_j(\psi(\lambda,t),t)$ for $j=1,2,3$. This yields

$$\gamma_1(\lambda,t) := \big(\kappa_1(\lambda,t),\,\kappa_2(\lambda,t),\,\kappa_3(\lambda,t)\big)$$

as the contact point between $\mathcal{L}_t$ and $\mathcal{Q}_\lambda$ for all lines $\mathcal{L}_t$ with $t\in[-2\pi/3,\,2\pi/3]$. Due to the symmetry of $\mathcal{O}$ with respect to the plane $z=0$, we have

$$\gamma_2(\lambda,t) := \Big(\kappa_1\Big(\lambda,\,\frac{4\pi}{3}-t\Big),\;\kappa_2\Big(\lambda,\,\frac{4\pi}{3}-t\Big),\;-\kappa_3\Big(\lambda,\,\frac{4\pi}{3}-t\Big)\Big)$$

if $t\in(2\pi/3,\,2\pi]$. Obviously,

$$\gamma_2(\lambda,2\pi/3) = \gamma_1(\lambda,2\pi/3) \quad\text{and}\quad \gamma_2(\lambda,2\pi) = \gamma_1(\lambda,-2\pi/3)$$

for every $\lambda\in\mathbb{R}$. □

Examples with contacting quadric and touching curves are shown in the Figures 2 and 3.

Remark 2. In the special case $\lambda = 1/2$ one gets the equations of the central inscribed ellipsoid, as mentioned in [3, p. 115, Eq. (27)].

3. Properties of the touching curves $\mathcal{C}_\lambda$

From Corollary 3 follows that all touching curves $\mathcal{C}_\lambda$ are symmetric with respect to the planes $x=0$ and $z=0$. We denote by X_1, X_2 the intersection points of $\mathcal{C}_\lambda$ with the plane $x=0$, and by Z_1, Z_2 those with the plane $z=0$, and find

$$\left.\begin{aligned} X_1(\lambda) &= \gamma(\lambda,\,0) = \Big(0,\,-\frac{3(1-\lambda)}{2(1+\lambda)},\,\frac{\sqrt{3}\,\lambda}{1+\lambda}\Big),\quad Z_1(\lambda) = \gamma\big(\lambda,\,\tfrac{2\pi}{3}\big) = \Big(\frac{\sqrt{3}\,(1-\lambda)}{2-\lambda},\,\frac{3\lambda}{2(2-\lambda)},\,0\Big),\\ X_2(\lambda) &= \gamma\big(\lambda,\,\tfrac{4\pi}{3}\big) = \Big(0,\,-\frac{3(1-\lambda)}{2(1+\lambda)},\,-\frac{\sqrt{3}\,\lambda}{1+\lambda}\Big),\; Z_2(\lambda) = \gamma(\lambda,2\pi) = \Big(-\frac{\sqrt{3}\,(1-\lambda)}{2-\lambda},\,\frac{3\lambda}{2(2-\lambda)},\,0\Big). \end{aligned}\right\} \tag{3}$$

One easily finds a parametrization $\mathcal{T}_\lambda$ of the tangent to $\mathcal{C}_\lambda$ at the point $\gamma(\lambda,t)$:

$$\mathcal{T}_\lambda(t) = \left\{\begin{array}{ll} \big\{(x,y,z)\in\mathbb{R}^3\,\big|\,x=\tau_1(\lambda,t,\mu),\;y=\tau_2(\lambda,t,\mu), & \\ \quad z=\tau_3(\lambda,t,\mu);\;\mu\in\mathbb{R}\big\} & \text{if}\quad t\in[-2\pi/3,\,2\pi/3],\\ \big\{(x,y,z)\in\mathbb{R}^3\,\big|\,x=\tau_1(\lambda,\,\tfrac{4\pi}{3}-t,\,\mu),\;y=\tau_2(\lambda,\,\tfrac{4\pi}{3}-t,\,\mu), & \\ \quad z=-\tau_3(\lambda,\,\tfrac{4\pi}{3}-t,\,\mu);\;\mu\in\mathbb{R}\big\} & \text{if}\quad t\in(2\pi/3,\,2\pi], \end{array}\right\} \tag{4}$$

with

$$\tau_j(\lambda,t,\mu) = \kappa_j(\lambda,t) + \mu\,\dot\kappa_j(\lambda,t)\,,\quad \dot\kappa_j = \frac{\mathrm{d}\kappa_j(\lambda,t)}{\mathrm{d}t}\quad\text{for}\quad j=1,2,3\,,$$

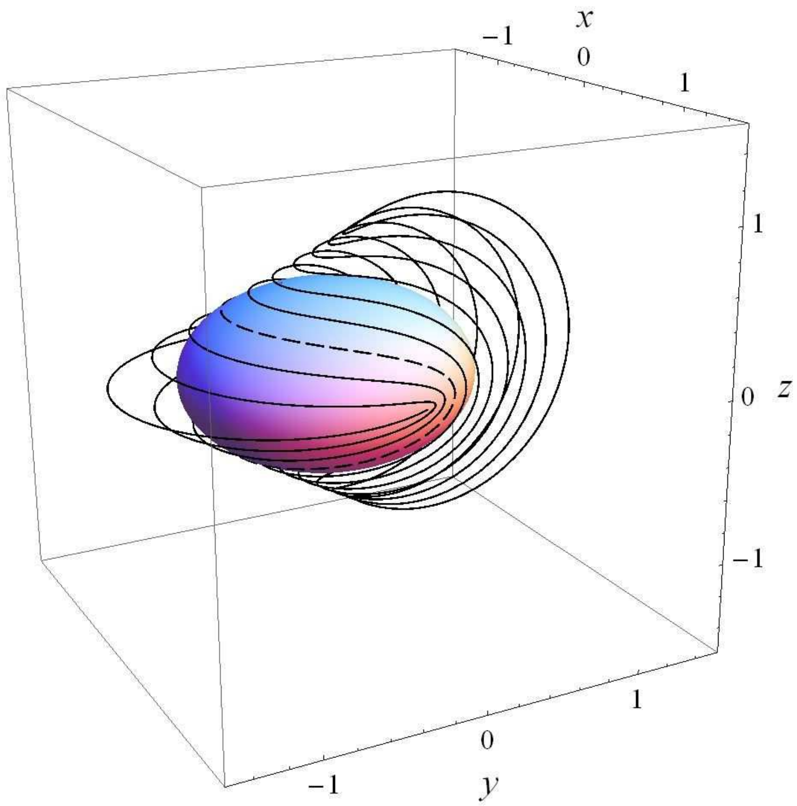

Figure 2: Touching curves $\mathcal{C}_\lambda$ for $\lambda = 0,\, 0.1,\, 0.2, \ldots,\, 0.9,\, 1$, and the ellipsoid $\mathcal{Q}_{0.3}$ in the box $-1.5 \le x, y, z \le 1.5$; the dashed line belongs to $\mathcal{C}_{0.3}$.

where

$$\dot{\kappa}_1(\lambda, t) = \frac{(1-\lambda)(\lambda + \cos t)}{(1+\lambda\cos t)^2}\,, \quad \dot{\kappa}_2(\lambda, t) = \frac{(1-\lambda+\lambda^2)\sin t}{(1+\lambda\cos t)^2}\,,$$
$$\dot{\kappa}_3(\lambda, t) = \frac{\lambda[\lambda(1+\cos t) - 1]\sin t}{(1+\lambda\cos t)^2\sqrt{1+2\cos t}}\,.$$

Now we consider the function $1 + \lambda\cos t$ in the denominators of $\kappa_1, \kappa_2, \kappa_3$. It vanishes in the interval $[-2\pi/3,\, 2\pi/3]$ for $t = \pm\arccos(-1/\lambda)$ if $\lambda \in \mathbb{R}\setminus(-1, 2)$. (For $\lambda = -1$, we have $t = 0$; and for $\lambda = 2$, $t = \pm 2\pi/3$.) $1+\lambda\cos t$ has no zeros in $[-2\pi/3,\, 2\pi/3]$ if $\lambda \in (-1, 2)$. Therefore, $\kappa_1, \kappa_2, \kappa_3$ are continuous functions if $\lambda \in (-1, 2)$; they are not continuous if $\lambda \in \mathbb{R}\setminus(-1, 2)$. So we have to distinguish the following cases:

Case 1, $-1 < \lambda < 2$: Since $\kappa_1, \kappa_2, \kappa_3$ are continuous functions, and

$$\gamma_2(\lambda, 2\pi/3) = Z_1(\lambda) = \gamma_1(\lambda, 2\pi/3), \quad \gamma_2(\lambda, 2\pi) = Z_2(\lambda) = \gamma_1(\lambda, -2\pi/3),$$

the curve $\mathcal{C}_\lambda$ is closed (see Figure 2).

Case 2, $\lambda \in \mathbb{R}\setminus[-1, 2]$: The parametrization $\gamma(\lambda, t)$ of $\mathcal{C}_\lambda$ has poles for

$$t_1 = -\arccos\left(-\frac{1}{\lambda}\right),\; t_2 = \arccos\left(-\frac{1}{\lambda}\right),\; t_3 = \frac{4\pi}{3} - \arccos\left(-\frac{1}{\lambda}\right),\; t_4 = \frac{4\pi}{3} + \arccos\left(-\frac{1}{\lambda}\right).$$

Therefore, $\mathcal{C}_\lambda$ consists of four branches. We calculate the asymptotes of $\mathcal{C}_\lambda$ at the poles. For $t \in [-2\pi/3,\, 2\pi/3]$ the tangent $\mathcal{T}_\lambda(t)$ intersects the plane $x = 0$ at the point with the

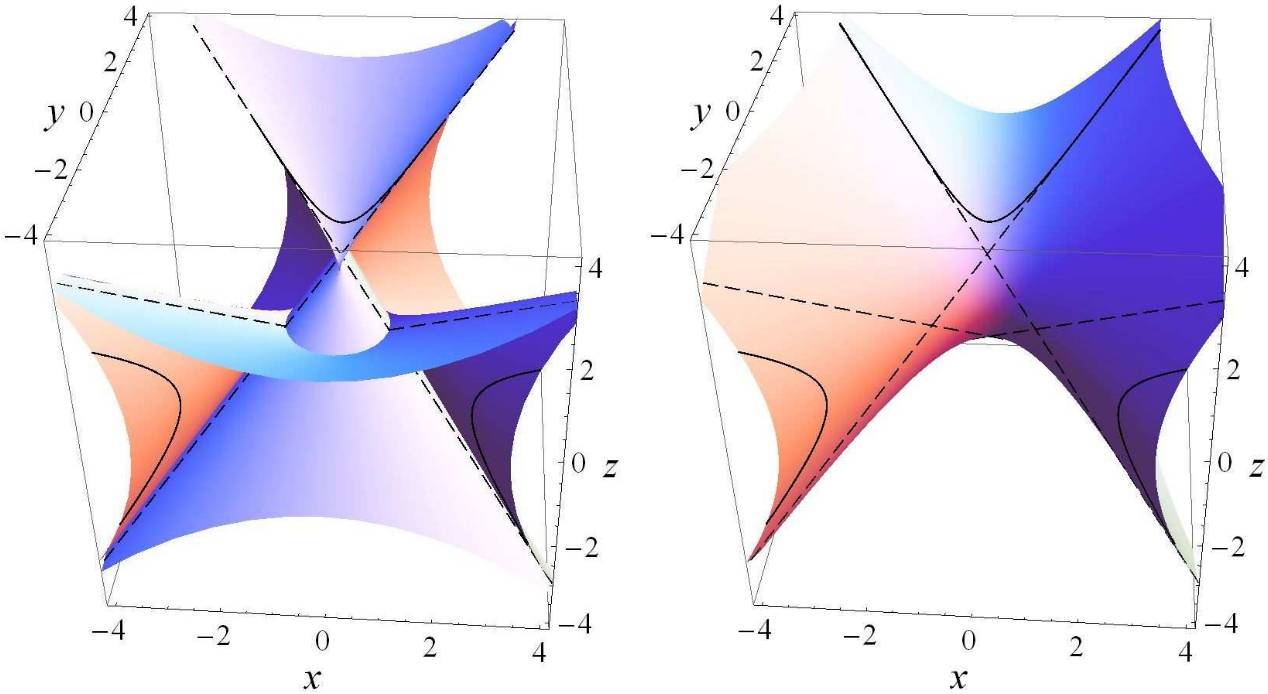

Figure 3: Extended oloid $\mathcal{O}$ (left) and hyperboloid $\mathcal{Q}_4$ (right) together with the touching curve $\mathcal{C}_4$ (solid lines) and the four common generators $\mathcal{G}_j(4)$, $j = 1, 2, 3, 4$, (dashed lines) within the box $-4 \le x, y, z \le 4$.

coordinates

$$y = \kappa_2(\lambda, t) - \frac{\dot\kappa_2(\lambda, t)}{\dot\kappa_1(\lambda, t)}\, \kappa_1(\lambda, t) = \frac{\lambda - 2 + (2\lambda - 1)\cos t}{2(\lambda + \cos t)}\,,$$

$$z = \kappa_3(\lambda, t) - \frac{\dot\kappa_3(\lambda, t)}{\dot\kappa_1(\lambda, t)}\, \kappa_1(\lambda, t) = \frac{\lambda(1 + \cos t + \cos^2 t)}{(\lambda + \cos t)\sqrt{1 + 2\cos t}}\,,$$

and $z = 0$ at

$$x = \kappa_1(\lambda, t) - \frac{\dot\kappa_1(\lambda, t)}{\dot\kappa_3(\lambda, t)}\, \kappa_3(\lambda, t) = \frac{(\lambda - 1)(1 + \cos t + \cos^2 t)}{[\lambda(1 + \cos t) - 1]\sin t}\,,$$

$$y = \kappa_2(\lambda, t) - \frac{\dot\kappa_2(\lambda, t)}{\dot\kappa_3(\lambda, t)}\, \kappa_3(\lambda, t) = -\frac{1 + \lambda + (2 - \lambda)\cos t}{2[\lambda(1 + \cos t) - 1]}\,.$$

For $t = t_1$ one finds that the asymptote $\mathcal{T}_\lambda(t_1)$ intersects the plane $z = 0$ at the point

$$(x_1, y_1, z_1) = \left(\frac{1 - \lambda + \lambda^2}{2 + \lambda - \lambda^2} \sqrt{1 - \frac{1}{\lambda^2}}\,,\ \frac{2 - 2\lambda - \lambda^2}{2\lambda(\lambda - 2)}\,,\ 0 \right),$$

and the plane $x = 0$ at

$$(\tilde x_1, \tilde y_1, \tilde z_1) = \left(0\,,\ \frac{1 - 4\lambda + \lambda^2}{2(\lambda^2 - 1)}\,,\ \frac{1 - \lambda + \lambda^2}{\lambda^2 - 1} \sqrt{\frac{\lambda}{\lambda - 2}} \right).$$

Hence, a parametrization $\mathcal{A}_1(\lambda)$ of the asymptote $\mathcal{T}_\lambda(t_1)$ is

$$\mathcal{A}_1(\lambda) = \{(x, y, z) \in \mathbb{R}^3 \,|\, x = \tilde\tau_1(\lambda, \nu),\ y = \tilde\tau_2(\lambda, \nu),\ z = \tilde\tau_3(\lambda, \nu);\ \nu \in \mathbb{R}\}$$

with

$$\begin{aligned}\tilde{\tau}_1(\lambda,\nu) &:= x_1 + \nu\,(\tilde{x}_1 - x_1) = \frac{1-\lambda+\lambda^2}{2+\lambda-\lambda^2}\sqrt{1-\frac{1}{\lambda^2}}\;(1-\nu)\,,\\ \tilde{\tau}_2(\lambda,\nu) &:= \; y_1 + \nu\,(\tilde{y}_1 - y_1) = \frac{2-2\lambda-\lambda^2}{2\lambda(\lambda-2)} + \nu\,\frac{(1-\lambda+\lambda^2)^2}{\lambda(\lambda-2)(\lambda^2-1)}\,,\\ \tilde{\tau}_3(\lambda,\nu) &:= \; z_1 + \nu\,(\tilde{z}_1 - z_1) = \nu\,\frac{1-\lambda+\lambda^2}{\lambda^2-1}\sqrt{\frac{\lambda}{\lambda-2}}\,.\end{aligned}$$

In order to abbreviate the notation, we set

$$\mathcal{A}_1(\lambda) = \{(\tilde{\tau}_1(\lambda,\nu),\ \tilde{\tau}_2(\lambda,\nu),\ \tilde{\tau}_3(\lambda,\nu))\,|\,\nu\in\mathbb{R}\}\,.$$

The remaining asymptotes are

$$\begin{aligned} t = t_2 :&\ \mathcal{A}_2(\lambda) = \{(-\tilde{\tau}_1(\lambda,\nu),\ \tilde{\tau}_2(\lambda,\nu),\ \tilde{\tau}_3(\lambda,\nu))\ |\ \nu\in\mathbb{R}\}\,,\\ t = t_3 :&\ \mathcal{A}_3(\lambda) = \{(-\tilde{\tau}_1(\lambda,\nu),\ \tilde{\tau}_2(\lambda,\nu),\ -\tilde{\tau}_3(\lambda,\nu))\,|\,\nu\in\mathbb{R}\}\,,\\ t = t_4 :&\ \mathcal{A}_4(\lambda) = \{(\ \tilde{\tau}_1(\lambda,\nu),\ \tilde{\tau}_2(\lambda,\nu),\ -\tilde{\tau}_3(\lambda,\nu))\ |\ \nu\in\mathbb{R}\}\,.\end{aligned}$$

Figure 4 shows as an example two projections of the touching curve $\mathcal{C}_{-1.4}$ (thick lines). The projection onto the plane $x = 0$ (see left-hand picture) is part of a hyperbola with the center $(y = 71/238 \approx 0.298,\ z = 0)$.

<u>Case 3, $\lambda = \infty$:</u> For $\gamma^*(t) := \lim_{\lambda\to\infty}\gamma(\lambda,t)$ one easily finds

$$\gamma^* :\ [-2\pi/3,\ 2\pi] \to \mathbb{R}^3\,,\quad t \mapsto \gamma^*(t) = \begin{cases}\gamma_1^*(t) & \text{if } t\in[-2\pi/3,\ 2\pi/3]\,,\\ \gamma_2^*(t) & \text{if } t\in(2\pi/3,\ 2\pi]\,,\end{cases}$$

with

$$\gamma_1^*(t) = (\kappa_1^*(t),\ \kappa_2^*(t),\ \kappa_3^*(t))\ ,\quad \gamma_2^*(t) = \left(\kappa_1^*\left(\tfrac{4\pi}{3}-t\right),\ \kappa_2^*\left(\tfrac{4\pi}{3}-t\right),\ -\kappa_3^*\left(\tfrac{4\pi}{3}-t\right)\right)\ ,$$

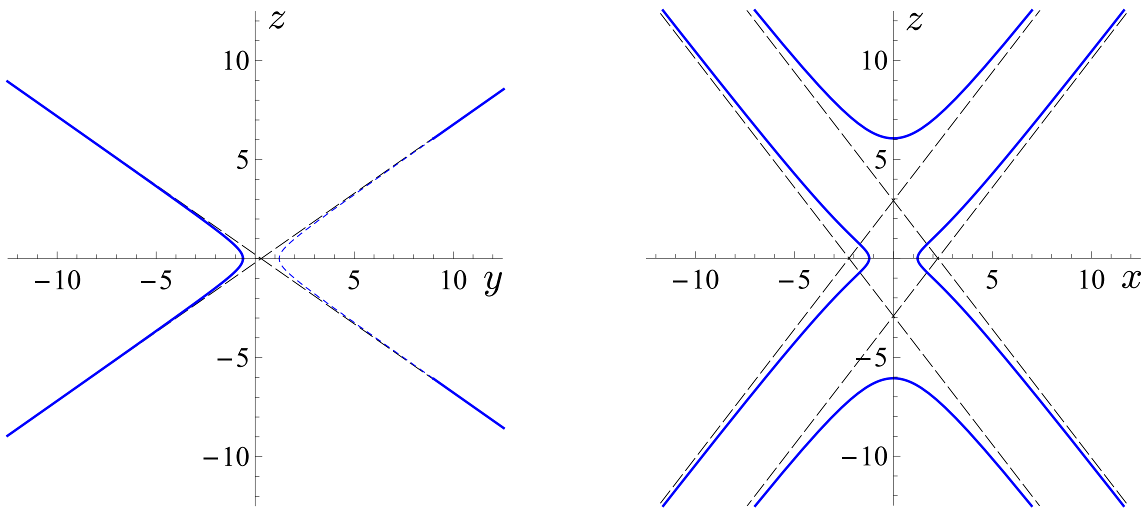

Figure 4: Projections of $\mathcal{C}_{-1.4}$ and its asymptotes $\mathcal{A}_k(-1.4)$, $k = 1,2,3,4$, onto the planes $x = 0$ and $y = 0$, within the box $-12 \le x,y,z \le 12$.

where

$$\kappa_1^*(t) = \lim_{\lambda\to\infty} \kappa_1(\lambda, t) = -\tan t, \quad \kappa_2^*(t) = \lim_{\lambda\to\infty} \kappa_2(\lambda, t) = \frac{1}{2} + \frac{1}{\cos t},$$
$$\kappa_3^*(t) = \lim_{\lambda\to\infty} \kappa_3(\lambda, t) = \frac{\sqrt{1+2\cos t}}{\cos t}.$$

The parametrization $\gamma^*(t)$ of $\mathcal{C}_\infty$ has poles for

$$t_1 = -\frac{\pi}{2}, \quad t_2 = \frac{\pi}{2}, \quad t_3 = \frac{4\pi}{3} - \frac{\pi}{2} = \frac{5\pi}{6}, \quad t_4 = \frac{4\pi}{3} + \frac{\pi}{2} = \frac{11\pi}{6},$$

and therefore, $\mathcal{C}_\infty$ consists of four branches (see Figure 5).

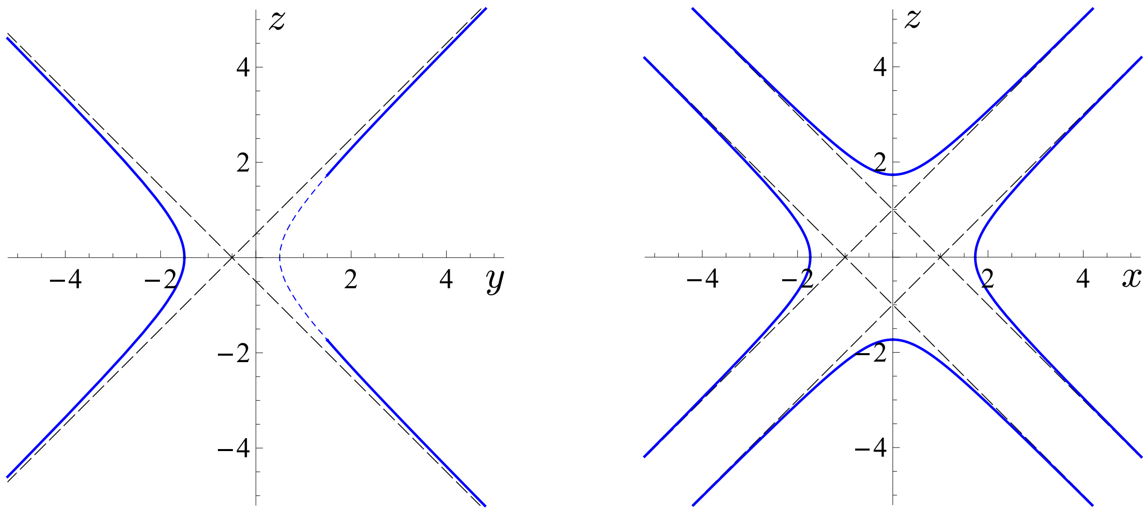

Figure 5: Projections of $\mathcal{C}_\infty$ and its asymptotes $\mathcal{A}_k(\infty)$, $k = 1, 2, 3, 4$, onto the planes $x = 0$ and $y = 0$ within the box $-5 \le x, y, z \le 5$.

For the asymptotes of $\mathcal{C}_\infty$ we find

$$\lim_{\lambda\to\infty} \mathcal{A}_1(\lambda) = \{(\nu-1,\ \nu-1/2,\ \nu) \mid \nu \in \mathbb{R}\}, \quad \lim_{\lambda\to\infty} \mathcal{A}_2(\lambda) = \{(1-\nu,\ \nu-1/2,\ \nu) \mid \nu \in \mathbb{R}\},$$
$$\lim_{\lambda\to\infty} \mathcal{A}_3(\lambda) = \{(1-\nu,\ \nu-1/2,\ -\nu) \mid \nu \in \mathbb{R}\}, \quad \lim_{\lambda\to\infty} \mathcal{A}_4(\lambda) = \{(\nu-1,\ \nu-1/2,\ -\nu) \mid \nu \in \mathbb{R}\}.$$

By virtue of (3), the intersection points with the planes of symmetry are

$$\lim_{\lambda\to\infty} X_1(\lambda) = \left(0,\ 3/2,\ \sqrt{3}\right), \qquad \lim_{\lambda\to\infty} Z_1(\lambda) = \left(\sqrt{3},\ -3/2,\ 0\right),$$
$$\lim_{\lambda\to\infty} X_2(\lambda) = \left(0,\ 3/2,\ -\sqrt{3}\right), \quad \lim_{\lambda\to\infty} Z_2(\lambda) = \left(-\sqrt{3},\ -3/2,\ 0\right).$$

From the parametrization $\gamma^*(t)$ of $\mathcal{C}_\infty$ we have

$$x^2 = \tan^2 t = \frac{1-\cos^2 t}{\cos^2 t}, \quad y = \frac{1}{2} + \frac{1}{\cos t}, \quad z^2 = \frac{1+2\cos t}{\cos^2 t}.$$

After the elimination of $\cos t$ we find the following algebraic equations of the projections of $\mathcal{C}_\infty$ onto the planes $x = 0$, $y = 0$, and $z = 0$:

$$x = 0: \quad (y+1/2)^2 - z^2 = 1, \tag{5}$$
$$y = 0: \quad (x^2 - z^2)^2 - 2(x^2 + z^2) = 3, \tag{6}$$
$$z = 0: \quad (y-1/2)^2 - x^2 = 1. \tag{7}$$

The Eqs. (5) and (7) define hyperbolas with the respective centers $(0,\, -1/2,\, 0)$ and $(0,\, 1/2,\, 0)$. The actual projection of $\mathcal{C}_\infty$ onto the plane $x = 0$ (plotted with thick lines) is part of the hyperbola (5).

Case 4, $\lambda \in \{-1, 2\}$: $\mathcal{C}_\lambda$ consists of two branches. From (3) follows for $\lambda = -1$

$$\lim_{\varepsilon\to 0} X_1(-1-\varepsilon) = (0,\infty,\infty)\,, \qquad \lim_{\varepsilon\to 0} X_1(-1+\varepsilon) = (0,-\infty,-\infty)\,,$$
$$\lim_{\varepsilon\to 0} X_2(-1-\varepsilon) = (0,\infty,-\infty)\,, \quad \lim_{\varepsilon\to 0} X_2(-1+\varepsilon) = (0,-\infty,\infty)\,,$$
$$Z_1(-1) = \big(2/\sqrt{3},\, -1/2,\, 0\big), \qquad Z_2(-1) = \big(-2/\sqrt{3},\, -1/2,\, 0\big),$$

and for $\lambda = 2$,

$$X_1(2) = \big(0,\, 1/2,\, 2/\sqrt{3}\,\big), \qquad X_2(2) = \big(0,\, 1/2,\, -2/\sqrt{3}\,\big),$$
$$\lim_{\varepsilon\to 0} Z_1(2-\varepsilon) = (-\infty,\infty,0)\,, \quad \lim_{\varepsilon\to 0} Z_1(2+\varepsilon) = (\infty,-\infty,0)\,,$$
$$\lim_{\varepsilon\to 0} Z_2(2-\varepsilon) = (\infty,\infty,0)\,, \qquad \lim_{\varepsilon\to 0} Z_2(2+\varepsilon) = (-\infty,-\infty,0)\,.$$

4. The edge of regression

In the following we denote by $\mathcal{R}$ the edge of regression of the developable surface $\mathcal{O}$ (see [8, pp. 119–125], and Figure 1).

Theorem 5. *For $t \in [-2\pi/3,\, 2\pi/3]$, the parametric equations of $\mathcal{R}$ are given by*

$$x = g_1(t) = \frac{\sin t - \tan t}{3}\,, \quad y = g_2(t) = \frac{2 + 3\cos t - 3\cos^2 t - 2\cos^3 t}{6(1+\cos t)\cos t}\,,$$
$$z = \pm g_3(t) = \pm\frac{(1+2\cos t)^{3/2}}{3(1+\cos t)\cos t}\,.$$

Proof. $\mathcal{R}$ is the solution of the system of equations

$$f_\lambda(x,y,z) \ = 0\,, \quad f'_\lambda(x,y,z) = \frac{\partial}{\partial\lambda} f_\lambda(x,y,z) = 0\,, \quad f''_\lambda(x,y,z) = \frac{\partial^2}{\partial\lambda^2} f_\lambda(x,y,z) = 0$$

with variable λ (see [2, pp. 523–524]). Since the touching curve $\mathcal{C}_\lambda$ is the intersection curve of the quadric $\mathcal{Q}_\lambda$ and the quadric defined by $f'_\lambda(x,y,z) = 0$, the parametric functions κ_1, κ_2, κ_3 of $\mathcal{C}_\lambda$ are not only solutions of

$$f_\lambda(\kappa_1(\lambda,t),\kappa_2(\lambda,t),\kappa_3(\lambda,t)) = 0, \text{ but also of } \ f'_\lambda(\kappa_1(\lambda,t),\kappa_2(\lambda,t),\kappa_3(\lambda,t)) = 0\,.$$

Furthermore, one finds

$$f''_\lambda(x,y,z) = \frac{x^2}{(1-\lambda)^3} + \frac{3y^2(\lambda-1)\lambda - y\,(2-3\lambda-3\lambda^2+2\lambda^3)}{(1-\lambda+\lambda^2)^3} + \frac{z^2}{\lambda^3} - \frac{9(\lambda-1)\lambda}{(1-\lambda+\lambda^2)^3}\,.$$

We solve the equation $f''_\lambda(\kappa_1(\lambda,t),\kappa_2(\lambda,t),\kappa_3(\lambda,t)) = 0$ for λ and obtain

$$\lambda = \phi(t) := \frac{1+2\cos t}{(2+\cos t)\cos t}\,.$$

This yields $x = g_1(t)$, $y = g_2(t)$, $z = g_3(t)$, where $g_j(t) := \kappa_j\big(\phi(t),t\big)$ for $j = 1,2,3$. Due to the symmetry of $\mathcal{O}$ with respect to the plane $z = 0$, we also have $x = g_1(t)$, $y = g_2(t)$, $z = -g_3(t)$. □

Corollary 6. *In terms of the parameter λ, the edge of regression is given by*

$$\begin{aligned}\mathcal{R} = \ & \{(r_1(\lambda),\, r_2(\lambda),\, r_3(\lambda)) \mid \lambda \in \mathbb{R} \setminus [0,1]\} \cup \{(-r_1(\lambda),\, r_2(\lambda),\, r_3(\lambda)) \mid \lambda \in \mathbb{R} \setminus [0,1]\} \\ & \cup \{(-r_1(\lambda),\, r_2(\lambda),\, -r_3(\lambda)) \mid \lambda \in \mathbb{R} \setminus [0,1]\} \cup \{(r_1(\lambda),\, r_2(\lambda),\, -r_3(\lambda)) \mid \lambda \in \mathbb{R} \setminus [0,1]\}\ ,\end{aligned}$$

where

$$r_1(\lambda) = \frac{\sqrt{(\lambda-1)^3\,[2-\lambda+2\rho(\lambda)]}}{\lambda\,[2-\lambda+\rho(\lambda)]}\,, \quad r_2(\lambda) = \frac{\lambda^2+2\lambda-2+(\lambda-2)\,\rho(\lambda)}{2\lambda\,[2-\lambda+\rho(\lambda)]}\,,$$

$$r_3(\lambda) = \frac{\operatorname{sgn}(\lambda)\,\sqrt{\lambda\,[2-\lambda+2\rho(\lambda)]}}{2-\lambda+\rho(\lambda)}\,, \quad \rho(\lambda) = \operatorname{sgn}(\lambda)\,\sqrt{1-\lambda+\lambda^2}\,.$$

Proof. We consider the function

$$\phi\colon\ [-2\pi/3, 2\pi/3]\ \to\ \mathbb{R}\,, \quad t\ \mapsto\ \phi(t) = \frac{1+2\cos t}{(2+\cos t)\cos t}\,,$$

used in the proof of Theorem 5. We denote by ϕ_1 the restriction of ϕ to the interval $(-2\pi/3, 0)$, and by ϕ_2 the restriction of ϕ to $(0, 2\pi/3)$. One easily finds the respective inverse functions

$$\phi_1^{-1}(\lambda) = -\arccos\frac{1-\lambda+\rho(\lambda)}{\lambda}\,, \quad \phi_2^{-1}(\lambda) = \arccos\frac{1-\lambda+\rho(\lambda)}{\lambda}$$

with $\rho(\lambda) := \operatorname{sgn}(\lambda)\,\sqrt{1-\lambda+\lambda^2}$ and $\lambda \in \mathbb{R} \setminus [0,1]$, hence

$$r_1(\lambda) := \kappa_1\big(\lambda, \phi_1^{-1}(\lambda)\big) = \frac{\sqrt{(\lambda-1)^3\,[2-\lambda+2\rho(\lambda)]}}{\lambda\,[2-\lambda+\rho(\lambda)]}\,,$$

$$r_2(\lambda) := \kappa_2\big(\lambda, \phi_1^{-1}(\lambda)\big) = \frac{\lambda^2+2\lambda-2+(\lambda-2)\,\rho(\lambda)}{2\lambda\,[2-\lambda+\rho(\lambda)]}\,,$$

$$r_3(\lambda) := \kappa_3\big(\lambda, \phi_1^{-1}(\lambda)\big) = \frac{\operatorname{sgn}(\lambda)\,\sqrt{\lambda\,[2-\lambda+2\rho(\lambda)]}}{2-\lambda+\rho(\lambda)}\,,$$

and

$$\kappa_1\big(\lambda, \phi_2^{-1}(\lambda)\big) = -r_1(\lambda)\,, \quad \kappa_2\big(\lambda, \phi_2^{-1}(\lambda)\big) = r_2(\lambda)\,, \quad \kappa_3\big(\lambda, \phi_2^{-1}(\lambda)\big) = r_3(\lambda)\,.$$

This implies $x = \pm r_1(\lambda)$, $y = r_2(\lambda)$, $z = r_3(\lambda)$, and, due to the symmetry of $\mathcal{O}$ with respect to the plane $z = 0$, also $x = \pm r_1(\lambda)$, $y = r_2(\lambda)$, $z = -r_3(\lambda)$. □

5. The self-polar tetrahedron

Theorem 7. *The faces of the common self-polar tetrahedron $\mathcal{P}$ of the inscribed quadrics $\mathcal{Q}_\lambda$ are formed by the planes $x_1 = 0$, $x_3 = 0$, $2x_2 = \sqrt{3}\,\mathrm{i}x_0$, and $2x_2 = -\sqrt{3}\,\mathrm{i}x_0$.*

Proof. In a tangential system of quadrics there are four which degenerate to conics, and their planes are the faces of the common self-polar tetrahedron [6, pp. 205], [7, p. 254], or [1, p. 136]. $\mathcal{Q}_\lambda$ degenerates if in (1) one of the denominators in $\tilde{f}_\lambda(x_0, x_1, x_2, x_3)$ vanishes.

For $\lambda = 0$ follows $x_3^2 = 0$, and therefore the plane $x_3 = 0$ delivers the first face of the tetrahedron $\mathcal{P}$. For $\lambda = 1$ we have $x_1^2 = 0$, and therefore $x_1 = 0$ is the second face of $\mathcal{P}$.

The two remaining faces correspond to the roots of the equation $1 - \lambda + \lambda^2 = 0$, i.e., to $\lambda_1 = \big(1 + \mathrm{i}\sqrt{3}\,\big)/2$ and $\lambda_2 = \big(1 - \mathrm{i}\sqrt{3}\,\big)/2$. So we have

$$\big(x_2 + \big(\tfrac{1}{2} - \lambda_1\big)x_0\big)^2 = \big(x_2 - \tfrac{\sqrt{3}}{2}\,\mathrm{i}x_0\big)^2 = 0\,, \quad \big(x_2 + \big(\tfrac{1}{2} - \lambda_2\big)x_0\big)^2 = \big(x_2 + \tfrac{\sqrt{3}}{2}\,\mathrm{i}x_0\big)^2 = 0\,.$$

Hence, the planes $2x_2 = \sqrt{3}\,\mathrm{i}x_0$ and $2x_2 = -\sqrt{3}\,\mathrm{i}x_0$ are the remaining two faces of $\mathcal{P}$. □

The degenerate quadrics $\mathcal{Q}_0$ and $\mathcal{Q}_1$ are the circles k_A and k_B, respectively. The conics of the degenerate quadrics $\mathcal{Q}_{\lambda_1}$ and $\mathcal{Q}_{\lambda_2}$ have the equations

$$f_{\lambda_1}(x,y,z) = \frac{x^2}{\Big(\frac{1}{2} - \frac{\sqrt{3}}{2}\,\mathrm{i}\Big)^2} + \frac{z^2}{\Big(\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i}\Big)^2} - 1,\ f_{\lambda_2}(x,y,z) = \frac{x^2}{\Big(\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i}\Big)^2} + \frac{z^2}{\Big(\frac{1}{2} - \frac{\sqrt{3}}{2}\,\mathrm{i}\Big)^2} - 1.$$

6. Common generating lines of $\mathcal{Q}_\lambda$ and $\mathcal{O}$

Only two of the conic sections $\mathcal{Q}_0$, $\mathcal{Q}_1$, $\mathcal{Q}_{\lambda_1}$, $\mathcal{Q}_{\lambda_2}$ are real. Therefore (see [6, p. 206]), the quadrics $\mathcal{Q}_\lambda$ are divided into two sets; one of these sets consists of ruled surfaces, each having four common generating lines with the developable surface $\mathcal{O}$. Clearly, these ruled surfaces are the one-sheeted hyperboloids $\mathcal{Q}_\lambda$, $\lambda \in \mathbb{R} \setminus [0,1]$, and the hyperbolic paraboloid $\mathcal{Q}_\infty$.

Theorem 8. (i) *For fixed value of $\lambda \in \mathbb{R} \setminus [0,1]$, the four common generating lines of the one-sheeted hyperboloid $\mathcal{Q}_\lambda$ and the extended oloid $\mathcal{O}$ are*

$$\begin{aligned}
\mathcal{G}_1(\lambda) &= \{(\ \ \widetilde{\omega}_1(m,\lambda),\ \widetilde{\omega}_2(m,\lambda),\ \ \ \widetilde{\omega}_3(m,\lambda)) \mid m \in \mathbb{R}\}\,,\\
\mathcal{G}_2(\lambda) &= \{(-\widetilde{\omega}_1(m,\lambda),\ \widetilde{\omega}_2(m,\lambda),\ \ \ \widetilde{\omega}_3(m,\lambda)) \mid m \in \mathbb{R}\}\,,\\
\mathcal{G}_3(\lambda) &= \{(-\widetilde{\omega}_1(m,\lambda),\ \widetilde{\omega}_2(m,\lambda),\ -\widetilde{\omega}_3(m,\lambda)) \mid m \in \mathbb{R}\}\,,\\
\mathcal{G}_4(\lambda) &= \{(\ \ \widetilde{\omega}_1(m,\lambda),\ \widetilde{\omega}_2(m,\lambda),\ -\widetilde{\omega}_3(m,\lambda)) \mid m \in \mathbb{R}\}\,,
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{\omega}_1(m,\lambda) &= (1-m)\,\frac{\sqrt{(\lambda-1)(2-\lambda+2\rho(\lambda))}}{-|\lambda|}\,,\\
\widetilde{\omega}_2(m,\lambda) &= (1-m)\,\frac{\lambda-2-2\rho(\lambda)}{2\lambda} + m\,\frac{2\lambda-1-\rho(\lambda)}{2(1+\rho(\lambda))}\,,\\
\widetilde{\omega}_3(m,\lambda) &= m\,\frac{\operatorname{sgn}(\lambda)\,\sqrt{\lambda(2-\lambda+2\rho(\lambda))}}{1+\rho(\lambda)}\,,
\end{aligned}$$

where $\rho(\lambda) = \operatorname{sgn}(\lambda)\,\sqrt{1-\lambda+\lambda^2}$.

(ii) *$\mathcal{G}_1(\lambda)$, $\mathcal{G}_2(\lambda)$, $\mathcal{G}_3(\lambda)$, $\mathcal{G}_4(\lambda)$ are the tangents to $\mathcal{R}$, and to $\mathcal{C}_\lambda$, in the respective points*

$$\begin{aligned}
&P_1(\lambda) = (r_1(\lambda), r_2(\lambda), r_3(\lambda))\,, && P_2(\lambda) = (-r_1(\lambda), r_2(\lambda), r_3(\lambda))\,,\\
&P_3(\lambda) = (-r_1(\lambda), r_2(\lambda), -r_3(\lambda))\,, && P_4(\lambda) = (r_1(\lambda), r_2(\lambda), -r_3(\lambda))
\end{aligned}$$

with $r_j(\lambda)$ according to Corollary 6 for $j = 1,2,3$.

Proof. (i) As already known, the parametric equations of the generating lines of $\mathcal{O}$ are

$$\begin{aligned} x = &\quad \omega_1(m,t) = 1-m)\sin t, \\ y = &\quad \omega_2(m,t) = (1-m)\left(-\frac{1}{2}-\cos t\right) + m\left(\frac{1}{2}-\frac{\cos t}{1+\cos t}\right), \\ z = &\pm\omega_3(m,t) = \pm\frac{m\sqrt{1+2\cos t}}{1+\cos t}. \end{aligned}$$

We substitute $x = \omega_1(m,t)$, $y = \omega_2(m,t)$, $z = \omega_3(m,t)$ in $f_\lambda(x,y,z) = 0$ and solve this equation for t. Thus, we find

$$t = \pm\tilde{t}(\lambda) \quad \text{with} \quad \tilde{t}(\lambda) = \arccos\frac{1-\lambda\pm\sqrt{1-\lambda+\lambda^2}}{\lambda}.$$

Since we are only interested in real solutions, we can write

$$\tilde{t}(\lambda) = \arccos\frac{1-\lambda+\rho(\lambda)}{\lambda} \tag{8}$$

with the function ρ from Corollary 6. This implies

$$\begin{aligned} \omega_1(m,\pm\tilde{t}(\lambda)) &= \pm(1-m)\,\frac{\sqrt{(\lambda-1)(2-\lambda+2\rho(\lambda))}}{|\lambda|}, \\ \omega_2(m,\pm\tilde{t}(\lambda)) &= (1-m)\,\frac{\lambda-2-2\rho(\lambda)}{2\lambda} + m\,\frac{2\lambda-1-\rho(\lambda)}{2(1+\rho(\lambda))}, \\ \omega_3(m,\pm\tilde{t}(\lambda)) &= m\,\frac{\operatorname{sgn}(\lambda)\sqrt{\lambda(2-\lambda+2\rho(\lambda))}}{1+\rho(\lambda)}. \end{aligned}$$

We put $\widetilde{\omega}_j(m,\lambda) := \omega_j(m,-\tilde{t}(\lambda))$ for $j = 1,2,3$. This yields $\mathcal{G}_1(\lambda)$ and $\mathcal{G}_2(\lambda)$. Due to the symmetry of $\mathcal{Q}_\lambda$ and $\mathcal{O}$ with respect to the plane $z = 0$, the lines $\mathcal{G}_3(\lambda)$ and $\mathcal{G}_4(\lambda)$ follow.

(ii) By virtue of (4), the tangent

$$\mathcal{T}_\lambda(t) = \{(\tau_1(\lambda,t,\mu),\, \tau_2(\lambda,t,\mu),\, \tau_3(\lambda,t,\mu)) \mid \mu\in\mathbb{R}\}, \quad t\in[-2\pi/3,\, 2\pi/3],$$

to $\mathcal{C}_\lambda$ is a generating line of $\mathcal{Q}_\lambda$ for all values of t that are solutions of

$$f_\lambda\left(\tau_1(\lambda,t,\mu), \tau_2(\lambda,t,\mu), \tau_3(\lambda,t,\mu)\right) = 0.$$

One finds $t = \pm\tilde{t}(\lambda)$ with $\tilde{t}(\lambda)$ from (8). At first we consider only $t = -\tilde{t}(\lambda)$. Calculation shows that

$$\kappa_j(\lambda,-\tilde{t}(\lambda)) = r_j(\lambda), \quad j = 1,2,3.$$

Hence, the tangent $\mathcal{T}^{(1)}(\lambda) := \mathcal{T}_\lambda(-\tilde{t}(\lambda))$ touches $\mathcal{C}_\lambda$ at the point $P_1(\lambda) = (r_1(\lambda), r_2(\lambda), r_3(\lambda)) \in \mathcal{R}$. According to [2, p. 489], $\mathcal{T}^{(1)}(\lambda)$ is equal to the tangent to $\mathcal{R}$ at this point. The common generating lines are tangents to the edge of regression [6, p. 206]. Thus one finds

$$\widetilde{\omega}_j(\widehat{m}(\lambda),\lambda) = r_j(\lambda) \;\text{ for }\; j = 1,2,3, \;\text{ where }\; \widehat{m}(\lambda) := \frac{1+\rho(\lambda)}{2-\lambda+\rho(\lambda)}.$$

It follows that $\mathcal{T}^{(1)}(\lambda) = \mathcal{G}_1(\lambda)$. Due to symmetry with respect to the planes $x = 0$ and $z = 0$, with $\tilde{\tau}_j(\lambda,\mu) := \tau_j(\lambda,-\tilde{t}(\lambda),\mu)$ we also have

$$\begin{aligned} \mathcal{T}^{(2)}(\lambda) &:= \{(-\tilde{\tau}_1(\lambda,\mu),\, \tilde{\tau}_2(\lambda,\mu),\, \tilde{\tau}_3(\lambda,\mu)) \mid \mu\in\mathbb{R}\} = \mathcal{G}_2(\lambda), \\ \mathcal{T}^{(3)}(\lambda) &:= \{(-\tilde{\tau}_1(\lambda,\mu),\, \tilde{\tau}_2(\lambda,\mu),\, -\tilde{\tau}_3(\lambda,\mu)) \mid \mu\in\mathbb{R}\} = \mathcal{G}_3(\lambda), \\ \mathcal{T}^{(4)}(\lambda) &:= \{(\tilde{\tau}_1(\lambda,\mu),\, \tilde{\tau}_2(\lambda,\mu),\, -\tilde{\tau}_3(\lambda,\mu)) \mid \mu\in\mathbb{R}\} = \mathcal{G}_4(\lambda). \end{aligned}$$

□

As an example, Figure 3 shows the common generating lines $\mathcal{G}_1(4)$, $\mathcal{G}_2(4)$, $\mathcal{G}_3(4)$, $\mathcal{G}_4(4)$ of $\mathcal{Q}_4$ and $\mathcal{O}$.

7. The development of $\mathcal{O}$

Now we consider the development of the extended oloid $\mathcal{O}$ onto its tangent plane E. For this, we define a cartesian (ξ,η)-coordinate system in E as follows: Let E touch $\mathcal{O}$ along the generating line

$$\mathcal{L}_0 = \{\omega_1(m,0),\, \omega_2(m,0),\, -\omega_3(m,0) \mid m \in \mathbb{R}\}$$

(see (2) and the proof of Corollary 4). Then $\mathcal{L}_0$ is the η-axis, and the line perpendicular to $\mathcal{L}_0$ at the point $m=0$ is the ξ-axis.

Any curve $\mathcal{C} \subset \mathcal{O}$ is developed onto a plane curve $\mathcal{C}^* \subset E$. A parametrization of $\mathcal{C}^*$ by the arc length t of the double circular arc $\mathcal{C}_0$ can be obtained from the vector transformation in [3, p. 114, Theorem 4]. For the sake of brevity, we set in the following $c = \cos t$ and $s = \sin t$; while $\lfloor\cdot\rfloor$ denotes the integer part of '$\cdot$'.

Theorem 9. *The development of the touching curve $\mathcal{C}_\lambda$ into the plane E is the curve*

$$\mathcal{C}^*_\lambda = \{(\kappa^*_1(\lambda,t),\, \kappa^*_2(\lambda,t)) \mid t \in \mathbb{R}\}$$

with the parametrization

$$\kappa^*_1(\lambda,t) = \operatorname{sgn}(t)\cdot\left\lfloor \frac{3\,|t|}{4\pi}+\frac{1}{2}\right\rfloor\cdot\frac{4\pi}{3\sqrt{3}} + \operatorname{sgn}(h(t))\cdot\tilde{\kappa}_1(\lambda,h(t))\,,\quad \kappa^*_2(\lambda,t) = \tilde{\kappa}_2(\lambda,h(t))\,,$$

where

$$\tilde{\kappa}_1(\lambda,t) = \frac{2\sqrt{3}}{9}\left(\arccos\frac{\sqrt{2}\,c}{\sqrt{1+c}} + \frac{(1-2\lambda)\,|s|\,\sqrt{2(1+2c)}}{(1+\lambda c)\sqrt{1+c}}\right),$$

$$\tilde{\kappa}_2(\lambda,t) = \frac{\sqrt{3}}{9}\left(\ln\frac{2}{1+c} + \frac{4+7\lambda+(11\lambda-4)c}{1+\lambda c}\right),$$

$$h(t) = t - \operatorname{sgn}(t)\cdot\left\lfloor\frac{3\,|t|}{4\pi}+\frac{1}{2}\right\rfloor\cdot\frac{4\pi}{3}\,.$$

Proof. After the substitution $x=\kappa_1(\lambda,t)$, $y=\kappa_2(\lambda,t)$, $z=-\kappa_3(\lambda,t)$ (see Corollary 4) in the vector transformation [3, p. 114, Theorem 4], a straight-forward calculation yields

$$\xi = \tilde{\tilde{\kappa}}_1(\lambda,t) = \frac{2\sqrt{3}}{9}\left(\arccos\frac{\sqrt{2}\,c}{\sqrt{1+c}} + \frac{(1-2\lambda)\,s\,\sqrt{2(1+2c)}}{(1+\lambda c)\sqrt{1+c}}\right),$$

$$\eta = \tilde{\kappa}_2(\lambda,t) = \frac{\sqrt{3}}{9}\left(\ln\frac{2}{1+c} + \frac{4+7\lambda+(11\lambda-4)c}{1+\lambda c}\right).$$

$\tilde{\kappa}_2(\lambda,t)$ is valid for $t \in [-2\pi/3,\, 2\pi/3]$. The periodic continuation of $\tilde{\kappa}_2(\lambda,t)$ gives $\kappa^*_2(\lambda,t)$, which is valid for $t \in \mathbb{R}$.

$\tilde{\tilde{\kappa}}_1(\lambda,t)$ is valid only for $t \in [0, 2\pi/3]$. The restriction of $\kappa^*_1(\lambda,t)$ to $t \in [-2\pi/3,\, 2\pi/3]$ must be an odd function. Replacing $\sin t$ by $|\sin t|$ in $\tilde{\tilde{\kappa}}_1(\lambda,t)$, we get the even function $\tilde{\kappa}_1(\lambda,t)$,

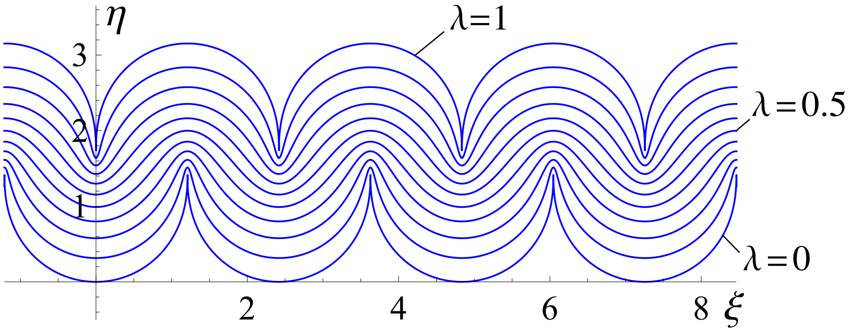

Figure 6: The curves $\mathcal{C}^*_\lambda$ for $\lambda = 0,\, 0.1,\, 0.2, \ldots,\, 0.9,\, 1$

$t \in [-2\pi/3,\, 2\pi/3]$. Now, $\kappa_1^\diamond(\lambda, t) := \operatorname{sgn}(t) \cdot \tilde{\kappa}_1(\lambda, t)$ is the required restriction of $\kappa_1^*(\lambda, t)$. We get

$$\kappa_1^\diamond(\lambda, 2\pi/3) - \kappa_1^\diamond(\lambda, -2\pi/3) = \frac{4\pi}{3\sqrt{3}}\,,$$

and therefore, using the step function

$$\operatorname{sgn}(t) \cdot \left\lfloor \frac{3\,|t|}{4\pi} + \frac{1}{2} \right\rfloor \cdot \frac{4\pi}{3\sqrt{3}}\,,$$

we have found $\kappa_1^*(\lambda, t)$, valid for $t \in \mathbb{R}$. □

Examples of curves $\mathcal{C}^*_\lambda$ with $\lambda \in [0, 1]$ are shown in Figure 6. The curve $\mathcal{C}^*_\infty$ and the development of the edge of regression $\mathcal{R}$ are shown in Figure 7.

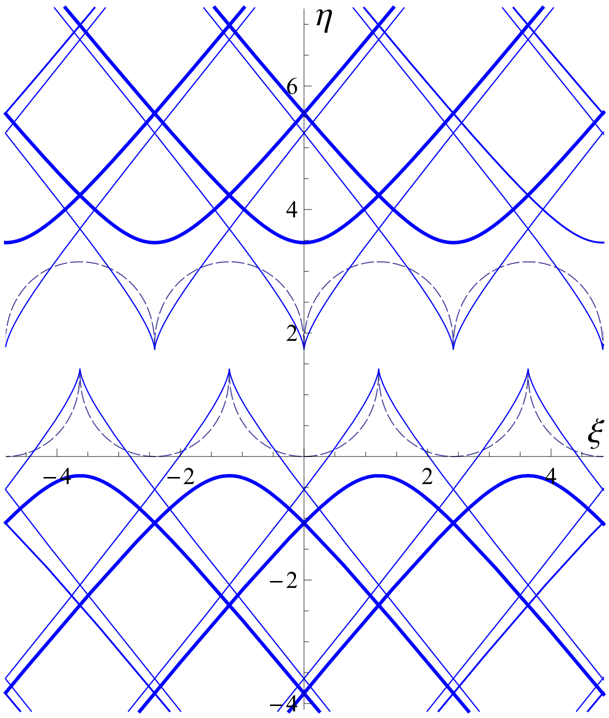

Figure 7: $\mathcal{C}^*_\infty$ (thick), development of $\mathcal{R}$ (thin), $\mathcal{C}^*_0$ and $\mathcal{C}^*_1$ (dashed)

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