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A Property of Liouville Surfaces and Manifolds

Călin-Şerban Bărbat

Dieselstraße 19, 80993 München, Germany
email: calin.barbat@web.de

Dedicated to my father on the occasion of his 75th birthday

Abstract. In this paper we use (from [6] or [12]) the definition of the energy of a curve on a surface and show that in a Liouville surface the energy integrals along the diagonals of a net rectangle (Liouville maps are conformal) are equal. This result allows a generalization to Liouville manifolds, which is stated and proved in this paper. A series of different surfaces with an induced Liouville metric are given in Euclidean spaces. One example is given in the pseudo-Euclidean (Minkowski) plane. The material presented here also relates to my previous article [1].

Key Words: Energy/Action of curve, Liouville metric, Liouville surface, Liouville manifold.
MSC 2010: 53A05, 53A07

1. Length and energy of a curve in a manifold

The length $L(p)$ and the energy $E(p)$ of a curve $p\colon [0,1] \to M$ in a Riemannian manifold M are given by the following expressions (see [6, Chapter 9, p. 194]), and they are related by the Schwarz inequality (with equality if and only if $\|\dot{p}\|$ is constant, that means p is parametrized proportionally to arc length):

$$\begin{aligned} L(p) &= \int_0^1 \|\dot{p}\| dt = \int_0^1 \sqrt{\langle \dot{p}, \dot{p} \rangle} dt, \\ E(p) &= \int_0^1 \|\dot{p}\|^2 dt = \int_0^1 \langle \dot{p}, \dot{p} \rangle dt, \\ & L^2(p) \leq E(p), \end{aligned}$$

where $\dot{p}$ is the tangent vector of the curve, and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the scalar product in the tangent space TM.

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Another representation of the length L and energy E of a curve $\varphi(\gamma(t))\colon\ [0,1] \xrightarrow{\gamma} \mathbb{R}^m \xrightarrow{\varphi} M \subset \mathbb{R}^n$ in a manifold M is given by the following expressions, which are related by the Schwarz inequality:

$$\begin{aligned} L(\varphi(\gamma(t))) &= \int_0^1 \sqrt{\sum_{i,j=1}^{m} g_{ij}(\gamma(t))\,(x_i\circ\gamma'(t))\,(x_j\circ\gamma'(t))\,dt}, \\ E(\varphi(\gamma(t))) &= \int_0^1 \left(\sum_{i,j=1}^{m} g_{ij}(\gamma(t))\,(x_i\circ\gamma'(t))\,(x_j\circ\gamma'(t))\right) dt, \\ &L^2(\varphi(\gamma(t))) \le E(\varphi(\gamma(t))). \end{aligned}$$

2. The Liouville line element as a special case of the Stäckel line element

W. Blaschke [3] gives the following formula (1) for the *Stäckel line element* in three dimensions,

$$ds^2 = \begin{vmatrix} U & V & W \\ U_1 & V_1 & W_1 \\ U_2 & V_2 & W_2 \end{vmatrix} \left(\frac{du^2}{\begin{vmatrix} V_1 & W_1 \\ V_2 & W_2 \end{vmatrix}} - \frac{dv^2}{\begin{vmatrix} U_1 & W_1 \\ U_2 & W_2 \end{vmatrix}} + \frac{dw^2}{\begin{vmatrix} U_1 & V_1 \\ U_2 & V_2 \end{vmatrix}} \right),$$

where U, U_1, U_2 are functions of u alone, etc., and he shows that the Stäckel line element is a necessary and sufficient condition for Ivory's theorem, when one considers the geodesic diagonals and distances [3, p. 662]. This answers a question posed by Stachel [11].

K. Zwirner, a student of Blaschke, considers in his dissertation [15] a special form of Stäckel line element,

$$\begin{aligned} ds^2 &= (U-V)(U-W)du^2 + (V-U)(V-W)dv^2 + (W-U)(W-V)dw^2 \\ &= \begin{vmatrix} U^2 & V^2 & W^2 \\ U & V & W \\ 1 & 1 & 1 \end{vmatrix} \left(\frac{du^2}{V-W} - \frac{dv^2}{U-W} + \frac{dw^2}{U-V} \right). \end{aligned}$$

An example of such a *Zwirner line element* are the ellipsoidal coordinates (using Jacobi elliptic functions) from [4, eq. (2.28)] or [7, p. 211]. This line element also appears in [13], the dissertation of another student of Blaschke.

In the case

$$\begin{vmatrix} V_1 & W_1 \\ V_2 & W_2 \end{vmatrix} = \begin{vmatrix} U_1 & W_1 \\ U_2 & W_2 \end{vmatrix} = \begin{vmatrix} U_1 & V_1 \\ U_2 & V_2 \end{vmatrix}$$

the Stäckel line element reduces to a *Liouville line element*

$$ds^2 = (U-V+W)\left(du^2 - dv^2 + dw^2\right).$$

Liouville parametrizations are isothermal and therefore conformal. The sphere provided in the [4, eq. (3.51)]) is Liouville parametrized by Jacobi elliptic functions.

If in the Liouville line element two of the functions U, V, W vanish, say $V = W = 0$, then we get a *Clairaut line element*

$$ds^2 = U\left(du^2 - dv^2 + dw^2\right).$$

The *Mylar balloon* in [8, (eq. (4.12)] is Clairaut parametrized by Jacobi elliptic integrals.

As BLASCHKE mentioned in [3], the generalization of these line elements to n dimensions is straightforward.

3. A property of Liouville surfaces

We can state and prove the following theorem about Liouville surfaces:

Theorem 1. *Let $A = (a_1, a_2)^t$ and $C = (c_1, c_2)^t$ be two points in the plane $\mathbb{R}^2$. Construct the rectangle $ABCD$, with the points $B = (c_1, a_2)^t$ and $D = (a_1, c_2)^t$. The diagonals of the rectangle are then (for $t \in [0, 1]$): $\gamma_1(t) = A + t(C - A)$ and $\gamma_2(t) = B + t(D - B)$.*
Consider now the image curves $p_1(t) = \varphi(\gamma_1(t))$ and $p_2(t) = \varphi(\gamma_2(t))$, where $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^n$ is a Liouville map with

$$g_{11} = \langle \varphi_{x_1}, \varphi_{x_1} \rangle = f_1(x_1) + f_2(x_2) = \langle \varphi_{x_2}, \varphi_{x_2} \rangle = g_{22}, \quad g_{12} = 0 = g_{21},$$

and the Liouville line element

$$ds^2 = (f_1(x_1) + f_2(x_2))(dx_1^2 + dx_2^2).$$

Then the energies of these diagonals are equal, i.e., $E(p_1(t)) = E(p_2(t))$.

Proof. For the energy of a diagonal we have

$$E(\varphi(\gamma(t))) = \int_0^1 g_{11}(\gamma(t)) \left((x_1 \circ \gamma'(t))^2 + (x_2 \circ \gamma'(t))^2 \right) dt.$$

It turns out that for both diagonals $\gamma_1(t) = A + t(C - A)$ and $\gamma_2(t) = B + t(D - B)$ the expression $\left((x_1 \circ \gamma'(t))^2 + (x_2 \circ \gamma'(t))^2 \right)$ is actually the same constant with respect to t. To see this, we first differentiate the diagonals with respect to t and get $\gamma_1'(t) = C - A$ and $\gamma_2'(t) = D - B$. Then

$$\begin{aligned} &\left((x_1 \circ \gamma_1'(t))^2 + (x_2 \circ \gamma_1'(t))^2 \right) = (c_1 - a_1)^2 + (c_2 - a_2)^2 \\ = \;& (a_1 - c_1)^2 + (c_2 - a_2)^2 = \left((x_1 \circ \gamma_2'(t))^2 + (x_2 \circ \gamma_2'(t))^2 \right). \end{aligned}$$

Because of this identity it suffices to show that

$$\int_0^1 g_{11}(\gamma_1(t)) dt = \int_0^1 g_{11}(\gamma_2(t)) dt.$$

In detail, we have

$$\begin{aligned} &\int_0^1 f_1(a_1 + t(c_1 - a_1)) + f_2(a_2 + t(c_2 - a_2)) dt \\ = &\int_0^1 f_1(c_1 + t(a_1 - c_1)) + f_2(a_2 + t(c_2 - a_2)) dt, \end{aligned}$$

which is obviously true. This completes the proof. □

Remark 1. All surface examples considered in this article are Liouville surfaces. Therefore we can apply this theorem to show that the energies of the diagonals are equal and do not need complicated calculations. For this reason all examples are corollaries which follow from this theorem.

Remark 2. In all the following examples I use the rectangle $ABCD$ and its diagonals $\gamma_1(t) = A + t(C - A)$ and $\gamma_2(t) = B + t(D - B)$ (for $t \in [0, 1]$).

4. An example of a Liouville parametrized sphere

In the article [4] we find a sphere (formula (3.51)) which is Liouville parametrized by Jacobi elliptic functions (a reference for Jacobi elliptic functions and elliptic integrals is [5]). We can prove the following corollary:

Corollary 2. *Consider the diagonals* $p_1(t) = \varphi(\gamma_1(t))$ *and* $p_2(t) = \varphi(\gamma_2(t))$*, where*

$$\varphi\colon [-2K, 2K] \times (K + i[0, 2K']) \to \mathbb{E}^3, \quad \varphi(x, y) = \begin{pmatrix} k\,\mathrm{sn}(x)\,\mathrm{sn}(y) \\ i\,\frac{k}{k'}\,\mathrm{cn}(x)\,\mathrm{cn}(y) \\ \frac{1}{k'}\,\mathrm{dn}(x)\,\mathrm{dn}(y) \end{pmatrix}.$$

Then the energies along the diagonals are equal, i.e., $E(p_1) = E(p_2)$.

Proof. This sphere has a Liouville line element

$$ds^2 = k^2(-\mathrm{sn}^2(x) - (-\mathrm{sn}^2(y)))(dx^2 - dy^2). \qquad \square$$

Here we used the short hand notation $\mathrm{sn}(x) = \mathrm{sn}(x, k)$, etc. where k is the elliptic modulus, $k' = \sqrt{1 - k^2}$ and $K = \mathbf{K}(k)$, $K' = \mathbf{K}(k')$ are the quarter periods of the elliptic functions, with

$$\mathbf{K}(k) = \mathbf{F}(1; k) = \int_0^1 \frac{1}{\sqrt{1 - t^2}\,\sqrt{1 - k^2t^2}}\,dt.$$

This parametrization covers the whole sphere once for $(x, y) \in [-2K, 2K] \times (K + i[0, 2K'])$, see Figure 1 (made with wxMaxima).

5. An example of a Clairaut parametrized surface

By using elliptic integrals of the first and second kind

$$\mathbf{F}(z; k) = \int_0^z \frac{1}{\sqrt{1 - t^2}\sqrt{1 - k^2t^2}}dt \quad \text{and} \quad \mathbf{E}(z; k) = \int_0^z \frac{\sqrt{1 - k^2t^2}}{\sqrt{1 - t^2}}dt,$$

the author of [8] parametrizes a surface in eq. (4.12) called the *Mylar balloon* (see Figure 2). We can prove the following result concerning the Mylar balloon:

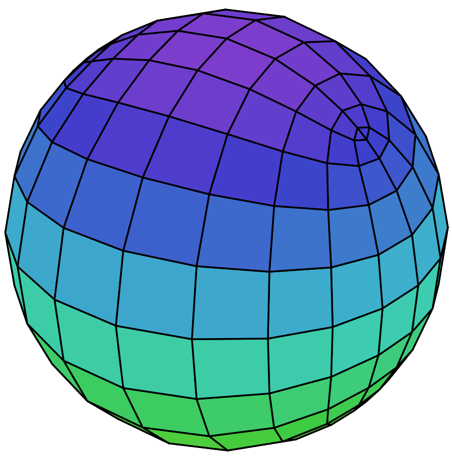

Figure 1: Elliptic coordinates on the sphere

Corollary 3. *Consider the diagonals* $p_1(t) = \varphi(\gamma_1(t))$ *and* $p_2(t) = \varphi(\gamma_2(t))$, *where* $\varphi\colon\ ]-\infty,\infty[\,\times[0,2\pi[\ \to\ \mathbb{E}^3$ *with*

$$\varphi(x,y) = \begin{pmatrix} \dfrac{r\cos(y)}{\sqrt{\cosh 2x}} \\ \dfrac{r\sin(y)}{\sqrt{\cosh 2x}} \\ \sqrt{2}r\left(\mathbf{E}\left(\dfrac{\sqrt{2}\sinh(x)}{\sqrt{\cosh(2x)}}, \dfrac{1}{\sqrt{2}}\right) - \dfrac{1}{2}\mathbf{F}\left(\dfrac{\sqrt{2}\sinh(x)}{\sqrt{\cosh(2x)}}, \dfrac{1}{\sqrt{2}}\right)\right) \end{pmatrix}.$$

Then the lengths along these diagonals are equal as well as the energies, i.e.,

$$L(p_1) = L(p_2) \text{ and } E(p_1) = E(p_2).$$

Proof. This parametrization has the Clairaut line element

$$ds^2 = \frac{r^2}{\cosh(2x)}(dx^2 + dy^2),$$

whis is a special case of a Liouville line element. The statement about the lengths follows from the rotational symmetry of the surface which makes the two diagonals symmetric. □

6. A construction of Clairaut surfaces

Corollary 4. *Consider the diagonals* $p_1(t) = \varphi(\gamma_1(t))$ *and* $p_2(t) = \varphi(\gamma_2(t))$ *where* $\varphi\colon\ ]-\infty,\infty[\,\times[0,2\pi[\ \to\ \mathbb{E}^3$ *with a differentiable radius function* $r\colon \mathbb{R} \to \mathbb{R}$ *and*

$$\varphi(x,y) = \begin{pmatrix} r(x)\cos(y) \\ r(x)\sin(y) \\ \int_{x_0}^{x} \sqrt{r^2(t) - r'^2(t)}dt \end{pmatrix}.$$

Then the lengths along the diagonals are equal, as well as the energies, i.e.,

$$L(p_1) = L(p_2) \text{ and } E(p_1) = E(p_2).$$

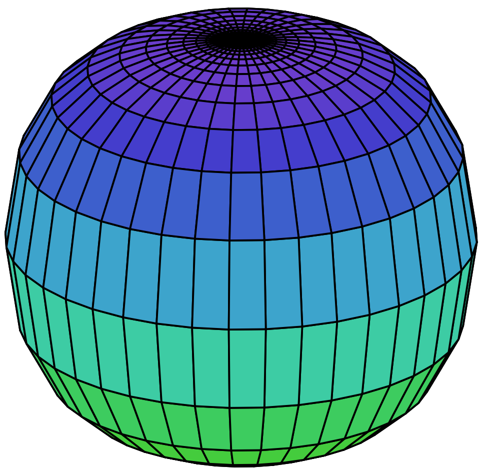

Figure 2: Parametrization of the *Mylar balloon* with elliptic functions

Proof. This surface has a Clairaut line element

$$ds^2 = r^2(x)(dx^2 + dy^2),$$

and because of the rotational symmetry also the lengths are equal. □

7. Some minimal Liouville surfaces in $\mathbb{E}^3$

From [2, eq. (4.3)] we know that every minimal Liouville surface in $\mathbb{E}^3$ can be written (we set $z = x + iy$, $v_1 = \frac{i}{\sqrt{2}}(e_1 + ie_2)$ and $v_2 = e_3$, and the other constants are a complex number $\alpha = \alpha_1 + i\alpha_2 \neq 0$ and real numbers $r \neq 0$, $\lambda_1 > 0$, $\lambda_1 \neq \lambda_2$, $\lambda_3 = 2\lambda_2 - \lambda_1$) as

$$\varphi(z) = \mathrm{re}\left(\alpha \int re^{\lambda_1 z} v_1 + i\sqrt{2}\, e^{\lambda_2 z} v_2 + \frac{1}{r}\, e^{\lambda_3 z}\, \overline{v}_1\; dz\right).$$

For $\lambda_p \neq 0$, $p = 1, 2, 3$, we get

$$\varphi(z) = \mathrm{re}\left(\alpha\left(\frac{re^{\lambda_1 z}}{\lambda_1}\, v_1 + \frac{i\sqrt{2}\, e^{\lambda_2 z}}{\lambda_2}\, v_2 + \frac{e^{\lambda_3 z}}{r\lambda_3}\, \overline{v}_1\right)\right) = \mathrm{re}\left(\alpha\begin{pmatrix} \dfrac{i\left(-\lambda_1 e^{\lambda_3 z} + r^2\lambda_3\, e^{\lambda_1 z}\right)}{\sqrt{2}\, r\lambda_1\lambda_3} \\ -\dfrac{\lambda_1 e^{\lambda_3 z} + r^2\lambda_3\, e^{\lambda_1 z}}{\sqrt{2}\, r\lambda_1\lambda_3} \\ \dfrac{i\sqrt{2}\, e^{\lambda_2 z}}{\lambda_2} \end{pmatrix}\right).$$

The other cases are either $\lambda_2 = 0$ or $\lambda_3 = 0$. We do not consider them here, but the proofs are similar.

Corollary 5. *Consider the diagonals $p_1(t) = \varphi(\gamma_1(t))$ and $p_2(t) = \varphi(\gamma_2(t))$ for the parametrization $\varphi\colon \mathbb{R}^2 \to \mathbb{E}^3$ as computed above:*

$$\varphi(x, y) = \begin{pmatrix} \dfrac{\alpha_2\left(\lambda_1 e^{\lambda_3 x}\cos(\lambda_3 y) - \lambda_3 r^2 e^{\lambda_1 x}\cos(\lambda_1 y)\right) + \alpha_1\left(\lambda_1 e^{\lambda_3 x}\sin(\lambda_3 y) - \lambda_3 r^2 e^{\lambda_1 x}\sin(\lambda_1 y)\right)}{\sqrt{2}\, r\lambda_1\lambda_3} \\ \dfrac{\alpha_2\left(\lambda_1 e^{\lambda_3 x}\sin(\lambda_3 y) + \lambda_3 r^2 e^{\lambda_1 x}\sin(\lambda_1 y)\right) - \alpha_1\left(\lambda_1 e^{\lambda_3 x}\cos(\lambda_3 y) + \lambda_3 r^2 e^{\lambda_1 x}\cos(\lambda_1 y)\right)}{\sqrt{2}\, r\lambda_1\lambda_3} \\ -\dfrac{\sqrt{2} e^{\lambda_2 x}\left(\alpha_1 \sin(\lambda_2 y) + \alpha_2\cos(\lambda_2 y)\right)}{\lambda_2} \end{pmatrix}.$$

Then the energies along the diagonals are equal, as well as the lengths, i.e.,

$$E(p_1) = E(p_2) \text{ and } L(p_1) = L(p_2).$$

Proof. By construction, the surface has a Liouville line element. Therefore the energies are equal. The statement about the lengths follows by computation:

$$\|\dot{p}_1(t)\| = \sqrt{\frac{\left((a_1 - c_1)^2 + (a_2 - c_2)^2\right)\left(\alpha_1^2 + \alpha_2^2\right)}{2\, r^2}\, e^{-((1+t)\, a_1 + t\, c_1)\,\lambda_1 - 2\, t\, a_1\, \lambda_2} \cdot \left(e^{2\, t\, c_1\, \lambda_2 + 2\, a_1\, (t\,\lambda_1 + \lambda_2)} + e^{2\, t\, c_1\, \lambda_1 + 2\, a_1\, (\lambda_1 + t\,\lambda_2)}\, r^2\right)}\,.$$

Having done the same computations for $p_2(t)$, we can calculate the difference as

$$\begin{aligned}\|\dot{p}_1(t)\| - \|\dot{p}_2(t)\| \;=\; & -\frac{\sqrt{(a_1-c_1)^2+(a_2-c_2)^2}\sqrt{\alpha_1^2+\alpha_2^2}}{\sqrt{2}\, e^{(a_1+c_1)((1+2t)\lambda_1+2t\lambda_2)}r} \\ & \cdot \big(e^{c_1(3t\lambda_1+2\lambda_2)+a_1((1+t)\lambda_1+4t\lambda_2)} - e^{a_1(3t\lambda_1+2\lambda_2)+c_1((1+t)\lambda_1+4t\lambda_2)} \\ & + e^{c_1((2+t)\lambda_1+2t\lambda_2)+a_1((1+3t)\lambda_1+2t\lambda_2)}r^2 - e^{a_1((2+t)\lambda_1+2t\lambda_2)+c_1((1+3t)\lambda_1+2t\lambda_2)}r^2\big).\end{aligned}$$

By integration we obtain the desired result for the lengths,

$$L(p_1) - L(p_2) = \int_0^1 \|\dot{p}_1(t)\| - \|\dot{p}_2(t)\| dt = 0. \qquad \square$$

8. A minimal Liouville surface in $\mathbb{E}^4$

We refer to the example of a minimal Liouville surface in $\mathbb{E}^4$ provided in [2, p. 39, Section 5] and multiply with -1, which does not change the property of being a minimal Liouville surface. This yields

Corollary 6. *Consider the diagonals $p_1(t) = \varphi(\gamma_1(t))$ and $p_2(t) = \varphi(\gamma_2(t))$ where*

$$\varphi\colon \mathbb{R}^2 \to \mathbb{E}^4, \quad \varphi(x,y) = \sqrt{2}\begin{pmatrix} -\cos(y)\sinh(x) \\ \sin(y)\sinh(x) \\ \sin(x)\sinh(y) \\ \cos(x)\sinh(y) \end{pmatrix}.$$

Then the energies along the diagonals are equal, i.e., $E(p_1) = E(p_2)$.

Proof. This is a minimal Liouville surface by construction and has therefore a Liouville line element. □

9. An example of a Liouville parametrized Minkowski plane

The example discussed here is inspired by [9], where we find on page 21 an example (second example: parabolic coordinates) in the Minkowski plane which has a Liouville line element (see Figure 3).

Corollary 7. *Consider the diagonals $p_1(t) = \varphi(\gamma_1(t))$ and $p_2(t) = \varphi(\gamma_2(t))$ with*

$$\varphi(x,y) = \frac{1}{4}\begin{pmatrix} 2(x-y)^2 + (x+y-1) \\ 2(x-y)^2 - (x+y+1) \end{pmatrix},$$

where $\varphi\colon \mathbb{R}^2 \to \mathbb{M}^2$ is a map into the Minkowski plane. Then the energies along the diagonals are equal, i.e., $E(p_1) = E(p_2)$.

Proof. With $\langle v, w\rangle = v_1w_1 - v_2w_2$ we have

$$\begin{aligned} g_{11} &= \langle \varphi_x, \varphi_x\rangle = x - y, \\ g_{12} &= \langle \varphi_x, \varphi_y\rangle = 0, \\ g_{22} &= \langle \varphi_y, \varphi_y\rangle = -(x-y) = -g_{11}. \end{aligned}$$

The line element of $f(x,y)$ is Liouville,

$$ds^2 = (x-y)(dx^2 - dy^2). \qquad \square$$

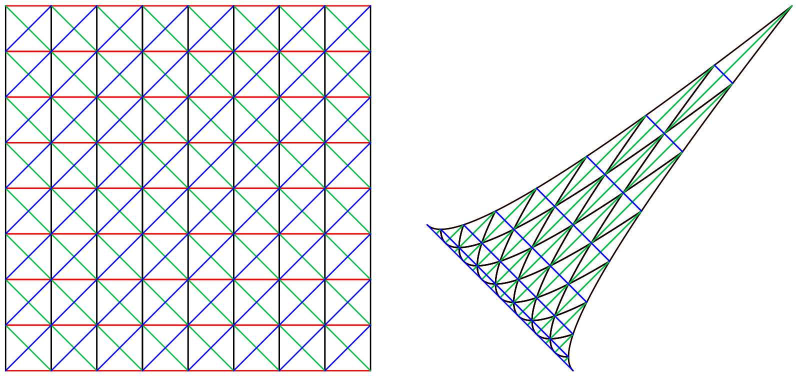

Figure 3: Map $f(x, y)$ from the Euclidean to the Minkowski plane

10. Some facts about orthotopes

An *m-orthotope* is the higher dimensional analogue of a rectangle; a rectangle can be called 2-orthotope. An m-orthotope in $\mathbb{R}^m$ can be specified by two points $P^0 = (p_1^0, p_2^0, \ldots, p_m^0)^t$ and $P^1 = (p_1^1, p_2^1, \ldots, p_m^1)^t$ with the property $p_i^0 \neq p_i^1$ for all $1 \leq i \leq m$. It has 2^m vertices (corner points) and 2^{m-1} main diagonals. For the number $B_k^m$of k-cell facets, $0 \leq k \leq m$, we have $B_k^m = 2^{m-k}\binom{m}{k}$ (see [14]). Each k-cell facet is a k-orthotope having 2^{k-1} main diagonals.

As an example, consider $m = 3$: For the sake of brevity, let's write 000 for P^0 and 111 for P^1. By counting from 000 to 111 in the binary system, we get all corners of the 3-orthotope: 000, 001, ..., 110, 111. We are interested in the main diagonals of the 3-orthotope. Start with the point 000. Its opposite point is 111, therefore 000–111 is a main diagonal. It is easy to see that the point ijk has as opposite the point $(1-i)(1-j)(1-k)$, where $i, j, k \in \{0, 1\}$. To get all diagonals, we must start with the vertices belonging to an 2-cell facet, for example all binary values starting with 0: 000, 001, 010, 011. By connecting these vertices with their respective opposites we get the $2^{3-1} = 4$ main diagonals as 000–111, 001–110, 010–101, and 011–100.

This example can be generalized and the procedure can be applied to every k-cell facet of the m-orthotope to get the main diagonals of that k-cell facet for $0 \leq k \leq m$.

11. A property of Liouville manifolds

We can state and prove the following main theorem of this paper.

Theorem 8. *Let $P^0 = (p_1^0, p_2^0, \ldots, p_m^0)^t$ and $P^1 = (p_1^1, p_2^1, \ldots, p_m^1)^t$ be two points in $\mathbb{R}^m$. Construct the uniquely determined l-orthotope, $0 \leq l \leq m$, which has a main diagonal $\gamma_1(t) = P^0 + t(P^1 - P^0)$ with $t \in [0, 1]$. Construct all main diagonals $\gamma_k(t)$, $1 \leq k \leq 2^{l-1}$, of this l-orthotope. Consider now the image curves $p_k(t) = \varphi(\gamma_k(t))$ of the main diagonals under the map $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ with Liouville line element*

$$ds^2 = \left(\sum_{i=1}^{m} f_i(x_i)\right)\left(\sum_{i=1}^{m} dx_i^2\right).$$

Then the energies of all image curves of the diagonals are equal,

$$E(p_1(t)) = E(p_2(t)) = \cdots = E(p_k(t)) = \cdots = E(p_{2^{l-1}}(t)).$$

We actually have a stronger result: in each h-cell facet, $0 \leq h \leq l$, of the l-orthotope the images of the main diagonals of the h-cell facet under φ have the same energy.

Proof. For the energy of a diagonal we have

$$E(\varphi(\gamma(t))) = \left(\sum_{i=1}^{m} \int_0^1 f_i(x_i \circ \gamma(t))dt\right)\left(\sum_{i=1}^{m} (x_i \circ \gamma'(t))^2\right).$$

It turns out that for every $1 \leq i \leq m$ we have $(x_i \circ \gamma'(t))^2 = (p_i^1 - p_i^0)^2 = (p_i^0 - p_i^1)^2$ and

$$\int_0^1 f_i(x_i \circ \gamma(t))dt = \int_0^1 f_i(p_i^0 + t(p_i^1 - p_i^0))dt = \int_0^1 f_i(p_i^1 + t(p_i^0 - p_i^1))dt.$$

Therefore the energies of the images of the main diagonals of the l-orthotope are equal. □

Remark 3. For $l < m$ we have $p_i^0 = p_i^1$ for some i, and the corresponding expressions are $f_i(x_i \circ \gamma(t)) = f_i(p_i^0) = f_i(p_i^1)$, respectively $(x_i \circ \gamma'(t))^2 = 0$. The same applies for h-cell facets with $0 \leq h < m$.

12. Open problems and ideas

We have shown that in every h-cell facet of each l-orthotope in a Liouville manifold the energies of the main diagonals are the same. My intuition says that also the converse is true, that is, if in every h-cell facet of each l-orthotope the energies of the main diagonals are equal, then the map/manifold is Liouville. But I think that this is not easy to prove.

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