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Perspective Between Fiction and Function: Pattern Mutations Through Science and Art

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Abstract. This paper aims to focus on the *mutations* occurred in the perspective pattern as signs of changes not only in the projective knowledge but also in the same idea of space. On this purpose, perspective sets from treatises between the 15th and 19th century will be discussed and compared as case studies, which will reveal significant differences in terms of geometric approach and graphic procedures. As Robin Evans could say [8], the former seem to follow the method of Art, aiming to provide affordable constructions for *pictorial fiction*, while the latter seem to concern more closely the method of Science, mainly focusing on *projective function*. Therefore, a question about 'what' the two systems actually show will emerge: according to the spirit of the ages, starting from showing the *image of objects*, perspective seems to have gradually shown the *image of space itself* over time.

Key Words: perspective, homology, projective geometry, descriptive geometry, history of geometry

MSC 2010: 00A66, 51N05, 01A05, 97U99

1. Introduction

In terms of graphic diagrams, nowadays a Perspective manual normally starts showing 'the end of the story': an adamantine reference system made of points, lines, circles and subtended planes, namely the main point and main distance, the horizon and ground lines, the distance and visual circles (Figure 1b). As well as a Projective Geometry manual does (Figure 1c). We will see that these two latter schemes are significantly different from that described at 'the beginning of the story' proposed by Leon Battista Alberti in *"De Pictura proestantissima et numquam satis laudata"*, the first official treatise following the Filippo Brunelleschi's perspective code: a point and a line on the picture plane, namely the main point and a segment on the ground line, together with a separate side view of the system, superbly graphically combined by Piero Della Francesca in the book *"De Prospectiva Pingendi"* (Figure 1a).

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About five centuries elapsed between the first and the last scheme in Figure 1, and the three diagrams work as operational *menus* of three different 'releases' of the projective 'software', which are significantly different one another, although it might be hard to believe by simply comparing the mentioned diagrams. The Renaissance method, indeed, assumes a not-extended Euclidean space, still wanting the ideal elements, and only the centric point is handled; in technical terms, the receding elements are located according to a foreshortened squared ground plane, distorted in advance by using the intersection point appearing in the overlapping side view. The reference system is still affected by the *ballistic principle* generating the perspective method, and referred to a confined space. We will call it *oriented projective box* (Figure 1a). On the contrary, the current model lays on the hypothesis of an extended Euclidean space including the ideal elements, whose projections as vanishing elements furnish the key bases for the graphic representation. In technical terms, it uses the homology as a key algorithm to relate figurative and metrical information, and all the constructions can be drawn on the picture plane, at the point that the reference system itself is set up as a *homological system*. The horizon and the ground lines humanize this mathematic scheme, which shows an *oriented projective space* (Figure 1b). Meanwhile the Central Projections setup is completely independent of any human orientation, showing a pure extended projective space (Figure 1c).

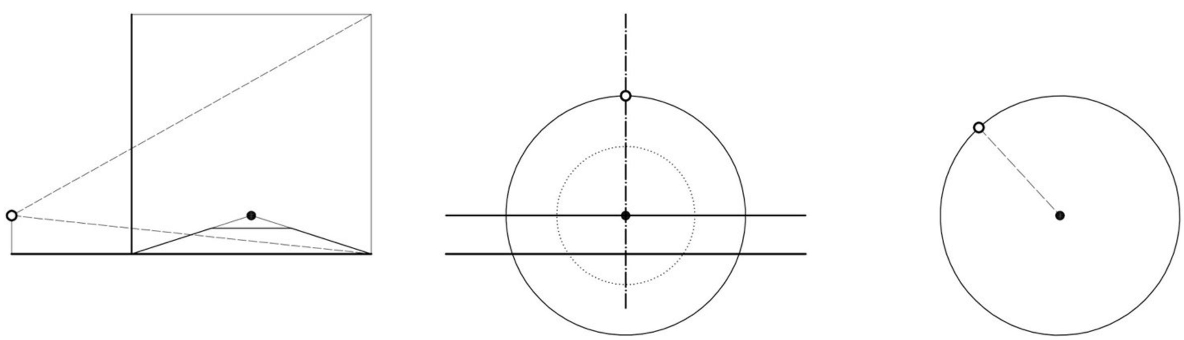

Figure 1: Perspective pattern mutations; a) left: early setup, 15th century (Piero Della Francesca), or the *oriented projective box*, b) midst: present setup, 17th to 19th century, or the *oriented projective space*, c) right: central projection setup, 19th to 20th century, or the *extended projective space* (drawn by the author).

2. The Diatribe: "Figure" and "Measure"

When speaking about *perspective*, people get normally fascinated by beauty and visual realism of the images. Quite differently, when dealing with *axonometric* and *orthogonal projections*, the metric sense tends to overcome the aesthetic feeling. In fact, over centuries we have been largely used to relating perspective to Art, while the others mentioned representative forms to Science and Technique. In this paper, we would like to remind that this is just a familiar habit. The fact is that since the beginning perspective, from which all the other forms of representation derive, we aimed to work on the relationships between geometry of images and geometry of space. It grew up relating the Euclidean visual principles of Optics and the Euclidean spatial principles of Elements (3rd century BC). The intrinsic connection between *figure* and *measure*, that is to say between Graphics and Geometry, should clearly arise from comparing the schemes proposed by Filippo Camerota (Figure 2) [2]. We know that Filippo

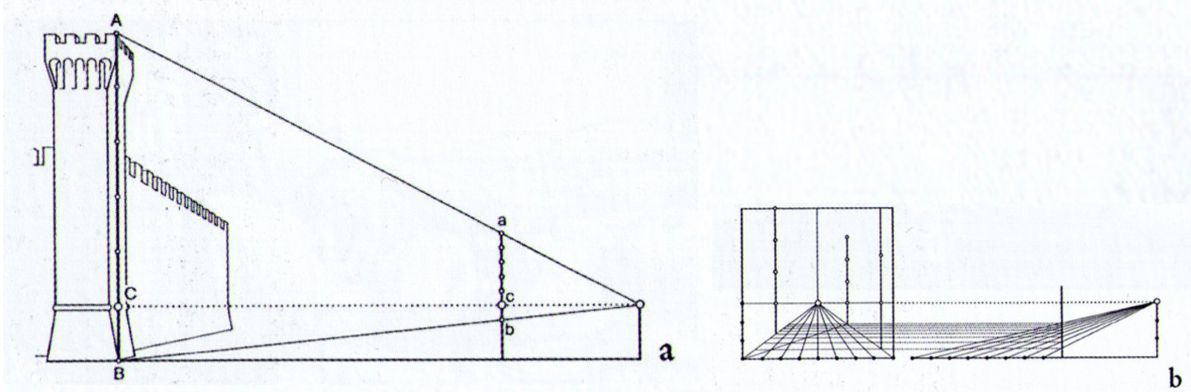

Figure 2: The double-dealing mode of the projection/section process: 'measuring' mode (a) and 'imaging' mode (b), according to Filippo Camerota [2].

Brunelleschi, author of the first *perspective demonstrations* (c. 1410) and better known as the father of perspective, was also an expert in the field of surveying. Indeed, the principle of *projection/section* he adopted, as described into the Leon Battista Alberti's treatise *"De Pictura"* (1475), can be used either for taking measures or redrawing images from life, as well as, and most of all, for generating images from imagination, which are visually and metrically consistent with the geometry of the real space.

Two other aspects can be discussed at this point. The first one concerns the wide use of perspective in the Renaissance painting. Compared to Science, Art appeared more advanced at that time, and painting seemed to be the most effective way to fix/produce/show 'verity', even more qualified by the use of proved geometric processes. As Erwin Panofsky stated [13], the perspective allowed an *'objectification of the subjectivity'*. In other words, we have good reasons to think that, if from a theoretical point of view Science was still incorporated in Philosophy, from an experimental point of view it was still embedded in Art. The second issue concerns the 19th century advancements in the field of Projective Geometry. Stressing the image of the real space by sometimes very strong projective distortions, indeed, Projective Geometry also stimulated studies and researches about new spaces and geometries. At the point that Arthur Cayley (1821–1895) stated that Projective Geometry itself would be *the* Geometry, being able to manage not only its own properties, but also those belonging to other geometric branches, as also clearly and simply proved by the great deal of helpful projective visualizations and representations concerning non-Euclidean spaces.

3. Early stage: ballistic setup

As space has three dimensions against the two of the picture plane, at the beginning the graphic perspective setup was based on at least two preparatory views, including at the same time the spatial location of the viewpoint, its projection on the picture plane, and a portion of space around the object to represent. Width and height could be directly and easily managed on the picture plane, while the receding depths needed to be determined with the help of a schematic side view apart, as Leon Battista Alberti recommended, which did not work in favor of the purity of the graphic constructions, since some measures had to be taken and manually translated on the perspective canvas. This obstacle might have suggested to combine the two views in one, as amazingly represented in Figure XIII by Piero Della Francesca, where the side view is finally rotated into the picture plane (Figure 3). We propose a digital

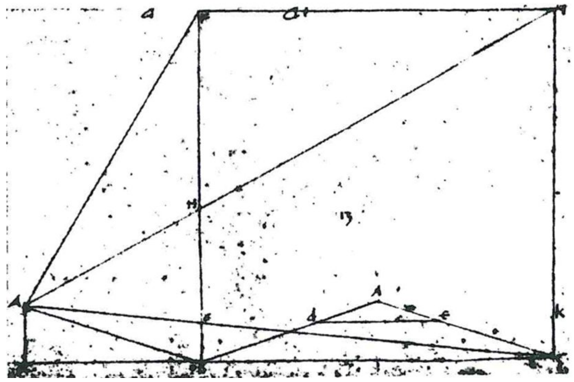

Figure 3: Piero Della Francesca's figure XIII. Perspective setup combining front view, side view, and perspective image (c. 1475) [22].

animation to show a possible expansion of the graphic constructions in the three-dimensional space (Figure 4). The wedding between Optics and Elements was finally celebrated and every geometric configuration could be easily shown in perspective only by drawing on the picture plane. Using a modern language, *metrics* and *position* became strictly related. In our opinion, this was the early crying towards the homology.

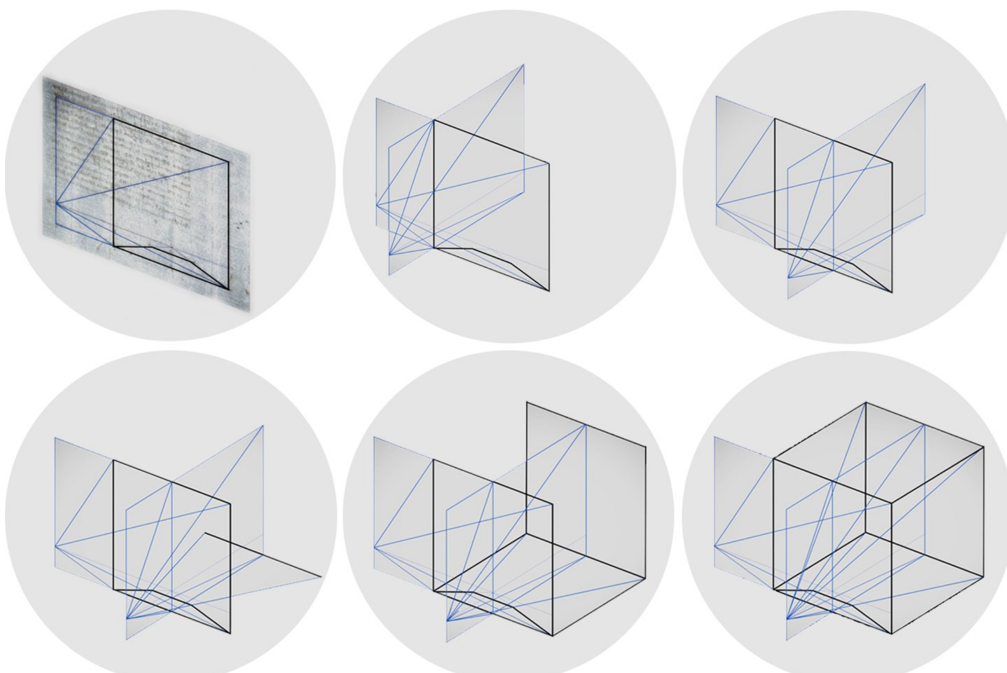

Figure 4: Perspective setup expanded in the three-dimensional field (set by the author, digital animation by K. Komarovskyi).

In *"De Prospectiva Pingendi"* (c. 1475) Piero Della Francesca also proposes two other methods for drawing perspective images of any possible shape, linear or curved, by using auxiliary orthogonal projections *(ante litteram).* The first of these methods graphically replays the *ballistic* process carried on by wires in the real space, as later shown in the famous images of Albrecht Dürer's treatise *"Unterweysung Der Messung . . . "* (1525). This part contains the most beautiful drawings of Piero's treatise, like the well-known perspective representations of human faces, but since we decided to follow the homology development, we

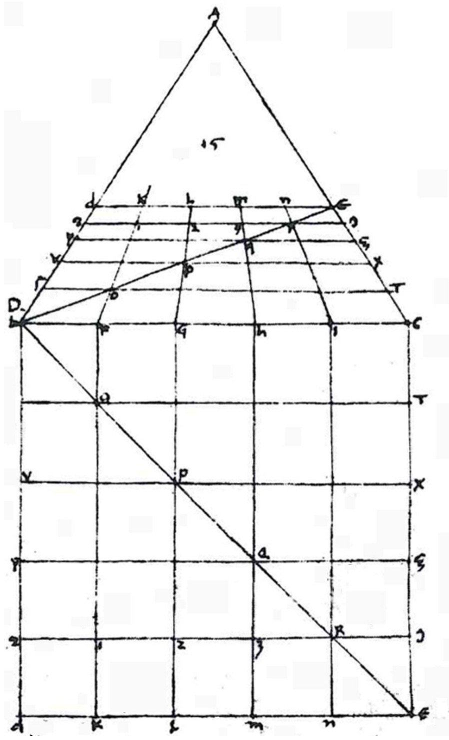

Figure 5: Piero's Figure XVa. Perspective and true shape of a tiled ground, graphically related [22].

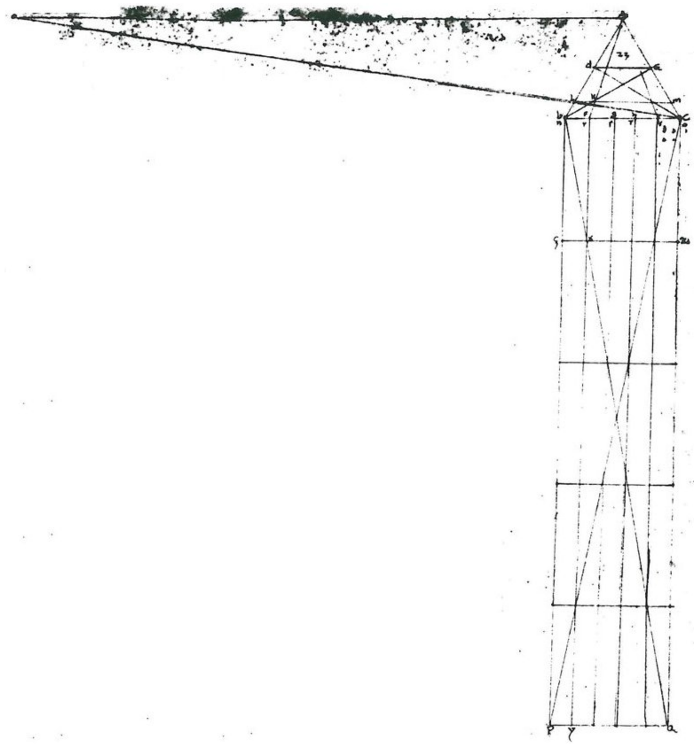

Figure 6: Piero's Figure XXIII. Perspective of a long corridor subdivided into squared areas by using the construction of the distance point [22].

will not talk about these methods. For the same reason, other practical procedures, like those using predefined foreshortened grids on the picture plane as in Paolo Uccello's (c. 1450) famous *sinopite*, will not be discussed as well, in spite of their wide diffusion over the European area at that time.

Then, going back to our story, once the perspective setup is defined, Piero shows us a new surprising image in Figure XVa, where the perspective of a tiled ground is related to its true-to-size representation, supposed rotated into the picture plane (Figure 5). In spite of the absence of the viewing distance, this drawing can be considered as the first 'ancestor' of a homological representation. Only depth lines are foreshortened towards the main point, while a diagonal is used as an auxiliary constructive element, as shown through the following several examples in the treatise. Even more surprising is Figure XXIII, where the diagonal is extended beyond the boundary of the foreshortened ground, up to the line of sight (Figure 6). It works as the vanishing point of the pencil of diagonals, but, differently from other authors, Piero uses it only for one diagonal, namely, the only one he utilizes in the following examples as an auxiliary construction line, and most of all he does not show this extended construction again in the treatise.

This is the reason why some theoreticians, like Giusta Nicco Fasola [7], think that it has been used for the first time as a properly said *distance point*, allowing to directly showing the viewing distance on the picture plane, as the distance of that point from the main point. It is evident that this construction has a theoretical value, as it will be clearly understood afterwards. Therefore, aiming at giving easy operational instructions to the painters, Piero shows many practical examples, referring them to the perspective of a square ground, in other words, working into an ideal perspective box.

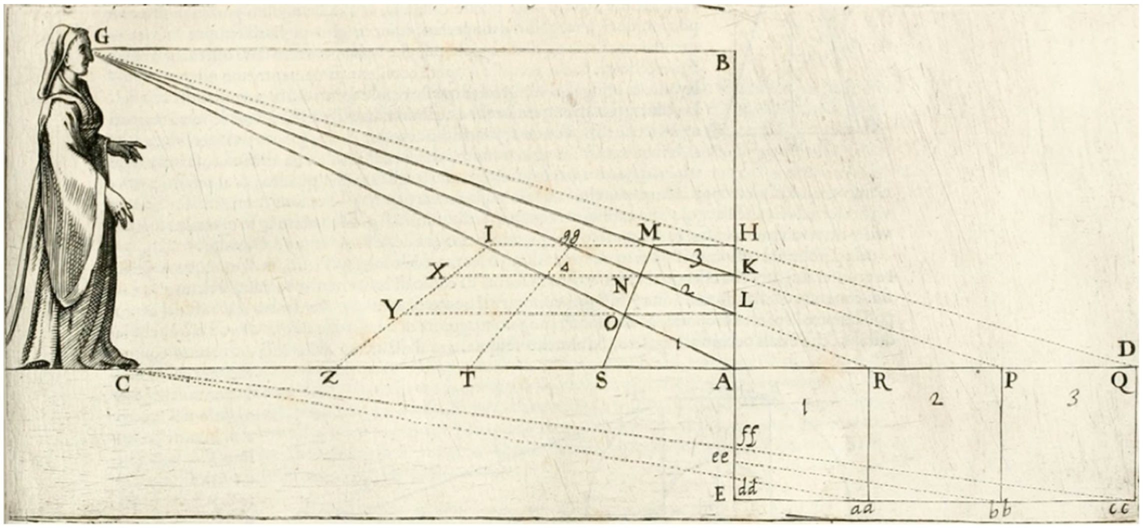

Figure 7: The *seconda regola* by Jacopo Barozzi da Vignola (1583). Double perspective function of the viewing distance [19].

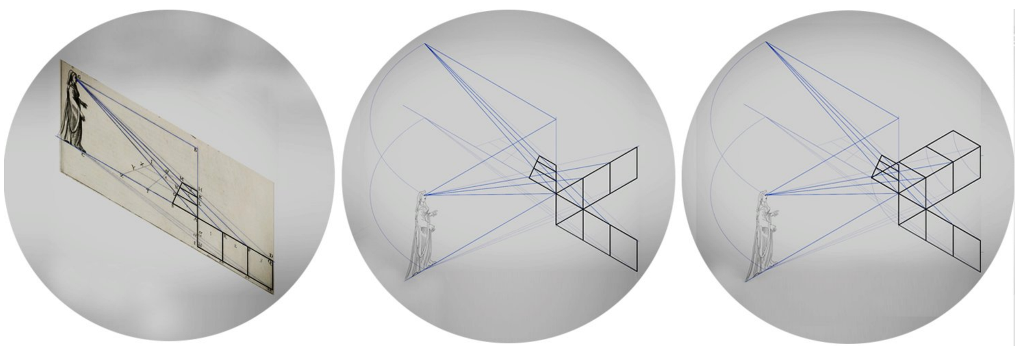

Figure 8: The *seconda regola* by Vignola: graphic constructions and spatial setup (set by the author, digital animation by K. Komarovskyi).

The double function of the viewing distance, as a way to mark receding points on the canvas, and to find the perspective meeting point of the related pencils of diagonal lines, is clearly illustrated in Jacopo Barozzi da Vignola's *seconda regola* (Figure 7). Again a digital animation helped us to understand the implicit spatial reasoning (Figure 8).

Combining plan and side view on the picture plane, Vignola shows the constructive principle of a tiled ground. Graphic features help him to explain the procedure. Looking closely at the image, indeed, projector lines (dotted lines) and construction lines (solid lines) are clearly distinguishable. Then, we can also see it as one of the first examples of the power of alliance between Geometry and Graphics for educational purposes. This diagram, coming one century later, can be considered as a further development of Piero's setup, either because of the insertion of the top view, or due to the clear use of the distance point also as a meeting point of the diagonals of square tiles. For decades, this point has been used as the only auxiliary point in the perspective constructions, not only in relation to horizontal but also vertical faces of the objects (Figure 9). However, while distance points on the horizon line were very frequently use, those related to vertical planes were very rarely adopted.

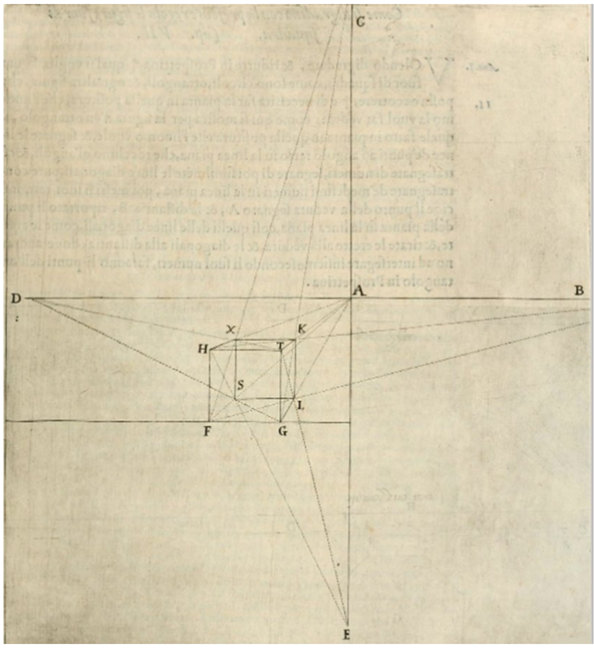

Figure 9: Extension of the notion of distance point to the vertical faces of a cube represented in perspective, from the *"Commentarii"* of Egnatio Danti to Vignola's treatise [19].

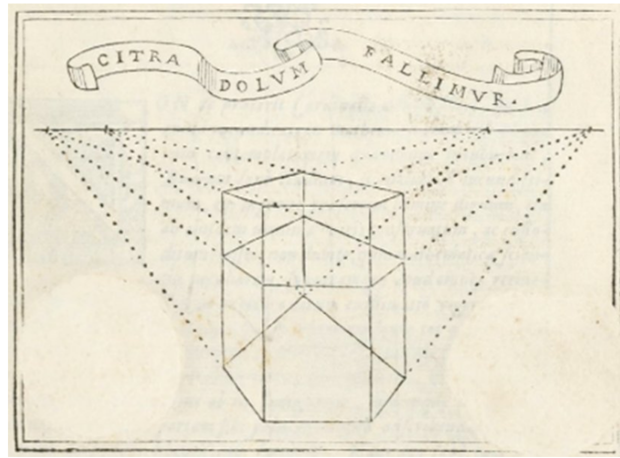

Figure 10: *"Perspectivae Libri Sex"* (1600). Frontispiece, showing several concurring bundles of horizontal lines [20].

4. Work in progress: the new vision

Between the 16th and 18th century, when mathematicians became more interested in perspective, the focus gradually changed, moving from the represented object to the representational structures. Guidubaldo Del Monte's treatise titled *"Perspectivae Libri Sex"* (1600) shows important advancements in this field. The demonstrations are clear and very well understandable. Before showing the perspective image of specific objects, a method is shown to represent in perspective any lines and planes however oriented in the space, as clearly anticipated in the image on the frontispiece of the book (Figure 10). The treatise also includes a chapter on scenes, which means, how to realize three-dimensional perspectives for theatrical representations.

Although the — properly said — notion of a vanishing point was still unknown because of the lack of the notion of a 'point at infinity', Guidubaldo Del Monte highlights the projective relationship between directions of bundles of lines in the real space and meeting points of the corresponding pencils of lines in the perspective image. These meeting points are called *puncta concursus*, that is, concurrent points, what we nowadays call vanishing points. At the end of the sixth book, an image is proposed where apart from the vertical lines no other lines are parallel to the picture plane, and the lines belonging to the sloping ground as well as all the other sloping lines parallel to them, meet at a concurrent point located above the horizon level (Figure 11a).

Concerning operational instructions, to make the constructions as clear as possible, in the first book not only graphic procedures on the picture plane are carefully shown, but also, *how* these procedures are generated according to the spatial location of the elements. This makes the treatise definitely modern (Figure 11b).

Anyway, once the connections between space and image are explained, a graphic method to

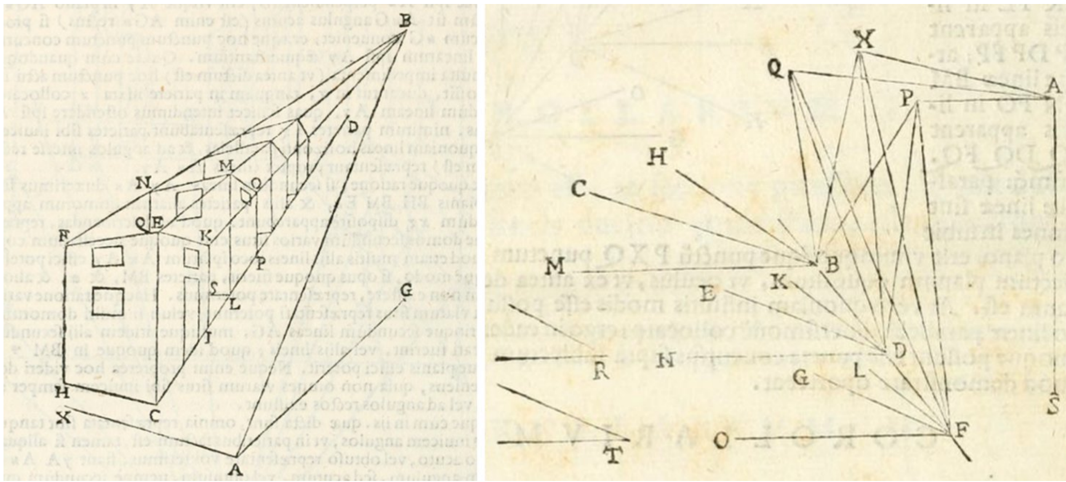

Figure 11: *"Perspectivae Libri Sex"*. Left (a): perspective representation including a sloping bundle of lines. Right (b): spatial relationships between real lines and their concurring perspective images [20].

draw perspectives is afforded. At the beginning of the second book he proposes a perspective construction based on the principle of the *rabatment* of a ground plane on the picture plane. Then, both the true figure and the viewing distance, together with sight height and visual rays are rotated into the picture plane, and used for constructing the perspective images, which of course, appear on the same picture plane.

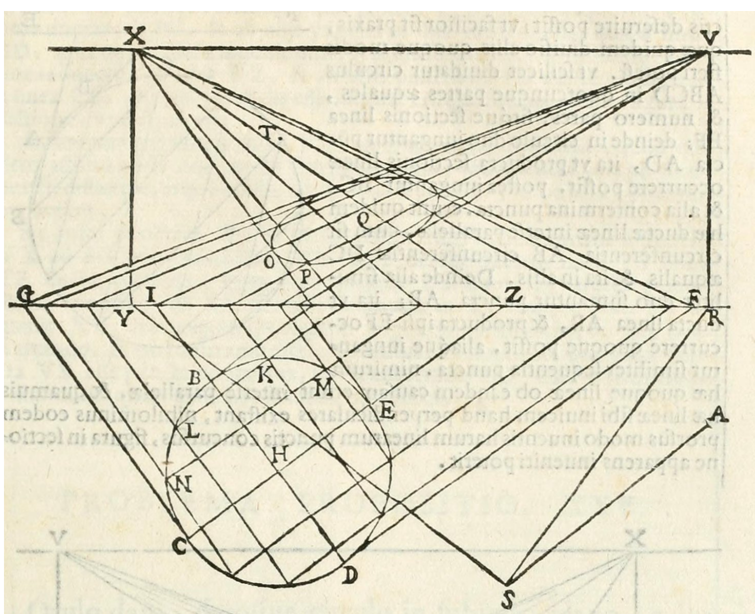

Figure 12: *"Perspectivae Libri Sex"*. Perspective of a circle, based on two orthogonal directions in the ground plane [20].

As an example, we have selected the construction XXIIII from the fourth volume of the treatise (Figure 12), focusing on the perspective of a circle, which is located on the ground plane. As we can see, the true shape and its perspective appear opposite by the effect of rotation about the ground line, below which, together with the true figure, in this case a circle, the sight point and the visual rays he used to finding the concurrent points, have been

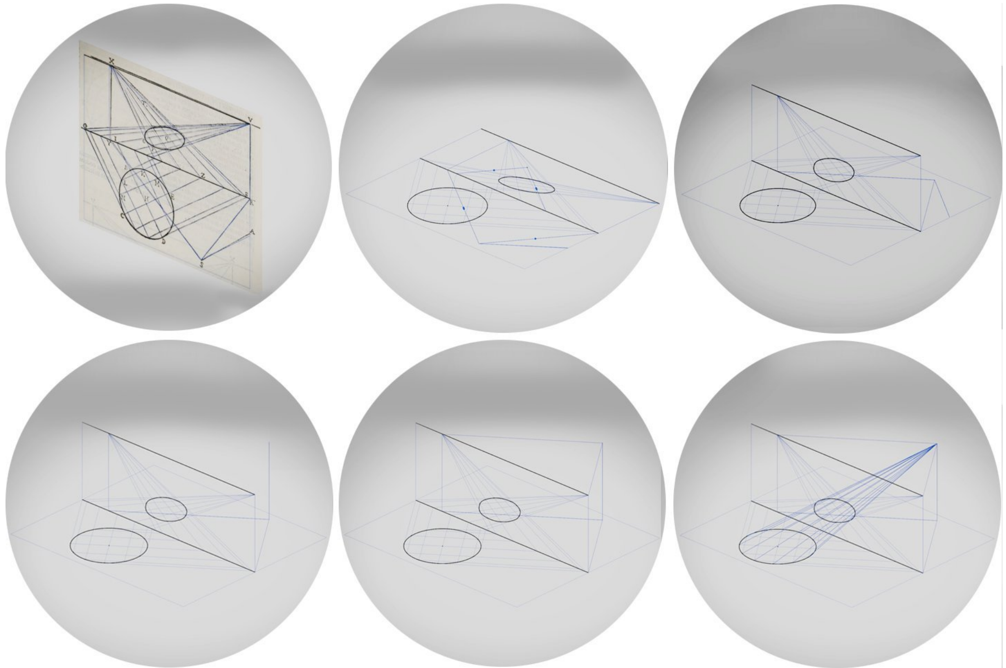

Figure 13: *"Perspectivae Libri Sex"*. The perspective setup unfolded in the digital space (set by the author, digital animation by K. Komarovskyi).

rotated as well. To find the right directions of the visual rays, the centre of the circle H and the two director lines HI and HZ need to be rotated, beforehand, above the ground line; for the sake of conciseness, in the figure only the point T, corresponding to H, is drawn above that line, while director lines are omitted. A possible sequence of the subtended geometric motions in the space has been developed through a digital animation (Figure 13).

Compared to the *rabatment* proposed by Piero Della Francesca, the metrical information about the sight point is here more explicit. However, although totally carried out in the picture plane, the constructions show some discontinuities and redundancies. Firstly, the abovementioned procedure for the recognition of the designated directions needs a double spatial rotation. Moreover, the horizon line is not used to find out, but only to show the meeting points of the pencils of perspective parallel lines. Indeed, the visual rays showing the corresponding real directions are drawn on the ground plane, that is, starting from the *foot* of the observer (point S) instead of starting from his *eye* (point A); consequently, the meeting points X and V are not directly found on the horizon line, but obtained by translating the points Y and R, which previously were defined on the ground line.

But the spatial setup was finally clear, and in the last part of the treatise, which is dedicated to scenography, the new knowledge and constructions are also applied to any possible location of elements in the space, including vertical planes (Figure 14).

Published in 1600, Guidubaldo Del Monte's treatise marks the threshold between two centuries and eras. All in all, it represents a great step towards the modern projective approach and, last but not least, towards the discovery of the projective links between true shapes and foreshortened images, that is, the basic ingredients of the homology. In the following decades of the 17th century, the theory of perspective could also take advantage of the new scientific advancements.

One of the most revolutionary ideas was proposed in the book *"Ad Vitellionem Paralipomena"* by Johannes Kepler (1604), who finally clearly defined the notion of point at

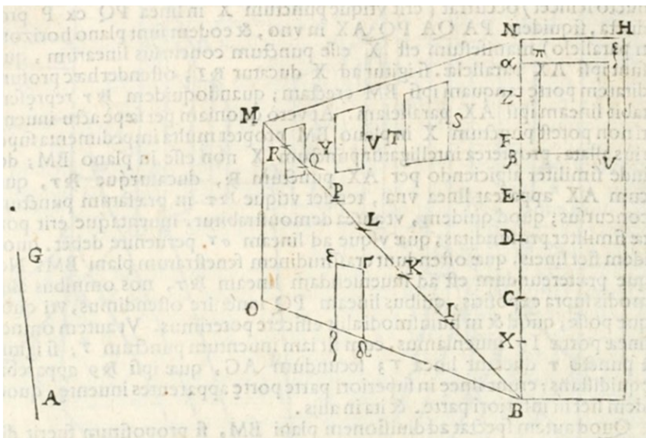

Figure 14: *"Perspectivae Libri Sex"*. Perspective construction of a vertical plane in a scenic space [20].

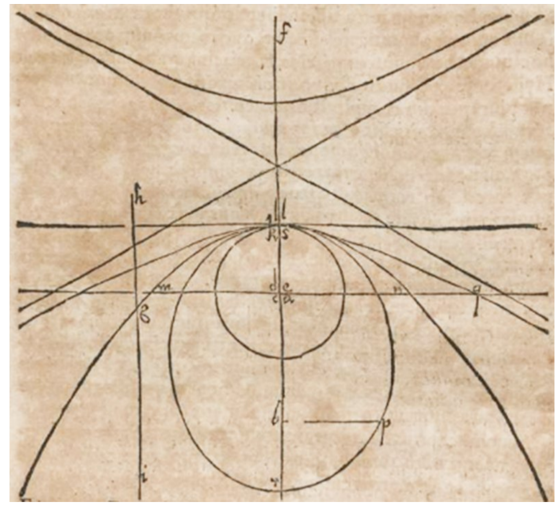

Figure 15: Johannes Kepler, *"de coni sectionibus"* (1604) [21].

infinity (Figure 15). He found it in relation to Optics, while discussing about the second focus of the parabola, which is, as we nowadays know, infinitely far. Then he also stated, for the first time, that bundles of parallel lines meet at points at infinity, so opening the way to the modern idea of a vanishing point, intended as a projective image of a point at infinity. Some years later, also Girard Desargues proposed new revolutionary ideas, like the projective analogy between conic and cylindrical projections, and the projective equivalency between straight lines, circles, and degenerate conics. From a strictly projective point of view, his theorem on corresponding triangles, which appeared in the *"Livrette de perspective adressé aux le théoriciens"* (1643, lost), drew the very fundamentals of the homology.

The spatial configuration resumed in Figure 16a shows the basic elements of a perspectivity between two planes, namely center, axis, corresponding elements. The plane scheme

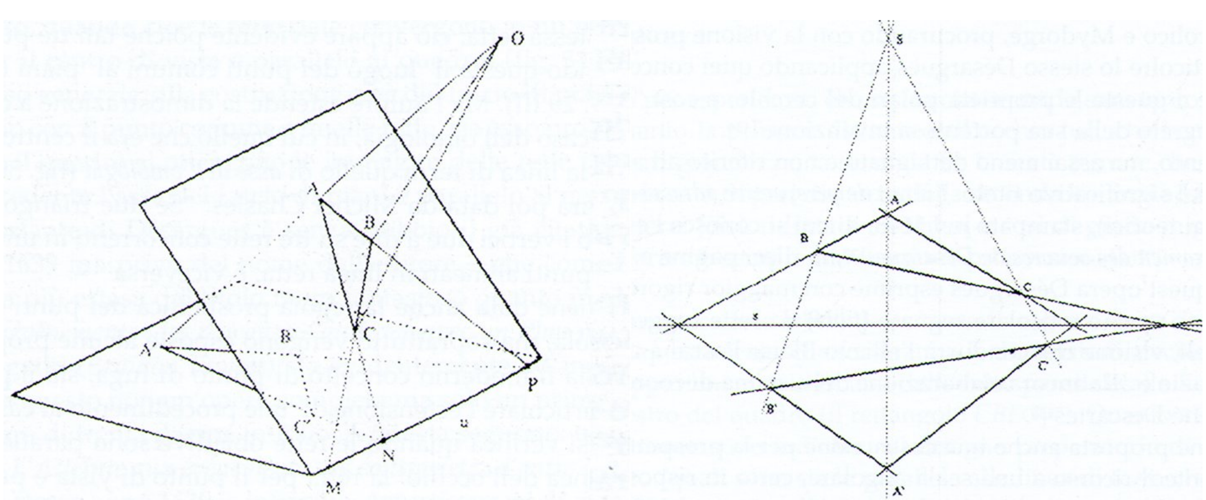

Figure 16: Girard Desargues, *"Livrette de perspective"* (1643). Scheme on the left (a): corresponding triangles, spatial version [6]. Scheme on the right (b): corresponding triangles, plane version [6].

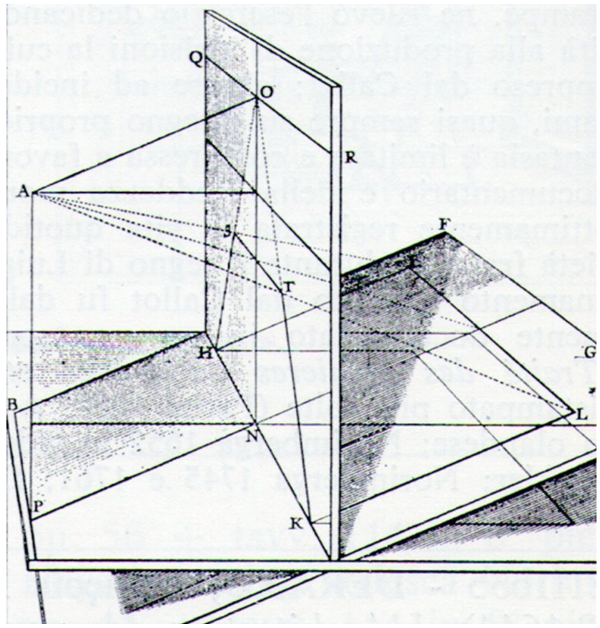

Figure 17: Etienne Migon, *"La Perspective Speculative et Pratique"* (1643) [17].

in Figure 16b shows, at the same time, a perspectivity between two figures belonging to the same plane, or better, to two overlapping planes. But, if we consider this configuration as a perspective projection of the configuration in Figure 16a, it shows a *homology*, in spite of the fact that this name will be used only about two centuries later, by Jean Victor Poncelet. Other important properties of space and conics were discovered and proposed by Desargues by using graphic methods originating from perspective, which from that moment on became to acquire an independent theoretical value, so preparing the way to the birth of Projective Geometry. As stated by Jean-Pierre Le Goff and referred by Anna Sgrosso [6], this new use of the projection shows one of the early heuristic applications of Perspective to Geometry. In other words, if in the past Geometry was used to enhance Perspective, from this moment on also Perspective could be used to enhance Geometry.

Anyway, a new theoretical sensitivity was now diffused, together with a new approach to perspective, taking into account or developing new geometric ideas not only into literary masterpieces but also into some minor works. Among these, we would mention the treatise *"Perspective Speculative et Pratique"* by Etienne Migon (1643), from which Figure 17 is taken, where not only lines, but also planes are finally acting — another step towards a modern approach.

We will close the review of this period with another interesting practical treatise, written by Giulio Troili, or Paradossi, whose intriguing title is *"Paradossi per praticare la prospettiva senza saperla (. . .)"* appeared in 1672, literally "paradoxes to practice perspective without knowing it (. . .)". Contrary to the expectations, the treatise is very accurate and the constructions are clearly represented and explained [23].

The remarkable news in the two Figures 18a and 18b is the distance circle, that is the circle drawn on the picture plane having the viewing distance as a radius, and connecting the four distance points represented by Egnatio Danti into Vignola's treatise (Figure 9), although the very first evidence of the distance circle has already appeared in the unpublished treatise *"Prospettiva pratica"* by Ludovico Cardi, or "Cigoli" (1612).

Anyway, the picture plane setup now shows all the modern ingredients: *ground line,*

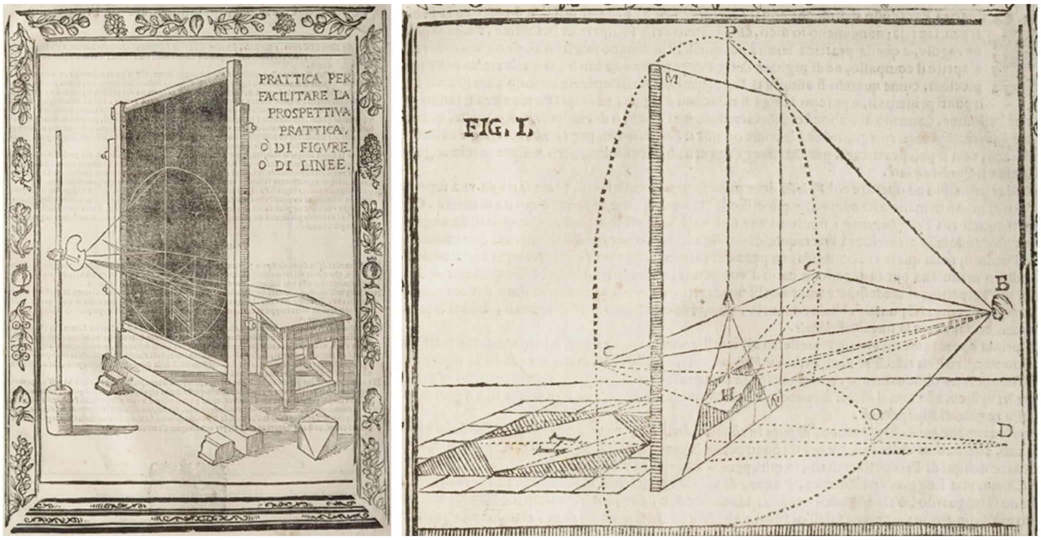

Figure 18: *"Paradossi per praticare la prospettiva senza saperla"* (1672). Left (a): Paradossi's perspective setup including the distance circle. Right (b): another image of Paradossi's perspective setup, showing the connection between picture plane and ground plane [23].

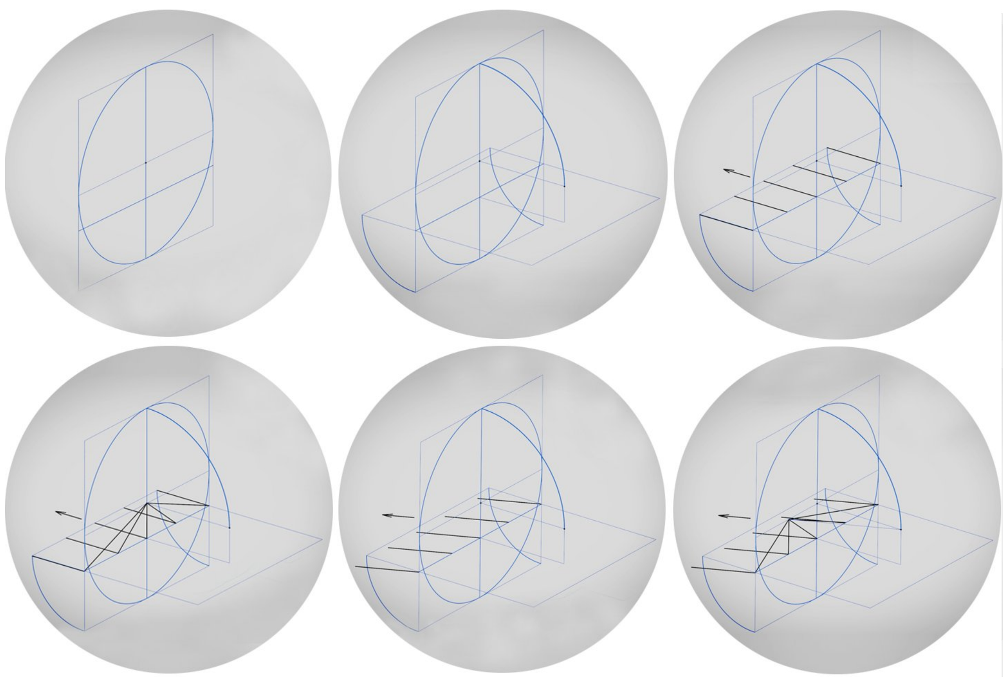

Figure 19: Spatial development of Paradossi's setup (set by the author, digital animation by K. Komarovskyi).

horizon line, main point and *distance circle*. As we see in the animation, apart from the horizon plane, not yet defined at that time, the Cardi Cigoli / Paradossi's spatial setup is very similar to that in use nowadays (Figure 19). All these elements would have been fully used in the following centuries. At the moment, in spite of the substantial theoretical advancements, the main goal seems to remain the correct graphic construction of figures, therefore also theory is developed and managed according to this purpose. But it is just in this period that the passage from the *fictional* era, aiming to provide images from the space by perspective, to the *functional* era, aiming to carry on researches on the space by perspective, had his early start.

5. The mature age: final algorithm

At the beginning of the 18th century, the connections between true shapes and their images on the picture plane were clearly and fully understood. Before taking its name, the *homology* was actually 'at work', either in the *construction* of perspective images (Figure 20a), or in the *reconstruction* of true shapes from perspective images (Figure 20b), as shown in the *"New principles of linear Perspective (...)"* by Brook Taylor, firstly appeared in 1715.

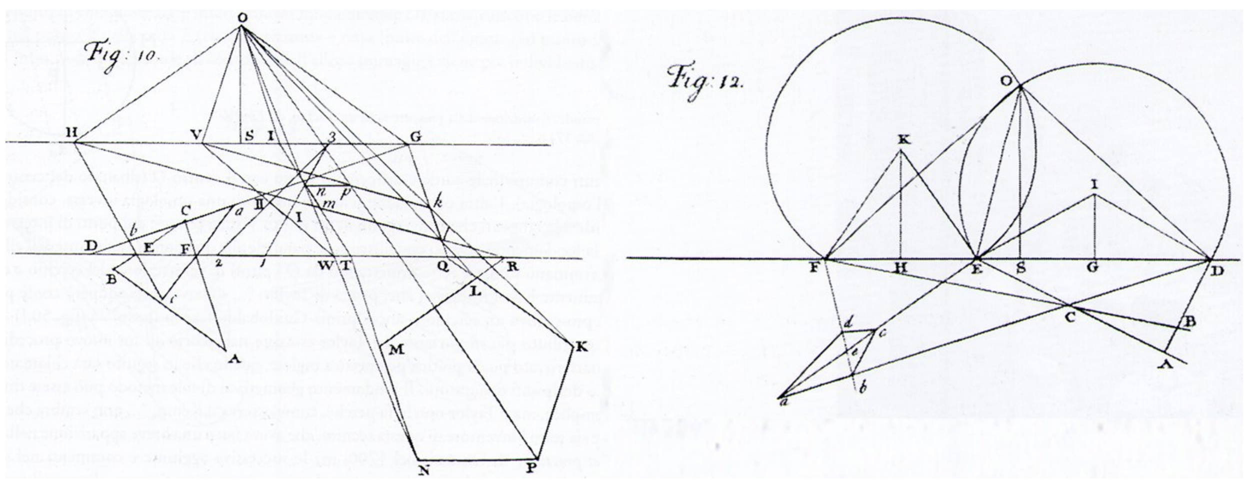

Figure 20: Brook Taylor, *"New principles of linear Perspective (...)"* (1749; 1st ed. 1715). Left (a): perspective drawing, starting from a given setup. Right (b): setup finding, starting from a given perspective [18].

The graphic procedures came from a clear understanding of the spatial setup of the geometric elements and of their transformations (Figure 21), while new theoretical investigations were driving perspective towards its final 'mutation' as a branch of Mathematics.

Following the theoretical line starting with Euclid and empowered by Desargues, the *"Traité des Propriétés projective des figures"*, by Jean Victor Poncelet, appeared in Paris in 1822, is the 'baptismal' work for a new geometric field based on the principles of perspective, namely the Projective Geometry, a so widely general field that Harold Scott MacDonald Coxeter stated that even Descriptive Geometry, the new brilliant field codified by Gaspard Monge at the end of the 18th century, has to be considered an *accident* of Projective Geometry. Getting rid of the 'human' visual restrains connected with perspective, the *projection/section* principle became a mathematical tool to fully 'inflect' in the abstract space, in search of new undiscovered properties (Figure 22). History testifies how important Projective Geometry would have been for the future advancements in Mathematics [1]. Focusing on the central projection, the projective invariants were finally defined as the bases of this new Geometry. Later on, thanks to Joseph Diez Gergonne (1847), also the fundamental principle of *duality* would have been found. With Poncelet' s treatise, finally *homology* took its name and acquired its official role, not only in perspective, but also and more generally in Geometry.

In a few years Projective and Descriptive Geometry increasingly interacted, like in the brilliant work of Wilhelm Fiedler. Since in *"Die Centralprojection als geometrisch ... "*, pub-

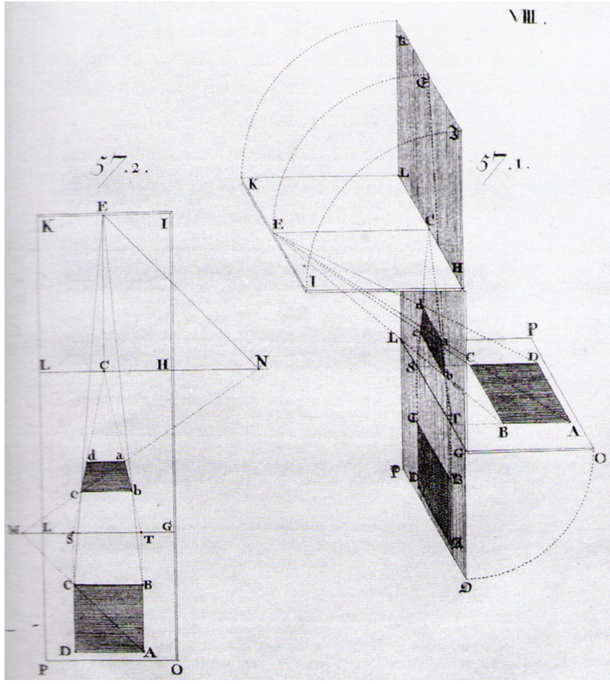

Figure 21: Joshua Kirby, *"Dr. Brook Taylor's Method of Perspective (...)"* (1754). Image (left) vs space (right): folding viewing and subject planes into the picture plane [11].

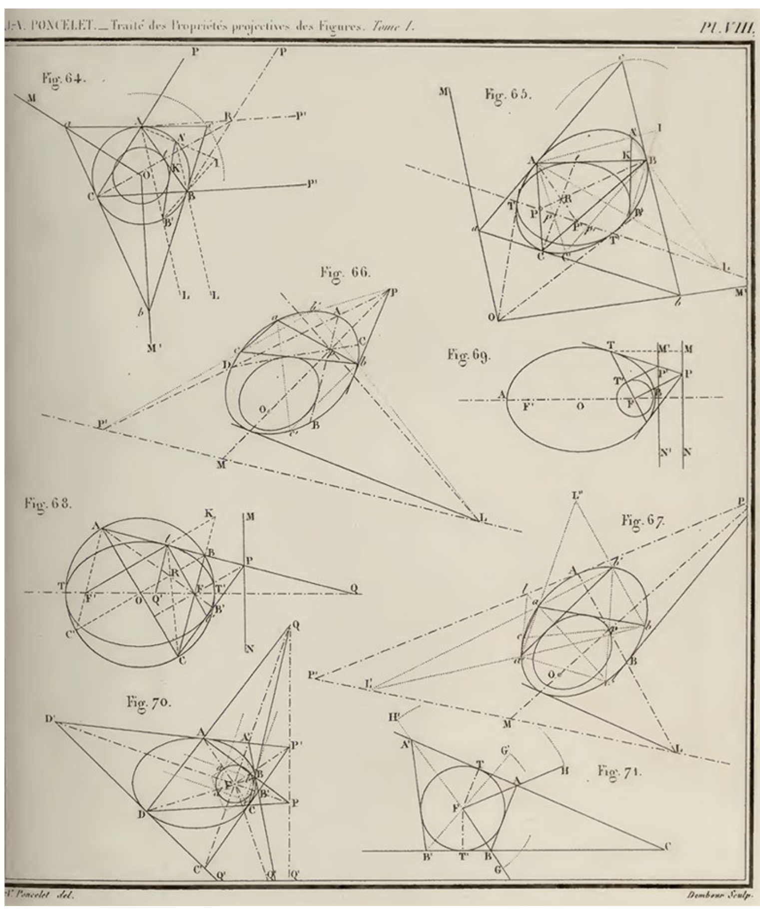

Figure 22: Jean Victor Poncelet, *"Traité des Propriétés projective des figures"* (1865; 1st ed. 1822), Planche VIII [24].

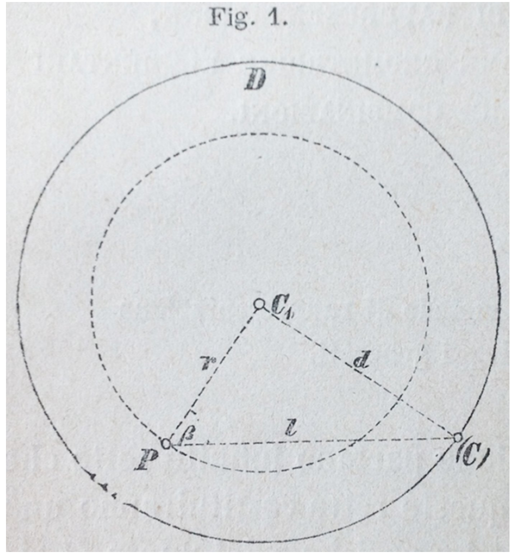

Figure 23: Wilhelm Fiedler, *"Die Darstellende Geometrie"* (1874, 1st ed. 1871), central projection, the graphic setup [9].

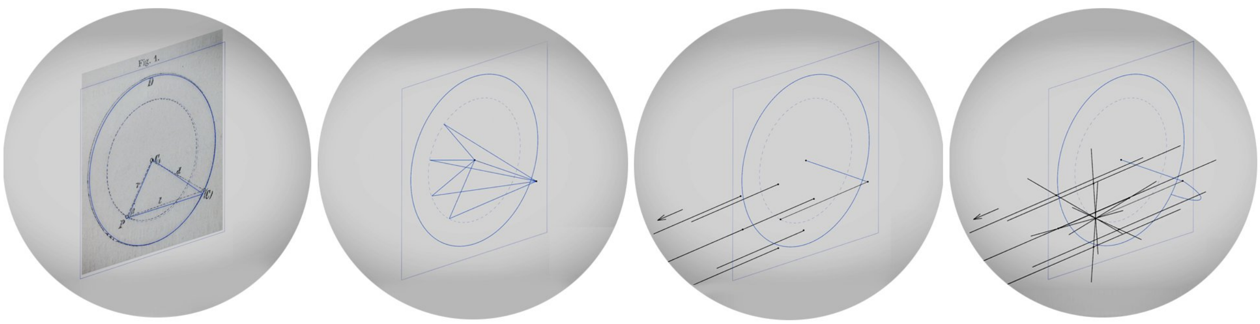

Figure 24: Relationship between ideal point and vanishing point, based on Fiedler's setup (set by the author, digital animation by K. Komarovskyi).

lished in 1860, he remarked that Descriptive Geometry should include more than Monge's projections. In fact, the first figure in his masterpiece, the treatise *"Die Darstellende Geometrie"* appeared in 1871, shows the graphic setup of a central projection (Figure 23), whose geometric principles form the fundamentals of all the other projective methods. Starting from this figure, projector lines and angles, conflicting since Euclid's theorem VIII, find a final appeasement, a clear definition and a reciprocal function. The animation shows the key points of the relationships between true spatial directions and vanishing points on the picture plane, according to the corresponding orientation of the visual rays (Figure 24).

A special place is deserved to the *homology*, that is, not only the one acting into the picture plane (supporting *drawing*), but also the spatial homology between three-dimensional configurations (supporting *modeling*), both available for scientific as well as for technical and artistic purposes, while in the last part of the book even the connections of this geometric branch with the algebraic procedures through projective coordinates are discussed. Last, the author does not forget to mention the aesthetic value of the projective methods, since he clearly states that by means of drawing (and models) Descriptive and Projective geometry combine *exactness* and *beauty*. The following four images show a wide field of possible homological applications, and the crucial role played by the graphic setup, that is, main point and

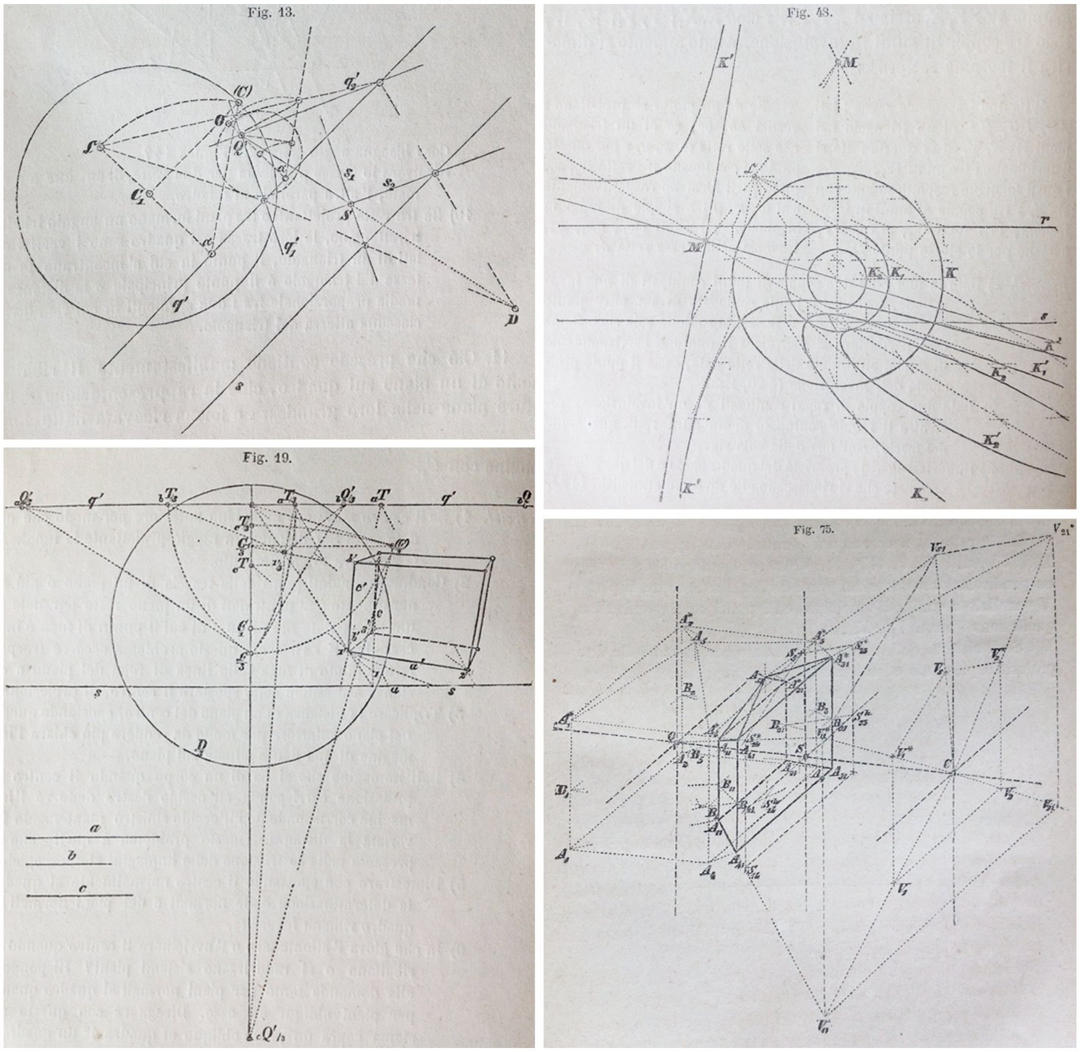

Figure 25: W. Fiedler. Above, left: homology, the graphic algorithm [9]. Above, right: homology in action in the abstract space [9]. Below, left: homology in action in a perspective drawing [9]. Below, right: explanation of homology in action in a three-dimensional configuration (perspective relief) [9].

distance circle, as reference elements for any transformation (Figure 25).

While mentioning these refined ‘visible’ images, we have to remind that a few years before, thanks to the pivotal work *“Geometrie der Lage”* by Karl Georg Christian von Staudt (1847), Projective Geometry had been sublimated as a pure mental structure *(geometry of position)*, independently of any metric contingency *(metrics geometry)* and of any graphic representation. Surprisingly, the lack of images works here as a powerful 'amplifier' for imagination, whose priority is in getting the universal properties of space before looking at real configurations. Which is also a recommendation that Staudt thoroughly addresses to teachers, not to put students too soon in the middle of metrical details, but to help them to understand what the space is and how it works. A recommendation resounding still valid nowadays.

We would like to complete our short excursus by mentioning Guido Hauck's treatise *“Die malerische Perspektive”*, published in Berlin in 1882, where the properties of the perspective space appear fully mastered through a symphony of graphic constructions and virtuosic fore-

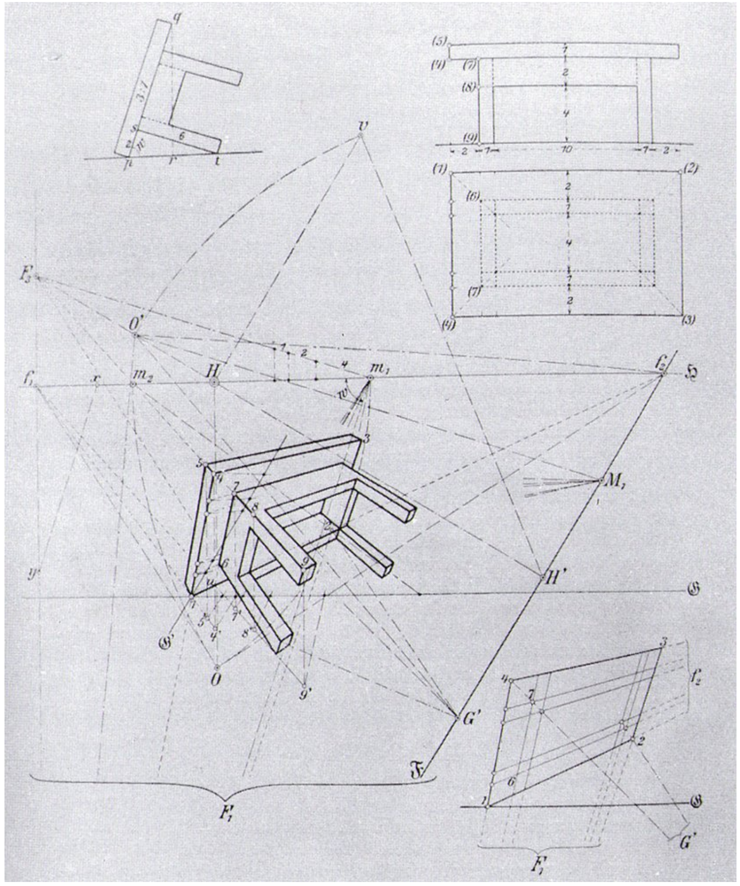

Figure 26: Guido Hauck, *"Die malerische Perspektive"* (1882). Figure and measure, final stage: the dominion of the space by means of the image [11].

shortenings (Figure 26). In this period, also the applications had significant advancements, even in connection with photography, or the 'automatic' perspective of the 19th century, as well as Hauck's work on the stereo-photogrammetric reconstructions from couples of photographs testifies. Of course, a new story to tell.

6. Conclusions

The importance of the search of a secure geometric connection between image and reality is clear at this point. This connection has been provided by the principle of *projection/section* officially introduced in the early Renaissance, then enhanced thanks to the development occurred in the field of Perspective, empowered by Descriptive Geometry, and finally widely generalized by Projective Geometry, which — taking into account *metrics* and *position* — also helped human beings to imagine new spaces and new geometries. And also clear is the brilliant intuition by Piero Della Francesca about the possibility to geometrically connecting, on the picture plane, the perspective image of a horizontal plane with its true shape. This is perfectly consistent with either Brook Taylor's constructions, or Fiedler's projective generalization, which a simple graphic overlap can easily demonstrate (Figure 27). Meanwhile the final projective setup is so flexible that any spatial orientation can be represented and controlled by means of appropriate homological constructions. In a way, we can virtually "surf" space by "surfing" projective images (Figure 28) [5]. The mutations, which

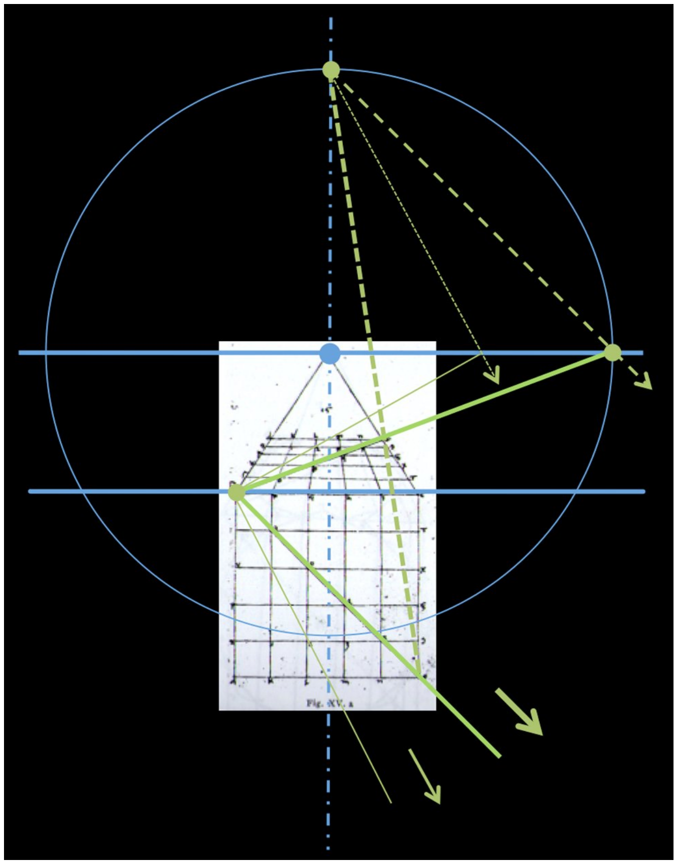

Figure 27: Logic and graphic consistency through the projective developments which historically occurred: overlapping diagrams (drawn by the author).

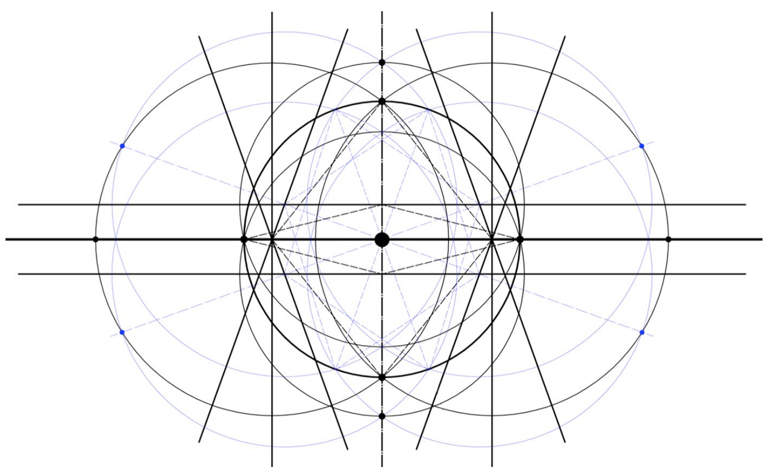

Figure 28: “Surfing” space by “surfing” images: spatial orientations according to viewing parameter (drawn by the author) [5]

occurred beneath the mutation of the perspective setup, reveal this story, ‘dually’ developed between Art and Science, dancing between *fiction* and *function*. A story that should not be forgotten in the digital era, when new ‘dualities’ between *fiction* and *function* have emerged and are still emerging, that could provide to revitalize both Geometry and Graphics. All in all, nowadays even into the digital screen is clearly recognizable the latest *pattern mutation* of the projective cast.

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