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Definition and Calculation of an Eight-Centered Oval which is Quasi-Equivalent to the Ellipse

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Abstract. Let $\mathcal{E}_b$ be an ellipse ($b = \dfrac{minor\ axis}{major\ axis}$). In this paper we consider different approximations by ovals, which are composed from circular arcs and have also two axes of symmetry. We study a) three four-centered ovals (quadrarcs) $O^a_{4,b}$, $O^c_{4,b}$, and $O^l_{4,b}$, which share the vertices with the ellipse $\mathcal{E}_b$. In addition, $O^a_{4,b}$ has the same surface area, $O^c_{4,b}$ has the minimum error of curvature at the vertices, and $O^l_{4,b}$ has the same perimeter length. b) Further, we investigate three eight-centered ovals $O^c_{8,b}$, $O^{c-a}_{8,b}$ and $O^{c-l}_{8,b}$, which also share the vertices with $\mathcal{E}_b$. The ovals $O^c_{8,b}$ have the same curvature at the vertices, and in addition, $O^{c-a}_{8,b}$ has the same surface area, and $O^{c-l}_{8,b}$ has the same perimeter length as $\mathcal{E}_b$.
As a conclusion, the eight-centered oval $O^{c-l}_{8,b}$ seems to be optimal and can therefore be called 'quasi-equivalent' to $\mathcal{E}_b$. We show that the difference of surface areas $\mathcal{A}_b = \mathcal{A}(O^{c-l}_{8,b}) - \mathcal{A}(\mathcal{E}_b)$ is rather small; the maximum value $\mathcal{A}_{0.1969} = 0.007085$ is achieved at $b = 0.1969$. The deformation error $E_b = E(\mathcal{E}_b, O^{c-l}_{8,b})$ has the maximum value 0.008970 which is achieved at $b = 0.2379$.

Key Words: Eight-centered oval, quadrarc, quasi-equivalent oval, ellipse.
MSC 2010: 51M04, 51N20

1. The ellipse and the eight-centered oval

Approximating ellipses by circular arcs has been a classic subject of study by geometers. This has long been used for a wide range of applications, for instance in geometry, astronomy, art, architecture. The reader can easily find a great deal of classical literature on these topics, in special for eight-centered ovals and four-centered ovals (also named *quadrarcs*). This kind of approximation, using eight-centered ovals, has recently been used to analyze architectural constructions as amphitheaters and military forts [5, 6, 7, 15]; also in astronomy, for analyzing

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orbits, was classically considered [4]. Moreover this subject of study is continued in modern research papers as [2, 8, 9, 10, 11, 12, 13, 14].

However, all these studies provide partial solutions only and specific constructions; and moreover, they do not show the global equations of the infinite possible infinite approximations depending on certain geometric properties. Here, in this paper, we study all such possible approximations, we provide their equations, and with them we look for an approximation which can qualify as being an 'equivalent' approximation.

In order to attain a similarity of results in physics or engineering applications, an approximation which is equivalent to the ellipse must have exactly coinciding geometric parameters. An eight-centered oval which is *equivalent* to an ellipse should have the same: center, axes, vertices, perimeter length, curvature at the vertices, and surface area; also, it should have little deformation in relation to the ellipse. Unfortunately, an eight-centered oval with all these exactly coinciding geometric parameters does not exist; it cannot have all the same geometric parameters and also the same surface area.

There are different methods of approximating curves, e.g., least squares, minimax, orthogonal family of polynomials. However, our approach is different; we look for the exact coincidence of single geometric parameters. We are going to to present exact analytical formulae for approximations of ellipses by eight-centered ovals and four-centered ovals. Further, we want to show the precise numerical calculations of these approximations. And, as a conclusion, we want to present not the 'equivalent' approximation because it does not exist, but the approximation of the ellipse by an eight-centered oval having the same center, axes, vertices, perimeter length, and curvature at the vertices as the ellipse, and also having a very similar surface area and showing little deformation in relation to the ellipse. We call this eight-centered oval 'quasi-equivalent' to the ellipse.

An oval is a curve resembling a flattened circle but, unlike the ellipse, it doesn't have a specific mathematical definition. Therefore, right now we must lay down the definitions and notations of this paper.

1.1. Definitions and notations

Let A, A', B, B', be the four vertices of an ellipse $\mathcal{E}$, where A, A' are the focal vertices and B, B' the transverse vertices. Without loss of generality, in the Euclidean plane $\mathbb{E}^2$, we can consider a Cartesian coordinate system $\mathcal{R}$ such that $A = (1, 0)$, $A' = (-1, 0)$, $B = (0, b)$, $B' = (0, -b)$, where $1 > b > 0$. We discard the cases $b = 1$ ($\mathcal{E}$ is a circle) and $b = 0$ ($\mathcal{E}$ is a straight segment), because the problem trivializes. In order to highlight the parameter b, we denote the ellipse by $\mathcal{E}_b$. Therefore, the parameter b is the hypothesis parameter which determines the problem.

We consider the infinite quantity of ovals with the vertices A, A', B, B'. Amongst them, we focus on the family of *eight-centered ovals*, which we denote by $O_{8,b}$. And finally, we consider the family of the *four-centered ovals* (quadrarcs) $O_{4,b}$. The second family is a subfamily of the first one.

An oval $O_{8,b}$ is made up by 8 circle arcs which are tangent to each other such that, in the system $\mathcal{R}$, they have the following 8 centers (see Figure 1):

$$
\begin{aligned}
&P_x = (x, 0),\ 0 < x < 1,\ \text{with } 1 - x < b, \quad P'_x = (-x, 0),\\
&P_y = (0, y),\ \text{with } y \le 0, \quad P'_y = (0, -y),\\
&P_3 = (x_3, y_3) \text{ with } x_3 \ge 0,\ y_3 \le 0,\\
&P'_3 = (-x_3, y_3), \quad P''_3 = (-x_3, -y_3), \quad P'''_3 = (x_3, -y_3).
\end{aligned}
$$

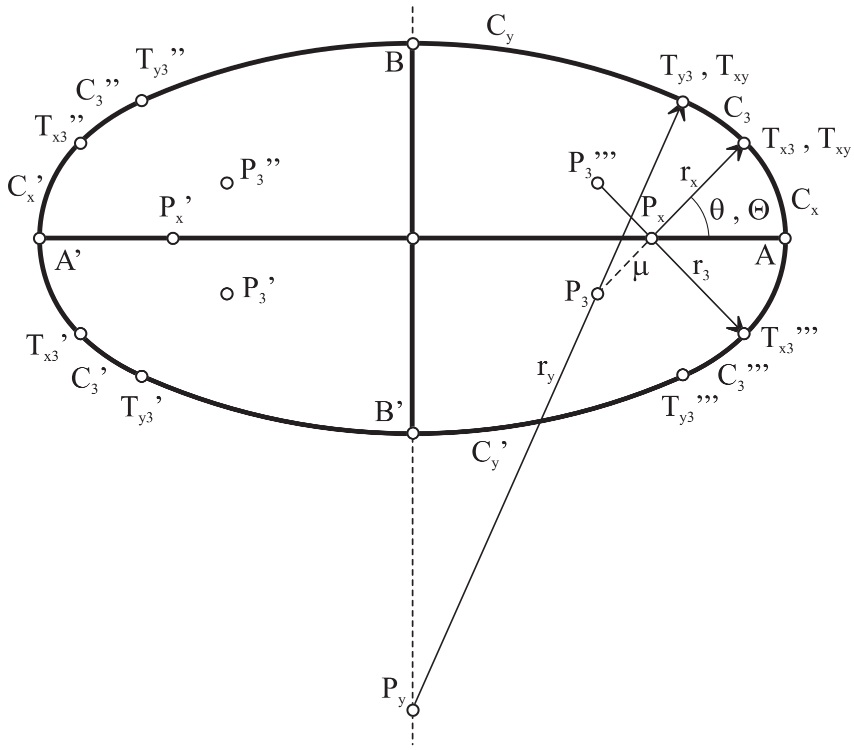

Figure 1: Elements of the 8-centered oval $O_{8,b}$ (quadrarc $O_{4,b}$ if $r_y = r_3$).

Moreover, the oval has radii r_x, r_y and r_3, where r_x is the radius of its arcs C_x, C'_x, with centers P_x, P'_x respectively; r_y is the radius of the arcs C_y, C'_y with respective centers P_y, P'_y; and r_3 is the radius of the arcs C_3, C'_3, C''_3, and C'''_3 with respective centers P_3, P'_3, P''_3, and P'''_3. The curvatures of these arcs are inverse to their radii, $k_x = \dfrac{1}{r_x}$, $k_y = \dfrac{1}{r_y}$, $k_3 = \dfrac{1}{r_3}$.

If $r_y = r_3$ then $P_y = P_3 = P'_3$ and $P'_y = P''_3 = P'''_3$. This special case of 8-centered oval $O_{8,b}$, noted as $O_{4,b}$, is called 4-centered oval. The segments AA', BB' are called major axis and minor axis of $O_{8,b}$, as well as at ellipses.

The oval $O_{8,b}$ has 8 contact points for its 8 arcs: point T_{x3} is the contact between C_x and C_3; point T_{y3} is the contact between C_y and C_3; points T'''_{x3}, T'''_{y3}, T''_{x3}, T''_{y3}, T'_{x3}, T'_{y3} are symmetrical to T_{x3}, T_{y3} with respect to the axes and center point, and they are the contact between C'''_3, C''_3, C'_3 and the arcs having their centers on the x-axis and the y-axis, respectively (see Figure 1).

In the case of $O_{4,b}$, the eight contact points are reduced to four, then $T_{x3} = T_{y3}$ (we call it T_{xy}), and similarly: $T'''_{xy} = T'''_{x3} = T'''_{y3}$, $T''_{xy} = T''_{x3} = T''_{y3}$, $T'_{xy} = T'_{x3} = T'_{y3}$.

1.2. Problem statements

Historically, several questions have been raised with regard to the construction of 4-centered ovals and 8-centered ovals. Nevertheless, in this paper we take a step forward in this classic subject which has applications in physics, engineering and architecture. The problem statements are:

Problem 1. Let $\mathcal{E}_b$ be an ellipse. Find — if they exist — the ovals $O_{8,b}$, $O_{4,b}$ having the same center, axes and vertices as $\mathcal{E}_b$; and also having the same

1. surface area,
2. curvature at the vertices,

3. perimeter length,
4. curvature at the vertices and surface area,
5. surface area and perimeter length, or
6. curvature at the vertices and perimeter length.

In the solutions of the problems, shown in the following section, we use the following notation (see Figure 1): θ is the angle $\angle(\overrightarrow{P_xA}, \overrightarrow{P_xT_{x3}})$ in the oval $O_{8,b}$, Θ is the angle $\angle(\overrightarrow{P_xA}, \overrightarrow{P_xT_{xy}})$ in the oval $O_{4,b}$, and μ is the distance $d(P_x, P_3)$ between P_x and P_3.

2. Analytical formulae

In this section we claim the uniqueness of ovals with some geometric properties, and we show their analytical formulae. We have proved all of them — formulae and uniqueness — with mathematical rigor, but for the sake of brevity we do not show the proofs in detail, since they are rather long, but straightforward and without intrinsic mathematical interest. Nevertheless, we will make some remarks about the proofs.

Theorem 1. *There is only one oval $O^a_{4,b}$ sharing the vertices with the ellipse $\mathcal{E}_b$, and also having the same surface area. The x-coordinate of the corresponding circle center $P_x = (x, 0)$ with $1 - x - b < 0$ is a zero of the function $\mathbb{H}^a_b$ as given below.*

$$\begin{aligned}\mathbb{H}^a_b(x) \;=\; & 2\left(\frac{(1-b)(1-b-2x)}{2(1-b-x)} + b\right)^2 \arctan\left(\frac{2x(1-b-x)}{(1-b)(1-b-2x)}\right) - \frac{x(1-b)(1-b-2x)}{1-b-x} \\ & +(1-x)^2\left(\pi - 2\arctan\left(\frac{2x(1-b-x)}{(1-b)(1-b-2x)}\right)\right) - \pi b\,.\end{aligned} \tag{1}$$

For the circle center $P_y = (0, y)$ of $O^a_{4,b}$ we obtain $y = -\dfrac{(1-b)(1-b-2x)}{2(1-b-x)}$. The point T_{xy} of transition is $T_{xy} = (\lambda x,\, y - \lambda y)$, where $\lambda = 1 + \dfrac{1-x}{\sqrt{x^2+y^2}}$.

Note that at the same time we obtain a trivial 8-centered oval $O^a_{8,b}$ having the same area, simply by considering $O^a_{8,b} = O^a_{4,b}$.

Remark 1. The proof of Theorem 1 is based on the calculation of the surface areas of the circular sectors using classical analytic geometry, with these surface areas depending on $\angle(\overrightarrow{P_yP_x}, \overrightarrow{P_yP'_x}) = 2\arctan\left(\dfrac{2x(1-b-x)}{(1-b)(1-b-2x)}\right)$.

Theorem 2. *There is no oval $O_{4,b}$ sharing the vertices with the ellipse $\mathcal{E}_b$ and also having the same curvature at the vertices, i.e., with $E^c_b(O_{4,b}) = 0$ in eq. (2).*
There is only one oval $O^c_{4,b}$ sharing the vertices with $\mathcal{E}_b$, and also having a minimum error $E^c_b(O^c_{4,b})$ of the curvature at the vertices with respect to $\mathcal{E}_b$. The x-coordinate of the circle center $P_x = (x, 0)$ of $O^c_{4,b}$ with $1 - x - b < 0$ is a zero of the function $\mathbb{H}^c_b$, as given below in eq. (3).

The quadratic error $E^c_b(O_{4,b})$ between the curvatures $k_A(\mathcal{E}_b)$ and $k_B(\mathcal{E}_b)$ of the ellipse $\mathcal{E}_b$ at the respective vertices A, B, and the curvatures $k_A(O_{4,b})$ and $k_B(O_{4,b})$ of an oval $O_{4,b}$ at the vertices A, B, is defined as

$$E^c_b(O_{4,b}) = \left(k_A(\mathcal{E}_b) - k_A(O_{4,b})\right)^2 + \left(k_B(\mathcal{E}_b) - k_B(O_{4,b})\right)^2. \tag{2}$$

The function $\mathbb{H}_b^c$ cited above is

$$\begin{aligned}\mathbb{H}_b^c(x) \;=\;& x^4\,(4b^5 - 12b^4 + 12b^3 - 4b^2 + 8)\\ &+ x^3\,(2b^7 - 4b^6 - 16b^5 + 52b^4 - 50b^3 + 36b^2 - 20)\\ &+ x^2\,(-6b^7 + 12b^6 + 24b^5 - 66b^4 + 78^2b^3 - 60b^2 + 18)\\ &+ x\,(6b^7 - 5b^6 - 16b^5 + 39b^4 - 54b^3 + 37b^2 - 7)\\ &+ (b^8 - 2b^7 + 4b^5 - 10b^4 + 14b^3 - 8b^2 + 1)\,.\end{aligned} \tag{3}$$

The minimum error $E_b^c(O_{4,b}^c)$, which is never null, is

$$E_b^{c\ \min}(O_{4,b}^c) = \frac{(b^2+x-1)^2}{(x-1)^2b^4} + \frac{(b-1)^2(2x+b+b^2-2)^2}{(2x+b^2-1)^2} \neq 0.$$

The coordinates for the circle center P_y and for the contact point T_{xy} of $O_{4,b}^c$ are the same as those for $O_{4,b}^a$.

Remark 2. The proof of Theorem 2 is based on the partial derivative

$$\begin{aligned}\frac{\partial}{\partial x}E_b^c(O_{4,b}) \;&=\; \frac{\partial}{\partial x}\left[(k_A(\mathcal{E}_b)-k_A(O_{4,b}))^2 + (k_B(\mathcal{E}_b)-k_B(O_{4,b}))^2\right]\\ &=\; \frac{\partial}{\partial x}\left[\left(\frac{1}{b^2}-\frac{1}{1-x}\right)^2 + \left(b - \frac{1}{b+\frac{1}{2}\frac{(1-b)(1-b-2x)}{1-b-x}}\right)^2\right] = 0.\end{aligned}$$

Theorem 3. *There is an infinite number of ovals $O_{8,b}^c(\theta)$ — not 4-centered ovals — sharing the vertices with $\mathcal{E}_b$, and having the same curvature at the vertices, i.e., with $E_b^c(O_{8,b}^c(\theta)) = 0$ in eq. (4).*
All of them have the same circle centers $P_x = (x,0)$ and $P_y = (0,y)$, where $x = 1-b^2$ and $y = b - \frac{1}{b}$, and each of these ovals $O_{8,b}^c(\theta)$ is determined by θ, i.e., the angle $\angle(\overrightarrow{P_xA}, \overrightarrow{P_xT_{x3}})$, as given in eq. (5), and $O_{8,b}^c(\theta)$ has the third circle center $P_3(\theta)$ as given in eq. (6).

The quadratic error $E_b^c(O_{8,b})$ between the curvatures $k_A(\mathcal{E}_b)$ and $k_B(\mathcal{E}_b)$ of the ellipse $\mathcal{E}_b$ at the vertices A, B and the curvatures $k_A(O_{8,b})$ and $k_B(O_{8,b})$ of an oval $O_{8,b}$ at the vertices A, B, is

$$E_b^c(O_{8,b}) = (k_A(\mathcal{E}_b)-k_A(O_{8,b}))^2 + (k_B(\mathcal{E}_b)-k_B(O_{8,b}))^2\,. \tag{4}$$

The infinite number of possible values of θ are those that satisfy

$$0 < \theta < \frac{\pi}{2} - \arctan\left(\frac{2b(b+1)}{2b+1}\right). \tag{5}$$

Each oval $O_{8,b}^c(\theta)$ has the circle center $P_3(\theta) = (x - \mu\cos\theta,\ -\mu\sin\theta)$ with

$$\mu = \frac{1}{2}\,\frac{b(b-1)}{b^2\cos\theta - b^2 + b\sin\theta + b\cos\theta - b + \sin\theta - 1}\,. \tag{6}$$

Proposition 4. *The geometric locus $\mathcal{L}_{b3}$ of the circle centers $P_3(\theta)$ is the arc of the ellipse $\mathcal{E}_{b3}$ with the equation*

$$\begin{aligned}&\frac{2b^2+2b+1}{(b-1)(b+1)}x^2 + \frac{b^2(2+2b+b^2)}{(b-1)(b+1)}y^2 - \frac{2b(b+1)}{b-1}xy\\ &+ (3b^2+4b+2)\,x - b(2b^2+4b+3)\,y + \frac{(b-1)(2b+1)(b+2)(2b^2+3b+2)}{4(b+1)} = 0\end{aligned}$$

and the endpoints

$$\left(0,\ \frac{(2b+1)(b-1)}{2b}\right) \quad \textit{and} \quad \left(\frac{(b+2)(1-b)}{2},\ 0\right).$$

Each of these ovals $O^c_{8,b}(\theta)$ *has the following transition points (see Figure 1):*

$$\begin{gathered} T_{x3} = \left(1-b^2+b^2\cos\theta,\ b^2\sin\theta\right),\ T_{y3} = \left(\lambda(1-b^2-\mu\cos\theta),\ y-\lambda y-\lambda\mu\sin\theta\right), \\ \textit{where } \lambda = 1+\frac{\mu+b^2}{\sqrt{y^2+\mu^2+(1-b^2)^2+2\mu\left(y\sin\theta-(1-b^2)\cos\theta\right)}}. \end{gathered} \tag{7}$$

Remark 3. In order to prove Theorem 3, we compute the y-coordinate of point P_y as

$$y = -\frac{1}{2}\frac{\Psi_1}{\Psi_2}, \text{ where } \Psi_1 = 2x\mu\cos\theta+2\mu-2\mu x-2\mu b+1+b^2-2b-2x+2xb$$
$$\text{and } \Psi_2 = -\mu\sin\theta-b+\mu+1-x.$$

Then

$$\begin{aligned} E^c_b(O_{8,b}) &= \left(k_A(\mathcal{E}_b)-k_A(O_{8,b})\right)^2+\left(k_B(\mathcal{E}_b)-k_B(O_{8,b})\right)^2 \\ &= \left(\frac{1}{b^2}-\frac{1}{1-x}\right)^2+\left(b-\frac{1}{b+\frac{1}{2}\frac{\Psi_1}{\Psi_2}}\right)^2 = 0. \end{aligned}$$

Therefore $x = 1-b^2$, and we have

$$2b^2\mu\cos\theta-2\mu b^2+2b\mu\sin\theta+2b\mu\cos\theta+2\mu\sin\theta-b^2+b-2\mu b-2\mu = 0.$$

Remark 4. In view of the proof of Proposition 4, we compute with the formulae of Theorem 3 the following five points:

$$P_3(0) = \left(\frac{(b+2)(1-b)}{2},\ 0\right) \text{ and } P_3\left(\frac{\pi}{2}-\arctan\left(\frac{2b(b+1)}{2b+1}\right)\right) = \left(0,\ \frac{(2b+1)(b-1)}{2b}\right),$$

the point satisfying

$$\frac{\partial(-\mu\sin\theta)}{\partial\theta} = 0 \quad \text{with} \quad P_3\left(\arccos\left(b\frac{b+1}{b+b^2+1}\right)\right) = (\Psi_3,\ \Psi_4), \text{ where}$$
$$\Psi_3 = -\frac{(b-1)(b+1)}{2}\,\frac{3b^2-2b\Phi_1+4b+2-2\Phi_1}{2b^2-b\Phi_1+2b-\Phi_1+1},\ \Psi_4 = \frac{1}{2}\Phi_1(b-1)\frac{b}{2b^2-b\Phi_1+2b-\Phi_1+1}$$
$$\text{and } \Phi_1 = \sqrt{2b^2+2b+1},$$

the point satisfying

$$\frac{\partial(-\mu\sin\theta)}{\partial\theta} = 0 \quad \text{with} \quad P_3\left(-\arccos\left(b\frac{b+1}{b+b^2+1}\right)\right) = (\Psi_5,\ \Psi_6), \text{ where}$$
$$\Psi_5 = -\frac{(b-1)(b+1)}{2}\,\frac{3b^2+2b\Phi_1+4b+2+2\Phi_1}{2b^2+b\Phi_1+2b+\Phi_1+1},\ \Psi_6 = -\tfrac{1}{2}\Phi_1(b-1)\frac{b}{2b^2+b\Phi_1+2b+\Phi_1+1},$$

and the point satisfying

$$\frac{\partial(x-\mu\cos\theta)}{\partial\theta} = 0 \quad \text{with} \quad P_3\left(\pi-\arcsin\frac{b+1}{b+b^2+1}\right) = (\Psi_7,\ \Psi_8), \text{ where}$$
$$\Psi_7 = -\frac{1}{2}(b-1)\frac{M}{N},\ \Psi_8 = \frac{1}{2}(b+1)\frac{b-1}{bN} \text{ with } \Phi_2 = \sqrt{b^2+2b+2},$$
$$M = 2b^3+2b^2\Phi_2+6b^2+8b+4b\Phi_2+3\Phi_2+4,\ N = b\Phi_2+2b+b^2+2+\Phi_2.$$

The ellipse $\mathcal{E}_{b3}$ presented in Proposition 4 is defined by these five points. Finally, by virtue of the formulae in Theorem 3, we confirm that all points $P_3(\theta)$ satisfy the equation of the ellipse $\mathcal{E}_{b3}$.

Theorem 5. *There is only one oval $O^l_{4,b}$ sharing the vertices with the ellipse $\mathcal{E}_b$ and having the same perimeter length. For $O^l_{4,b}$, the x-coordinate of the circle center $P_x = (x, 0)$ is the zero of the function $\mathbb{H}^l_b$, as given below, and satisfying $1 - x - b < 0$.*

$$\begin{aligned} \mathbb{H}^l_b(x) &= L - \pi(1+b) \sum_{n=0}^{n=\infty} \left(\frac{\sqrt{\pi}}{n!\,(1-2n)\,\Gamma\left(\frac{1}{2}-n\right)} \left(\frac{1-b}{1+b}\right)^n \right)^2, \quad \textit{where} \\ L &= 4(1-x)\Theta + 4\left(b + \frac{(1-b)(1-b-2x)}{2(1-b-x)}\right)\left(\frac{\pi}{2} - \Theta\right), \\ \Theta &= \frac{\pi}{2} - \arctan\frac{2x(1-b-x)}{(1-b)(1-b-2x)}, \quad \textit{and} \quad \Gamma(\alpha) = \textstyle\int_0^\infty t^{\alpha-1} e^{-t} dt. \end{aligned} \tag{8}$$

For $O^l_{4,b}$ the coordinates of the circle center P_y and the contact point T_{xy} of are the same as those for $O^a_{4,b}$. At the same time we obtain a trivial 8-centered oval $O^l_{8,b}$ having the same perimeter length, simply by setting $O^l_{8,b} = O^l_{4,b}$.

Remark 5. We point out that each 4-centered oval $O_{4,b}$ with the circle center $P_x = (x, 0)$ has the angle $\Theta = \frac{\pi}{2} - \arctan\frac{2x(1-b-x)}{(1-b)(1-b-2x)}$ (see Figure 1).

Remark 6. The proof of Theorem 5 is based on the Gauss-Kummer series [1, 3], on the remarks given above, and on the angle $\angle(\overrightarrow{P_yP_x}, \overrightarrow{P_yP'_x})$.

Theorem 6. *There is only one oval $O^{c-a}_{8,b}$ sharing the vertices with $\mathcal{E}_b$, having the same curvature at the vertices, i.e., with $E^c_b(O^{c-a}_{8,b}) = 0$ by eq. (4), and also the same surface area. For $O^{c-a}_{8,b}$ the analytic expressions for the circle centers $P_x = (x, 0)$, $P_y = (0, y)$, P_3, and for μ are the same as those for $O^c_{8,b}(\theta)$, but its θ is the zero of the function*

$$\begin{aligned} \mathbb{H}^{c-a}_b(\theta) &= \frac{b^2(b+1)^2(2b\sin\theta + 2b^2\cos\theta - 2b^2 - 1)^2}{2N^2}\left(\frac{\pi}{2} - \theta - \arctan(M)\right) \\ &\quad - \frac{\left((1-b^2) - \left(\frac{1}{b} - b\right)\right) y(y-1)}{N}\sin\theta + \frac{2}{b^2}\arctan(M) \\ &\quad - 2\left(\frac{1}{b} - b\right)^2 M + 2b^4\theta - \pi b, \ \textit{where} \\ M &= b + b^2\frac{\cos\theta - y\sin\theta}{b(2b^2+4b+2)\cos\theta + (3b^2+4b+2)(\sin\theta - 1) - b^2 - 2b^3} \quad \textit{and} \\ N &= (b^2+b)\cos\theta + (b+1)\sin\theta - b^2 - b - 1. \end{aligned} \tag{9}$$

Remark 7. The proof of Theorem 6 is based on the calculation of the surface areas of the circular sectors using classical analytic geometry and the formulae given in Theorem 3.

Theorem 7. *There is only one oval $O^{c-l}_{8,b}$ sharing the vertices with $\mathcal{E}_b$, having the same curvature at the vertices, i.e., with $E^c_b(O^{c-l}_{8,b}) = 0$ in eq. (4), and also the same perimeter length. For $O^{c-l}_{8,b}$ the analytical expressions for the circle centers $P_x = (x, 0)$, $P_y = (0, y)$, P_3,* *and for μ are the same as those given for $O^c_{8,b}(\theta)$, but its θ is a zero of the function*

$$
\begin{aligned}
\mathbb{H}^{c-l}_b(\theta) &= L - \pi(1+b)\sum_{n=0}^{n=\infty}\left(\frac{\sqrt{\pi}}{n!\,(1-2n)\Gamma\left(\frac{1}{2}-n\right)}\left(\frac{1-b}{1+b}\right)^n\right)^2, \text{ where} \\
L &= 4b^2\theta + \frac{4}{b}\theta_y + 4(b^2+\mu)\left(\frac{\pi}{2}-\theta-\theta_y\right), \quad \theta_y = \arctan\frac{\lambda(1-b^2-\mu\cos\theta)}{-\lambda y - \lambda\mu\sin\theta}, \\
\lambda &= 1 + \frac{\mu+b^2}{\sqrt{y^2+\mu^2+(1-b^2)^2+2\mu(y\sin\theta-(1-b^2)\cos\theta)}}, \text{ and} \\
\Gamma(\alpha) &= \textstyle\int_0^\infty t^{\alpha-1}e^{-t}dt.
\end{aligned}
\tag{10}
$$

Remark 8. The proof is based on the Gauss-Kummer series and the formulae given in Theorem 3.

Remark 9. We point out that there is no 8-centered oval $O_{8,b}$ having the same vertices, the same surface area and the same perimeter length as the ellipse $\mathcal{E}_b$. Therefore, there is also no 4-centered oval $O_{4,b}$ sharing the vertices, the surface area and the perimeter length with $\mathcal{E}_b$.

3. Numerical calculations

Except for the case $O^c_{4,b}$, all the above equations have implicit solutions only. Therefore, in order to calculate the circle centers of the ovals, we must use numerical methods. In the case $O^c_{4,b}$, by means of Ferrari's method [16] for the zeros of fourth degree polynomial equations, it is possible to provide an explicit solution. However, the obtained analytical expression does not provide any effective improvement in the calculation compared to the result obtained numerically.

The experienced reader can generate the corresponding numerical calculations, and therefore we do not show them here. Instead, we offer the graphs obtained by the calculations carried out for 1000 values of $b \in (0,1)$.

In Figure 2 we show the graphs of the functions $b(x)$ which are given implicitely by the equations $\mathbb{H}^a_b(x)=0$, $\mathbb{H}^l_b(x)=0$, $\mathbb{H}^c_b(x)=0$, which correspond to the 4-centered ovals $O^a_{4,b}$, $O^l_{4,b}$, and $O^c_{4,b}$, respectively. In Figure 3, the graphs $b(\theta)$ are displayed, which are given by the equations $\mathbb{H}^{c-a}_b(\theta)=0$ and $\mathbb{H}^{c-l}_b(\theta)=0$, respectively correspond to the 8-centered ovals $O^{c-a}_{8,b}$ and $O^{c-l}_{8,b}$.

Table 1 shows some numerical values resulting from these calculations

4. Conclusions

4.1. Deformation error

In order to solve problems related to physics, engineering and architecture, one needs to find an ellipse $\mathcal{E}_b$ or an 8-centered oval $O_{8,b}$ which fits to a given set of points. In these cases it is useful to know the deformation error between the ellipse and the oval:

Definition 1. For each point $p \in \mathcal{E}_b$, let $q_p \in O_{8,b}$ be the point, which is closest to p among all points of intersection between $O_{8,b}$ and the straight line perpendicular to the ellipse at p. The maximum value of the distance $d(p,\, q_p)$, when p moves along the ellipse $\mathcal{E}_b$, is called the *deformation error* $E(\mathcal{E}_b,\, O_{8,b})$ between the two curves.

This gives rise to the following problem statements:

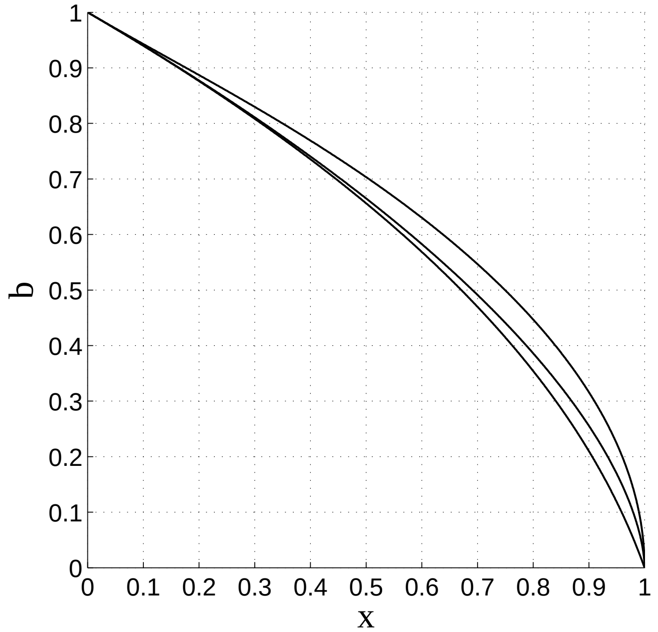

Figure 2: Graphs of the functions $b(x)$ given by the equations $\mathbb{H}_b^a(x) = 0$, $\mathbb{H}_b^l(x) = 0$, and $\mathbb{H}_b^c(x) = 0$, corresponding respectively to the 4-centered ovals $O_{4,b}^a$ (lower line), $O_{4,b}^l$ (center line), and $O_{4,b}^c$ (upper line).

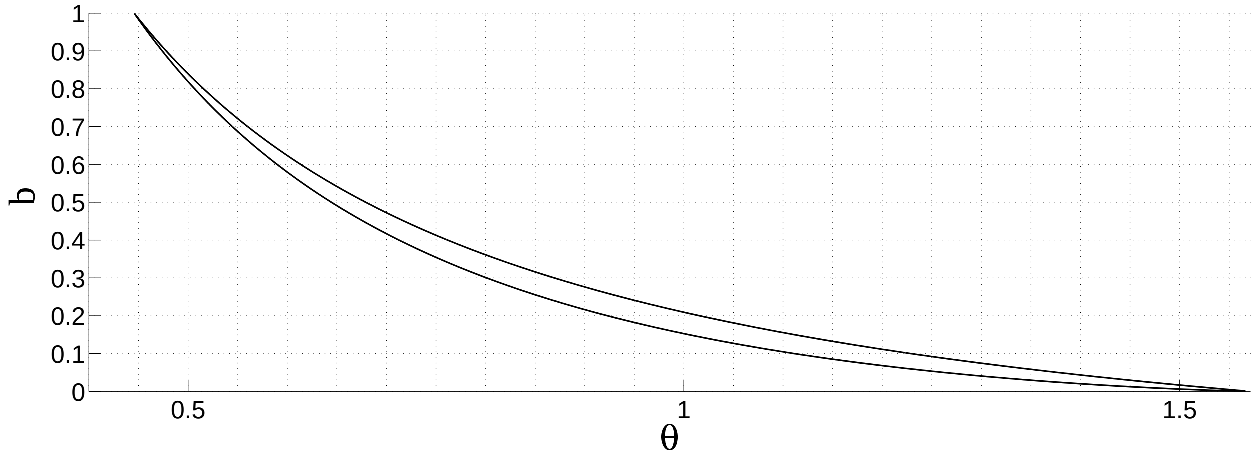

Figure 3: Graphs of the functions $b(\theta)$ given by the equations $\mathbb{H}_b^{c-a}(\theta) = 0$ and $\mathbb{H}_b^{c-l}(\theta) = 0$, corresponding respectively to the 8-centered ovals $O_{8,b}^{c-a}$ (upper line) and $O_{8,b}^{c-l}$ (lower line).

Problem 2. Let $\mathcal{E}_b$ be an ellipse. Find for the approximating ovals, which have been provided in the previous theorems, the

1. the deformation error $E(\mathcal{E}_b, O_{4,b}^a)$,
2. the deformation error $E(\mathcal{E}_b, O_{4,b}^l)$,
3. the deformation error $E(\mathcal{E}_b, O_{8,b}^{c-a})$,
4. the deformation error $E(\mathcal{E}_b, O_{8,b}^{c-l})$.

In order to solve these problems, one needs the formulae of the theorems given in the previous section. The experienced reader can perform these numerical calculations. We present here only the graphs obtained in calculations of 1000 values of $b \in (0, 1)$.

Table 1: Some numerical values of the graphs shown in the Figures 2 and 3.

b	$\mathbb{H}_b^a(x)=0$ x	$\mathbb{H}_b^l(x)=0$ x	$\mathbb{H}_b^c(x)=0$ x	$\mathbb{H}_b^{c-a}(\theta)=0$ θ	$\mathbb{H}_b^{c-l}(\theta)=0$ θ
0.9	0.163300	0.164402	0.176884	0.477881	0.473885
0.8	0.311867	0.316029	0.349982	0.515540	0.506506
0.7	0.445860	0.454590	0.504895	0.560040	0.544573
0.6	0.565470	0.579727	0.637996	0.613505	0.589739
0.5	0.670943	0.690982	0.749419	0.679087	0.644457
0.4	0.762585	0.787741	0.839885	0.761643	0.712633
0.3	0.840778	0.869131	0.909987	0.869138	0.800989
0.2	0.905993	0.933833	0.960001	1.015621	0.922757
0.1	0.958826	0.979508	0.989999	1.228623	1.111048

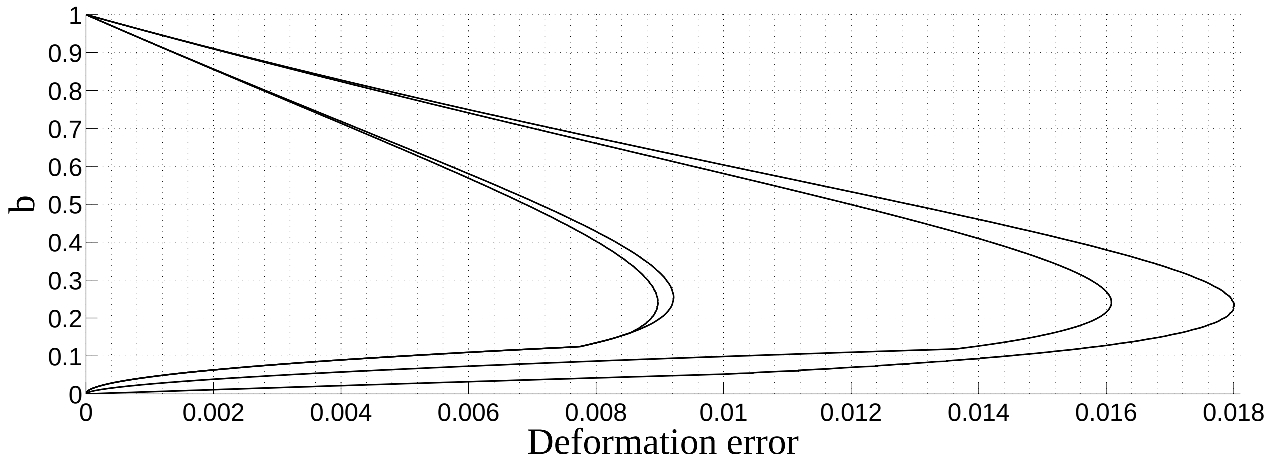

Figure 4: Graphs of the deformation errors $E(\mathcal{E}_b, O^a_{4,b})$ (far right line), $E(\mathcal{E}_b, O^l_{4,b})$ (inner right line), $E(\mathcal{E}_b, O^{c-a}_{8,b})$ (inner left line), and $E(\mathcal{E}_b, O^{c-l}_{8,b})$ (far left line).

In Figure 4 we show the graphs of deformation errors $E(\mathcal{E}_b, O^a_{4,b})$, $E(\mathcal{E}_b, O^l_{4,b})$, $E(\mathcal{E}_b, O^{c-a}_{8,b})$, and $E(\mathcal{E}_b, O^{c-l}_{8,b})$. Table 2 shows some numerical values resulting from these calculations.

4.2. An eight-centered oval which is quasi-equivalent to the ellipse

In this paper we presented the ovals $O^a_{4,b}$, $O^c_{4,b}$, $O^l_{4,b}$, $O^c_{8,b}$, $O^{c-a}_{8,b}$, and $O^{c-l}_{8,b}$, among them
a) three four-centered ovals (quadrarcs) having the same vertices as $\mathcal{E}_b$; further, $O^a_{4,b}$ has the same surface area, $O^c_{4,b}$ the minimum error of curvature at the vertices, and $O^l_{4,b}$ the same perimeter length as $\mathcal{E}_b$; and
b) three eight-centered ovals sharing the vertices with $\mathcal{E}_b$; further, $O^c_{8,b}$ has the same curvature at the vertices, and — in addition — $O^{c-a}_{8,b}$ has the same surface area and $O^{c-l}_{8,b}$ the same perimeter length as $\mathcal{E}_b$.

We have shown that, for any given ellipse $\mathcal{E}_b$, the smallest deformation error with respect to the ellipse $\mathcal{E}_b$ is reached with $O^{c-l}_{8,b}$. Also, we calculated the deformation error $E(\mathcal{E}_b, O^{c-l}_{8,b}) = E_b$ for all values of parameter b (see Figure 4), and we have found that $E_b < 0.008970$ for all b.

Table 2: Some numerical values of the graphs in Figure 4.

b	$E(\mathcal{E}_b, O^a_{4,b})$	$E(\mathcal{E}_b, O^l_{4,b})$	$E(\mathcal{E}_b, O^{c-a}_{8,b})$	$E(\mathcal{E}_b, O^{c-l}_{8,b})$
0.9	0.002245	0.002214	0.001383	0.001377
0.8	0.004692	0.004559	0.002811	0.002780
0.7	0.007326	0.007013	0.004271	0.004189
0.6	0.010103	0.009518	0.005723	0.005577
0.5	0.012913	0.011980	0.007105	0.006887
0.4	0.015532	0.014191	0.008312	0.008026
0.3	0.017494	0.015758	0.009101	0.008810
0.2	0.017851	0.015866	0.009010	0.008868
0.1	0.014456	0.010245	0.004999	0.004999

The maximum deformation error corresponds to the parameter $b = 0.2379$. In view of many practical problems, this is a small error, and the more the error obtained for other values of b is negligible.

Furthermore, we calculated the difference $\mathcal{A}_b = \mathcal{A}(O^{c-l}_{8,b}) - \mathcal{A}(\mathcal{E}_b)$ of their surface areas, and we showed it in Figure 5. Also, we found $\mathcal{A}_b < 0.007085$ for all b. The maximum surface area error corresponds to the parameter $b = 0.1969$. Again, for many practical problems this is a small error and negligible. Table 3 shows some numerical values resulting from the calculations.

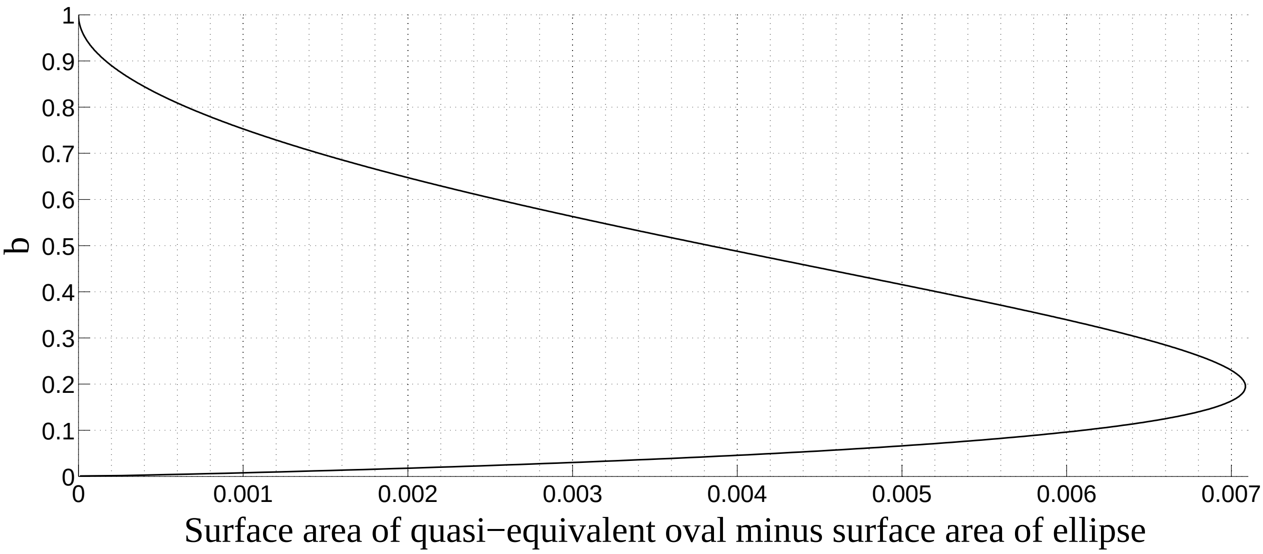

Figure 5: Graph of the difference $\mathcal{A}_b = \mathcal{A}(O^{c-l}_{8,b}) - \mathcal{A}(\mathcal{E}_b)$ of areas.

This leads to our final conclusion: Among the different approximations of the ellipse $\mathcal{E}_b$, as presented above, the eight-centered oval $O^{c-l}_{8,b}$ has not only the same vertices, perimeter length, and curvature at the vertices as $\mathcal{E}_b$, but also only a small difference of the surface areas and a small deformation error. This qualifies to call the oval $O^{c-l}_{8,b}$ *'quasi-equivalent'* to the ellipse $\mathcal{E}_b$.

Table 3: Some numerical values of the graph in Figure 5.

b	$\mathcal{A}(O_{8,b}^{c-l}) - \mathcal{A}(\mathcal{E}_b)$	b	$\mathcal{A}(O_{8,b}^{c-l}) - \mathcal{A}(\mathcal{E}_b)$
0.90	0.000165	0.80	0.000685
0.70	0.001461	0.60	0.002542
0.50	0.003833	0.40	0.005212
0.30	0.006447	0.20	0.007083
0.10	0.006100	0.01	0.001235

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