Journal for Geometry and Graphics
Volume 19 (2015), No. 2, 189–200.

Blending of Spheres by Rotation-Minimizing Surfaces

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Abstract. Computation of the blending surface of two given spheres is discussed in this paper. The blending surface (or skin), although not uniquely defined in the literature, is normally required to touch the given spheres in plane curves (i.e., in circles). The main advantage of the presented method over the existing ones is the minimization of unwanted distortions of the blending surface. This is achieved by the application of rotation-minimizing frames for the transportation of a vector along a given curve, which technique, beyond its theoretical interest, helps us to determine the corresponding points along the touching circles of the two spheres. Parametric curves of the blending surface are also defined by the help of the rotation minimizing transportation.

Key Words: sphere based surface design, skinning, blending surface, rotation minimizing frame

MSC 2000: 68U05, 65D17, 53A04

1. Introduction and related work

The problem of joining spheres by a surface being tangent to them is considered by several authors using various approaches. A common view of this problem in the literature is to find a blending surface or skin, touching the spheres along circles, where the tangent cone of the sphere along this circle contains the end tangents of the blending surface [7, 9, 10]. This problem is of essential importance in newly emerged animation design software tools, such as ZSpheres® or Spore™, where the point-based design is altered by sphere-based construction, but it also appears in medical applications [9]. Although with somehow similar results, the well-studied ringed surfaces and canal surfaces (see, e.g., [13, 14]) have different input data:

ISSN 1433-8157/$ 2.50

while our approach starts from a finite set of spheres, in case of ringed and canal surfaces a continuous function is required. The final skinning surface consists of patches connecting pairs of spheres [7], thus here we consider a subproblem of connecting two spheres where touching cones are also provided (cf. Figure 1), [8].

Dupin cyclides and their generalizations have also been applied to this problem (see, e.g., [8, 11, 16] and references therein) with a deep geometric view, but also with restrictions on the positions of the initial data: in case of a single cyclide surface the axes of the touching cones must be coplanar, while in case of two joining cyclides the geometric form of the blending surface has not always been satisfactory, since the two cyclides may form unnecessary large twist for some initial positions. Similar geometric restrictions to the initial data are supposed in [15], the axes of the given cylinders must be coplanar. In our approach the axes of the touching cones can be in arbitrary position.

An iterative method of skinning has been introduced in [10] and [9], where, starting from an initial skin, an energy function is minimized during the iterations in order to reach the optimal surface. This method has certain advantages from the viewpoint of optimization, but it is also quite sensitive for the initial position of the skin, and due to the iterative nature it is quite slow with no proven convergence.

Another field of classical geometry has been used in [7], where a set of spheres has been blended by surface patches. In this approach the touching circles are computed with the help of Apollonian circles, and the blending surface is formed by cubic Hermite interpolants.

Although this method works for almost all positions of the initial data, the final surface, just as in case of all the above mentioned methods, may suffer from distortions due to the unwanted rotation along the central (spine) curve. The source of this distortion is the fact that the pairing of corresponding points on the two touching circles is not solved in a sufficient way. To avoid the unnecessary rotation of the blending surface along the spine curve, a straightforward idea is to apply the recent results of rotation-minimizing frames. These frames, originally developed as an alternative of the well-known Frenet frames of parametric curves, were also known as relatively parallel adapted frames or Bishop frames [1]. They can provide a sufficient solution of minimizing this kind of distortion along a curve ([5, 6]). The rotation-minimizing frames have been also applied in constructing surfaces in [12], but that approach provides only an approximate solution, moreover it requires a continuous one-parameter set of spheres instead of discrete input data.

The aim of this paper is at providing a method to blend two spheres by a rotation-minimizing blending surface. The method works for a wide range of initial data and can be computed in seconds, although not necessarily in real time. The surface is rotation-minimizing in a sense that the parametric curves of the surface have no distortion comparing to the corresponding rotation-minimizing frame along the spine curve. In Section 2 of this paper the problem is precisely formulated, together with the overview of the method provided in [7]. The basic notations and the necessary computational methods are summarized in terms of rotation-minimizing frames in Section 3, where we also define the notion of rotation-minimizing transportation along a curve. The solution and the detailed computation are provided in Section 4.

2. The problem

Suppose that two spheres are given in $\mathbb{R}^3$. For our problem it is enough to consider the cones which touch the spheres along two arbitrary circles $(\mathbf{p}_i, r_i)$ and $(\mathbf{p}_f, r_f)$, having centers $\mathbf{p}_i$ and

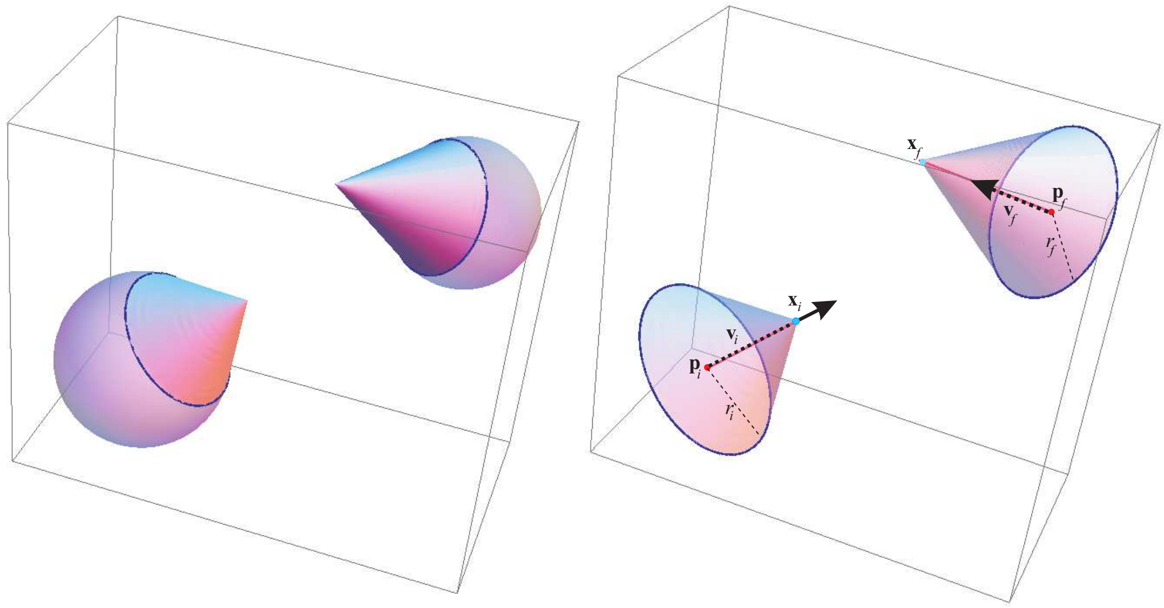

Figure 1: Visualization of the statement of the problem. The initial data contain two spheres and their touching cones.

$\mathbf{p}_f$ and radii r_i and r_f, respectively. Let $\mathbf{x}_i$ be the apex of the tangent cone touching the first sphere along the circle $(\mathbf{p}_i, r_i)$, while $\mathbf{x}_f$ is the apex of the tangent cone touching the second sphere along the circle $(\mathbf{p}_f, r_f)$ (see Figures 1 and 5).

Further, let $\mathbf{t}_i$ and $\mathbf{t}_f$ be vectors from the center of the given circles, parallel to the axes of the touching cones, that is $c_i\mathbf{t}_i = \mathbf{x}_i - \mathbf{p}_i$ and $c_f\mathbf{t}_f = \mathbf{x}_f - \mathbf{p}_f$ for some $c_i, c_f \in \mathbb{R} \setminus \{0\}$.

Our aim is to define a surface $\mathbf{s}(\theta, t)$ touching the given spheres at the given circles with the least possible distortion. It is constructed in a way that the parametric curve $\mathbf{s}(\theta^*, t)$ which belongs to a fixed value θ^* will connect a point $\mathbf{q}_i$ of the first circle and the corresponding point $\mathbf{q}_f$ of the second circle, while tangent vectors of the curve in these points will be parallel to the corresponding generatrix of the tangent cone, that is

$$\begin{aligned} \mathbf{s}(\theta^*, 0) &= \mathbf{q}_i, \quad \mathbf{s}(\theta^*, 1) = \mathbf{q}_f, \\ \frac{\partial \mathbf{s}}{\partial t}(\theta^*, 0) &= \mathbf{v}_i, \quad \frac{\partial \mathbf{s}}{\partial t}(\theta^*, 1) = \mathbf{v}_f \end{aligned}$$

where

$$\mathbf{v}_i = a_i\, \mathbf{x}_i - \mathbf{q}_i, \qquad \mathbf{v}_f = a_f\, \mathbf{x}_f - \mathbf{q}_f \tag{1}$$

with $a_i, a_f \in \mathbb{R}^+$.

The key question is: which point $\mathbf{q}_f$ of the second circle should belong to the initial point $\mathbf{q}_i$? In [7] a fast and simple algorithm is provided to compute the corresponding points. By defining an arbitrary direction $\mathbf{e}$ not parallel to $\mathbf{t}_i$ and $\mathbf{t}_f$, the two corresponding points are defined by the vectors $\mathbf{e} \times \mathbf{t}_i$ and $\mathbf{e} \times \mathbf{t}_f$, more precisely

$$\mathbf{q}_i = \mathbf{p}_i \pm r_i \frac{\mathbf{e} \times \mathbf{t}_i}{\|\mathbf{e} \times \mathbf{t}_i\|}, \qquad \mathbf{q}_f = \mathbf{p}_f \pm r_f \frac{\mathbf{e} \times \mathbf{t}_f}{\|\mathbf{e} \times \mathbf{t}_f\|},$$

where the sign of the latter terms depends on the measures of the angles between the vectors $\mathbf{t}_i$ and $\mathbf{p}_f - \mathbf{p}_i$, and $\mathbf{t}_f$ and $\mathbf{p}_f - \mathbf{p}_i$, respectively. If the angle is greater than $\pi/2$, then the

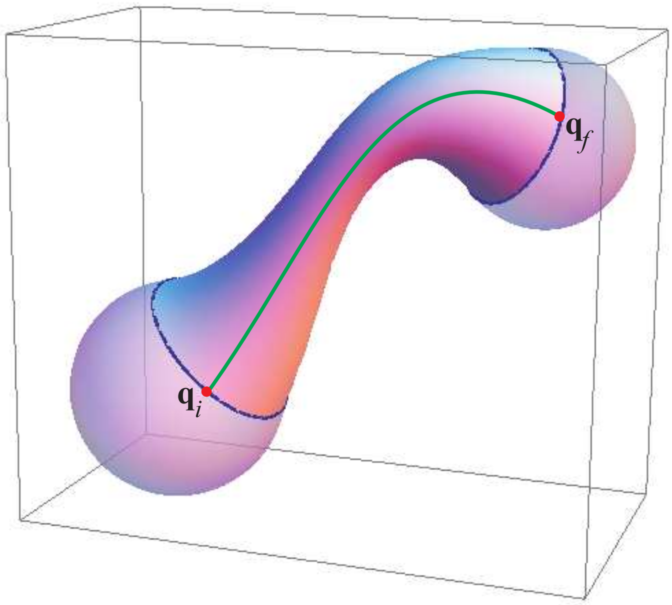

Figure 2: A surface blending the given spheres touching them at the given circles.

sign is negative, otherwise it is positive. Further pairs of points are obtained by the rotation of the original pairs along the circles.

Note, that the choice of $\mathbf{e}$ has not been uniquely defined in this algorithm. A similar, even more evident choice of the two corresponding points can be defined simply by the vector $\mathbf{t}_i \times \mathbf{t}_f$ as

$$\mathbf{q}_i = \mathbf{p}_i + r_i \frac{\mathbf{t}_i \times \mathbf{t}_f}{\|\mathbf{t}_i \times \mathbf{t}_f\|}, \qquad \mathbf{q}_f = \mathbf{p}_\mathbf{f} + r_f \frac{\mathbf{t}_i \times \mathbf{t}_f}{\|\mathbf{t}_i \times \mathbf{t}_f\|}.$$

This straightforward but somehow ad hoc selection of corresponding points of the circles works in a satisfactory way for the cases, when the vectors $\mathbf{t}_i, \mathbf{t}_f$ and $\mathbf{p}_f - \mathbf{p}_i$, that is the axes of the touching cones are parallel or intersecting lines. In most figures of [7] this is the case. However, if the positions of the touching circles differ from this case, that is the axes of the touching cones are skew lines, the above mentioned method of mapping of points cannot be justified. In fact, the blending surface is heavily affected by the choice of corresponding points and can have a kind of distortion in some cases, as it can be seen in Figure 3. Our aim is to provide a theoretically founded method of finding the most suitably corresponding pairs of points.

3. Rotation minimizing frames

In this section we briefly overview the basic concept of Rotation Minimizing Frames (RMF in short) based on [5, 6], and we define the rotation minimizing transportation of a vector along the curve. Throughout this session we use the well known Frenet-frame denoted by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$.

Definition 1. Given a regular curve $\boldsymbol{\alpha}\colon [0,1] \to \mathbb{R}^3$, a positively oriented orthonormal frame $\{\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2\}$ along the curve is called *rotation-minimizing frame* (RMF) if

1. $\mathbf{w}_0 = \mathbf{t}$,
2. $\mathbf{w}_1{}'$ is parallel to $\mathbf{t}$.

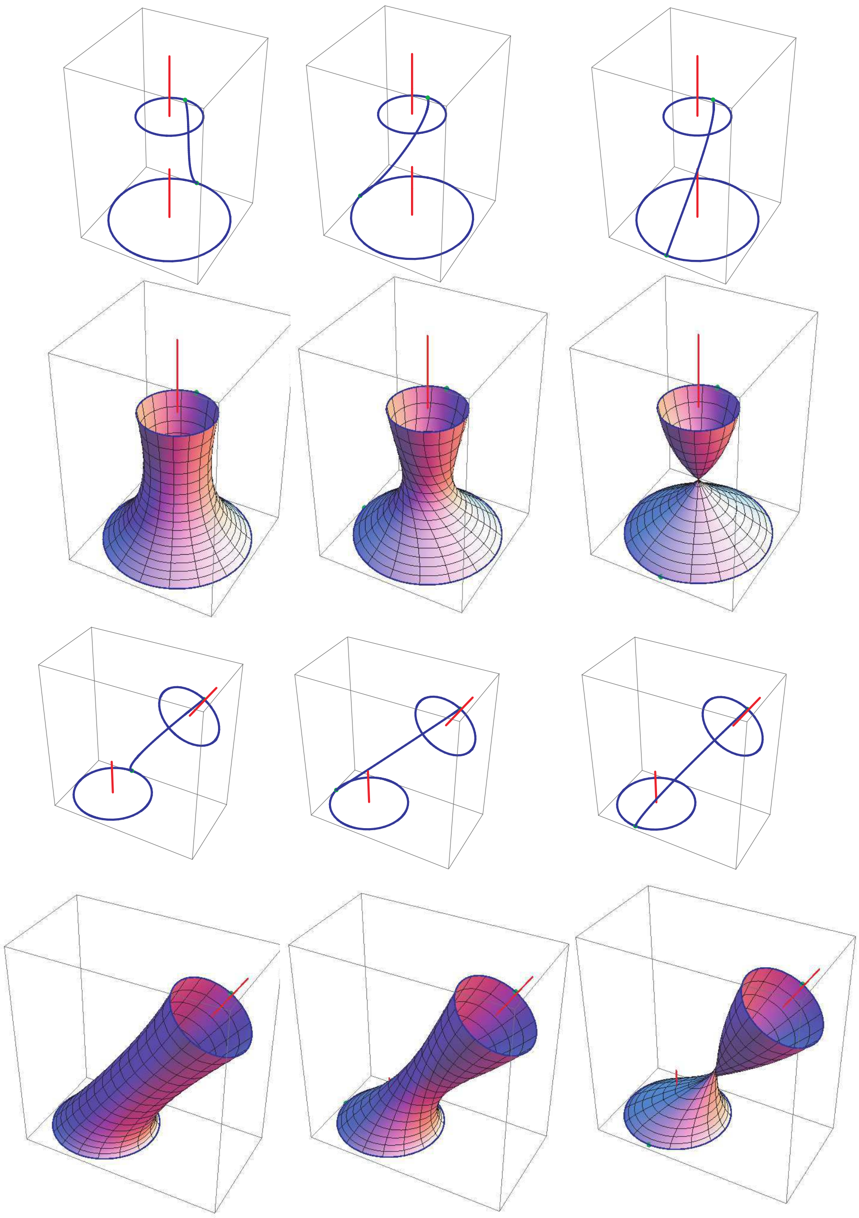

Figure 3: The effect of different choices of corresponding initial points on the circles deeply affect distortions of the surfaces.

Remark 1. The second condition is equivalent to ask for $\dot{\mathbf{w}}_1$ being parallel to $\mathbf{t}$, where the dot denotes arc-length derivative.

3.1. Transportation along a curve using a RMF

Given a unit vector $\mathbf{w}_i$, orthogonal to $\boldsymbol{\alpha}'(0)$, there is a unique rotation-minimizing frame along the curve $\boldsymbol{\alpha}$, denoted by $\{\mathbf{t}, \mathbf{w}_1, \mathbf{w}_2\}$, such that

$$\mathbf{w}_1(0) = \mathbf{w}_i\,.$$

Let us recall how to compute it.

$$\mathbf{w}_1(t) = \cos\theta(t)\,\mathbf{n}(t) - \sin\theta(t)\,\mathbf{b}(t), \tag{2}$$

where

$$\theta(t) = \theta_0 - \int_0^t \tau(s)\, ds, \tag{3}$$

θ_0 is the initial angle between $\mathbf{w}_i = \mathbf{q}_i - \boldsymbol{\alpha}(0)$ and the binormal vector to the curve $\boldsymbol{\alpha}(t)$ at $\boldsymbol{\alpha}(0)$, $\mathbf{b}(0)$, and τ denotes the torsion of the curve $\boldsymbol{\alpha}$. The arc-length parameter is denoted by s. If another parameter is used for the curve $\boldsymbol{\alpha}$, then

$$\int_0^t \tau(s)\, ds = \int_0^t \tau(u)||\boldsymbol{\alpha}'(u)||\, du.$$

The *rotation-minimizing transportation* (RM-transportation for short) to $\boldsymbol{\alpha}(1)$ of the vector $\mathbf{w}_i$ along the curve $\boldsymbol{\alpha}$ uses a rotation-minimizing frame as the vector $\mathbf{w}_1(1)$. Notice that $||\mathbf{w}_1(1)|| = 1$ and $\mathbf{w}_1(1) \perp \boldsymbol{\alpha}'(1)$.

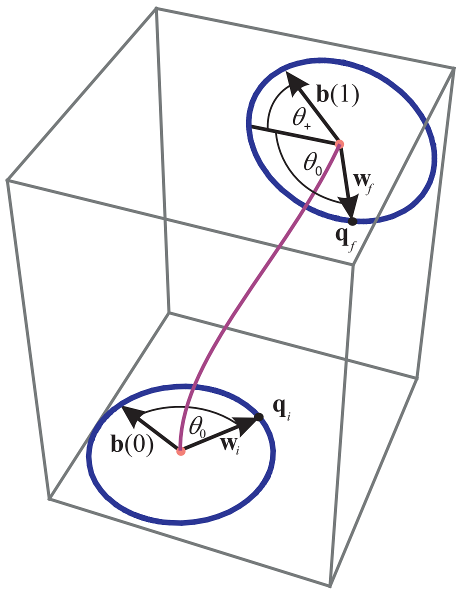

Figure 4: The rotation-minimizing map $q_i \to q_f$ between the perpendicular unit circles at the end points of the curve $\alpha(t)$.

Therefore, we can define a map between the perpendicular unit circles at the end points of the curve $\boldsymbol{\alpha}$. Given a point $\mathbf{q}_i$ in the unit circle on the normal plane to the curve at $\boldsymbol{\alpha}(0)$, let us consider the vector $\mathbf{w}_i = \mathbf{q}_i - \boldsymbol{\alpha}(0)$, then $\mathbf{q}_f = \boldsymbol{\alpha}(1) + \mathbf{w}_1(1)$. The angle θ_+ (see Figure 4) is defined as

$$\theta_+ = -\int_0^1 \tau(u)||\boldsymbol{\alpha}'(u)||\, du.$$

Notice that in this way we can control which point $\mathbf{q}_f$ in the second circle is associated with a given point $\mathbf{q}_i$ in the first circle, while the tangent vectors of the parameter curve of the surface joining $\mathbf{q}_i$ and $\mathbf{q}_f$ are inherited from the touching cones.

4. The solution

In order to apply the rotation-minimizing frame method, at first we define a spine curve $\boldsymbol{\alpha}(t)$ of the blending surface. For the notations see Figure 5.

Let $\boldsymbol{\alpha}\colon [0,1] \to \mathbb{R}^3$ be a regular curve satisfying the C^1-Hermite conditions:

$$\boldsymbol{\alpha}(0) = \mathbf{p}_i, \quad \boldsymbol{\alpha}(1) = \mathbf{p}_f,$$
$$\boldsymbol{\alpha}'(0) = \mathbf{t}_i, \quad \boldsymbol{\alpha}'(1) = \mathbf{t}_f.$$

Let $\mathbf{w}_i = \dfrac{\mathbf{q}_i - \mathbf{p}_i}{r_i}$ and let $\mathbf{w}_f$ be the RM-transportation to $\boldsymbol{\alpha}(1)$ of the vector $\mathbf{w}_i$.

Being r_f the radius of the second circle, let us define the corresponding point $\mathbf{q}_f$ as

$$\mathbf{q}_f = \mathbf{p}_f + r_f \mathbf{w}_f.$$

We construct the parametric curve $\mathbf{s}(\theta_0, t) = \boldsymbol{\beta}(t)\colon [0,1] \to \mathbb{R}^3$ such that

$$\begin{aligned} \boldsymbol{\beta}(0) &= \mathbf{q}_i, \quad \boldsymbol{\beta}(1) = \mathbf{q}_f, \\ \boldsymbol{\beta}'(0) &= \mathbf{v}_i, \quad \boldsymbol{\beta}'(1) = \mathbf{v}_f, \end{aligned} \tag{4}$$

hold, and further on the curve follows the RM-transportation, that is for any $t \in [0,1]$, the vector $\boldsymbol{\beta}(t) - \boldsymbol{\alpha}(t)$ is parallel to $\mathbf{w}_1(t)$, where $\mathbf{w}_1(t)$ is the actual position of the RM-transported vector.

The last condition implies that there exists a function $\lambda\colon [0,1] \to \mathbb{R}$ such that

$$\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(t) + \lambda(t)\mathbf{w}_1(t).$$

Therefore

$$\boldsymbol{\beta}'(t) = \boldsymbol{\alpha}'(t) + \lambda'(t)\mathbf{w}_1(t) + \lambda(t){\mathbf{w}_1}'(t) = ||\boldsymbol{\alpha}'(t)||\,(1 + \mu(t)\lambda(t))\,\mathbf{t}(t) + \lambda'(t)\mathbf{w}_1(t),$$

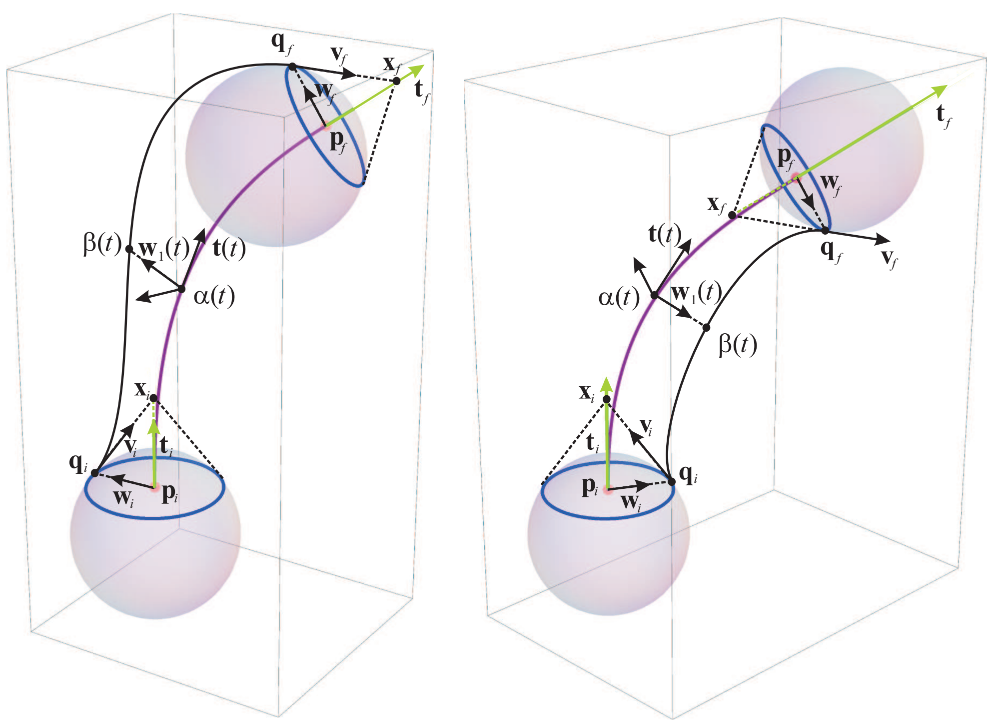

Figure 5: Visualization of various notations in two different positions of initial data (with unit spheres). The spine curve $\alpha(t)$ (purple) and one of the parameter curves $\beta(t)$ (black) can also be seen.

where we have used that, since $\{\mathbf{t}, \mathbf{w}_1, \mathbf{w}_2\}$ is a rotation-minimizing frame,

$$\dot{\mathbf{w}}_1(s) = \mu(s)\mathbf{t}(s)$$

or, equivalently,

$$\mathbf{w}_1{}'(t) = ||\boldsymbol{\alpha}'(t)||\mu(t)\mathbf{t}(t).$$

Let us compute explicitly the function $\mu(s) = \langle \dot{\mathbf{w}}_1(s), \mathbf{t}(s)\rangle$. From (2) we have

$$\begin{aligned} \dot{\mathbf{w}}_1 &= -\dot{\theta}\sin(\theta)\mathbf{n} + \cos(\theta)\dot{\mathbf{n}} - \dot{\theta}\cos(\theta)\mathbf{b} - \sin(\theta)\dot{\mathbf{b}} \\ &= -\dot{\theta}\sin(\theta)\mathbf{n} + \cos(\theta)(-\kappa\mathbf{t} - \tau\mathbf{b}) - \dot{\theta}\cos(\theta)\mathbf{b} - \sin(\theta)\tau\mathbf{n} \\ &= -\cos(\theta)\kappa\mathbf{t}, \end{aligned}$$

where we have applied that $\dot{\theta} = -\tau$ (see (3)). Therefore,

$$\mu(s) = -\cos(\theta(s))\kappa(s). \tag{5}$$

Let us write conditions (4) in terms of the function $\lambda(t)$.

$$\begin{aligned} \boldsymbol{\beta}(0) &= \boldsymbol{\alpha}(0) + \lambda(0)\mathbf{w}_1(0) \\ &= \mathbf{p}_i + \lambda(0)\mathbf{w}_i \\ &= \mathbf{p}_i + \lambda(0)\frac{\mathbf{q}_i - \mathbf{p}_i}{r_i}. \end{aligned}$$

Since we asked for $\boldsymbol{\beta}(0) = \mathbf{q}_i = \mathbf{p}_i + (\mathbf{q}_i - \mathbf{p}_i)$, then

$$\lambda(0) = r_i. \tag{6}$$

Analogously,

$$\lambda(1) = r_f. \tag{7}$$

More conditions on λ can be obtained through the derivative of the curve $\boldsymbol{\beta}$:

$$\begin{aligned} \boldsymbol{\beta}'(0) &= ||\boldsymbol{\alpha}'(0)||\,(1 + \mu(0)\lambda(0))\,\mathbf{t}(0) + \lambda'(0)\mathbf{w}_1(0) \\ &= (1 + \mu(0)r_i)\,\mathbf{t}_i + \lambda'(0)\frac{\mathbf{q}_i - \mathbf{p}_i}{r_i}. \end{aligned}$$

Since $\boldsymbol{\beta}'(0) = \mathbf{v}_i = a_i\,(\mathbf{q}_i - \mathbf{x}_i)$ (see Eqs. (4) and (1)), then

$$\lambda'(0) = -a_i r_i \tag{8}$$

and

$$1 + \mu(0)r_i = a_i c_i.$$

According to Eq. (5), $\mu(0) = -\cos(\theta_0)\kappa(0)$. Therefore,

$$a_i = \frac{1}{c_i}\big(1 - \cos(\theta_0)\kappa(0)r_i\big). \tag{9}$$

This means that the length of the initial tangent vector to the curve $\boldsymbol{\beta}$ depends on $\theta_0, \kappa(0)$ and r_i. Analogously,

$$\lambda'(1) = -a_f r_f \tag{10}$$

and

$$a_f = \frac{1}{c_f}\big(1 - \cos(\theta(1))\kappa(1)r_f\big). \tag{11}$$

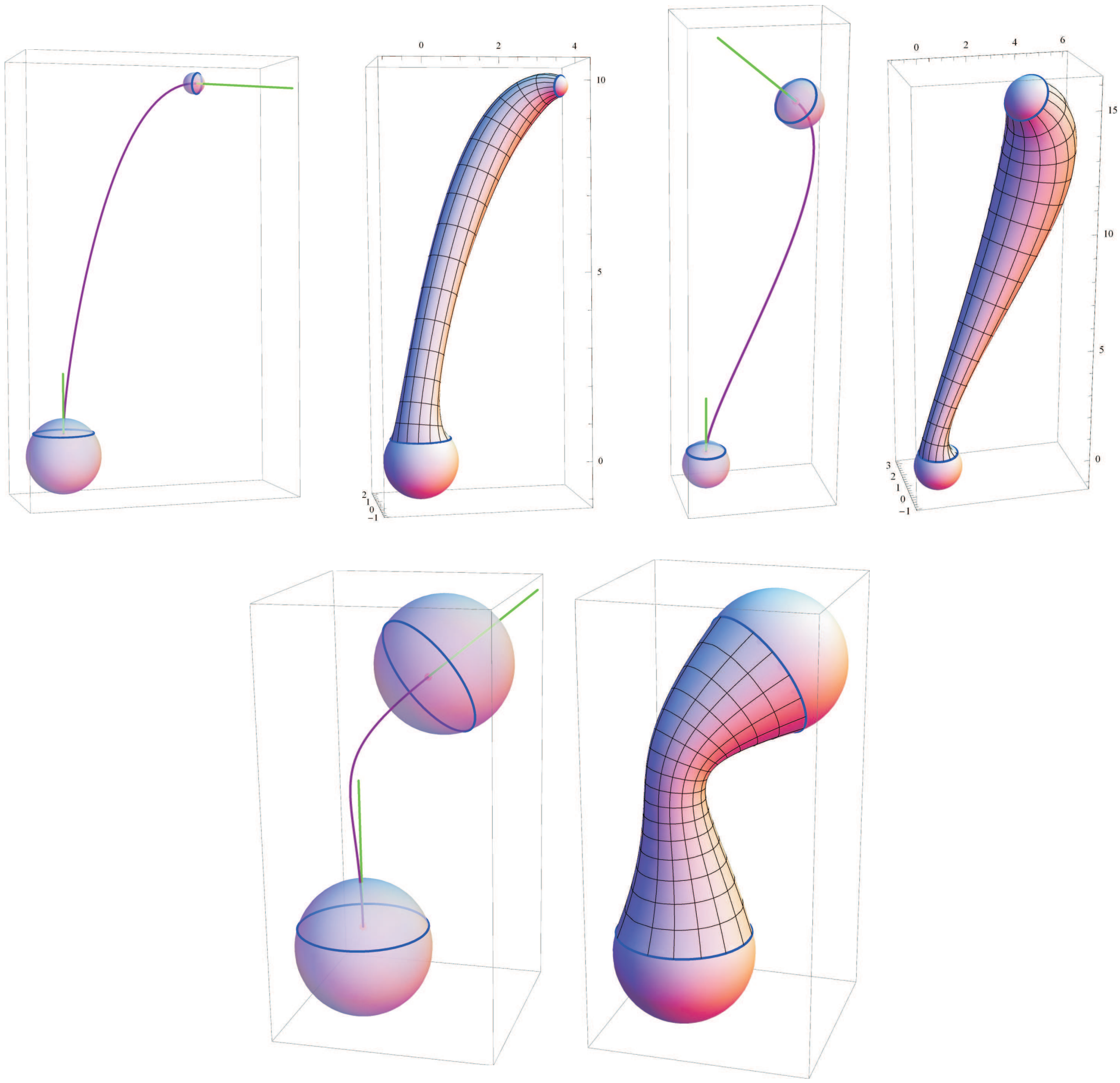

Figure 6: Three positions of initial data, each with the spine curve (left) and the computed surface (right).

Remark 2. One restriction must be considered: Since both a_i and a_f must be positive, the spine curve has to be such that

$$\kappa(0)r_i \leq 1, \quad \text{and} \quad \kappa(1)r_f \leq 1.$$

Such conditions can be satisfied through a convenient choice of the initial and final vectors, $\boldsymbol{\alpha}'(0) = \mathbf{t}_i$ and $\boldsymbol{\alpha}'(1) = \mathbf{t}_f$ of the spine curve. A simple computation shows that the curvature of the spine curve at the initial point is

$$\kappa(0) = 2\,\frac{||\mathbf{t}_i \times (3(\mathbf{p}_f - \mathbf{p}_i) - \mathbf{t}_f)||}{||\mathbf{t}_i||^3}.$$

Short length initial and final vectors could produce spine curves with large initial and final curvatures. Curvatures not satisfying the restrictions at the endpoints of the spine curve may yield self-intersections of the resulting surface, as it can be seen in Figure 7. In the special case where the center of one sphere is in the other sphere and the angle between the planes of touching circles is larger than 45° the necessarily restricted spine curve cannot be computed and thus our method cannot provide the rotation minimizing blending surface.

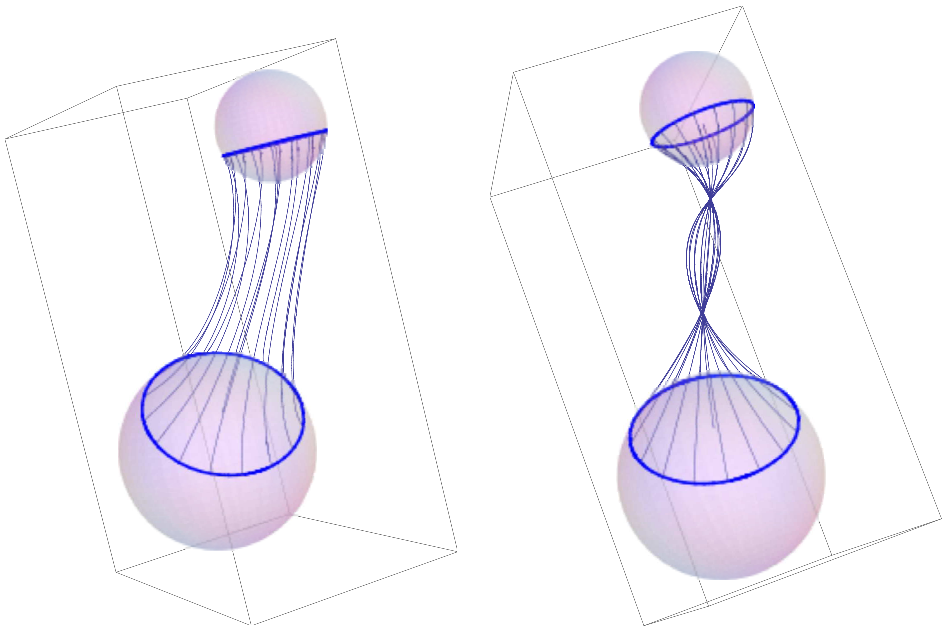

Figure 7: Curvatures not satisfying the restrictions at the endpoints of the spine curve may yield self-intersections of the resulting surface (right). Smooth blending surface can be obtained for the same initial data with an appropriate choice of spine curve (left).

The easiest solution for function λ is the Hermite solution of Eqs. (6), (7), (8), (10):

$$\lambda(0) = r_i, \quad \lambda(1) = r_f, \quad \lambda'(0) = -a_i r_i, \quad \lambda'(1) = -a_f r_f,$$

where a_i and a_f are given by Eqs. (9) and (11). This is

$$\lambda(t) = r_f t^2(3 - 2t + a_f(1 - t)) - r_i(-1 + t)^2(-1 + (-2 + a_i)t).$$

While the results of our method for various initial data can be seen in Figure 6, one can also observe the advantages of this technique in the comparison of the results of the method of [7] and the provided solution (Figure 8). The earlier method may yield unnecessary distortions of the surface, while our rotation minimizing surface has no such distortion. In the sense of rotation minimization, our blending surface is optimal.

5. Conclusion

A novel method of skinning of spheres has been presented where the key step is the rotation minimizing transportation of a vector along the spine curve. By the help of this technique the corresponding points of the touching circles are determined and thus the resulted surface has no unwanted distortions. Due to the time consuming integration of the torsion during the computation, a couple of seconds is required to compute the skin, that is, no real time computation is possible, in which sense further improvements may be necessary.

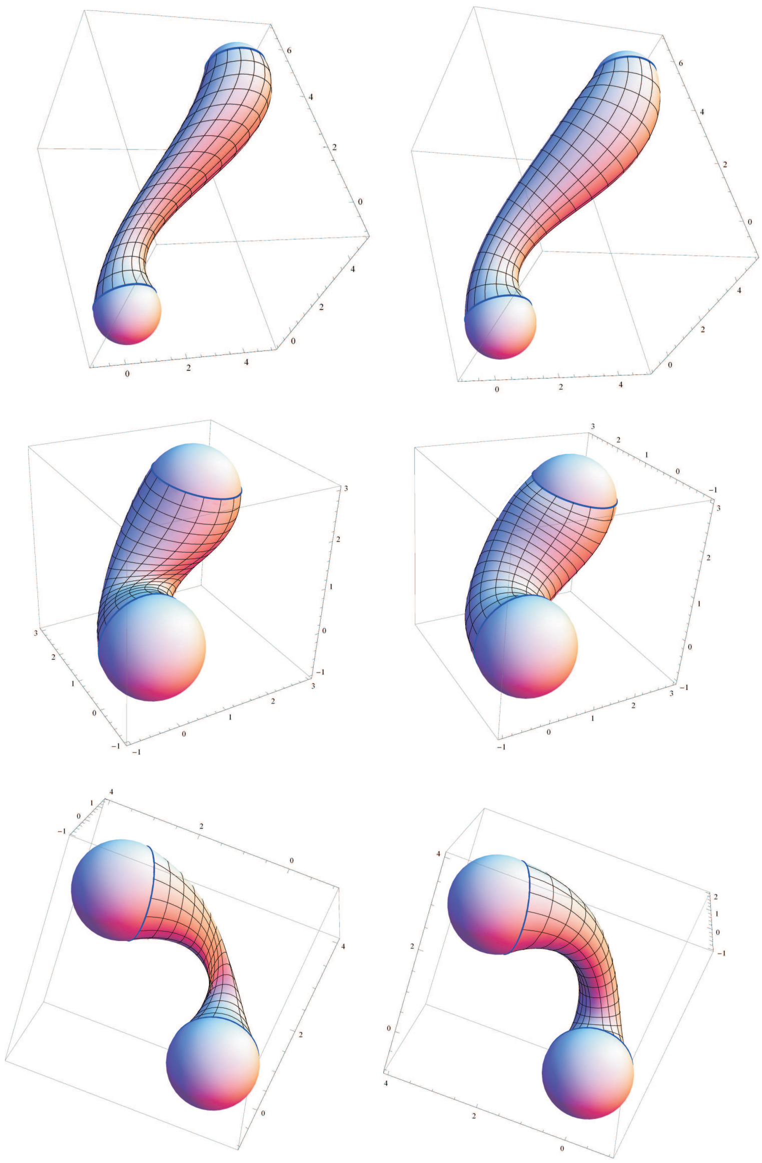

Figure 8: Comparison of results of the earlier method (left) and our method (right). Surfaces at the right side have visibly less distorted shape.

Acknowledgement

The first author would like to thank the Universitat de València for the warm hospitality during his stay. The second author has been partially supported by grant MTM2009-08933 from the Spanish Ministry of Science and Innovation.

References

[1] L.R. Bishop: *There is more than one way to frame a curve.* Amer. Math. Monthly **83**, 246–251 (1975).

[2] R.T. Farouki: *The conformal map $z \to z^2$ of the hodograph plane.* Comput. Aided Geom. Design **11**, 363–390 (1994).

[3] R.T. Farouki: *Pythagorean-Hodograph Curves. Algebra and Geometry inseparable.* Springer, Berlin 2008.

[4] R.T. Farouki, T. Sakkalis: *Pythagorean hodographs.* IBM Journal of Research and Development **34**/5, 736–752 (1990).

[5] R.T. Farouki, C.Y. Han: *Rational approximation schemes for rotation-minimizing frames on Pythagorean-hodograph curves.* Comput. Aided Geom. Design **20**, 435–454 (2003).

[6] W. Wang, B. Jüttler, D. Zheng, Y. Liu: *Computation of Rotation Minimizing Frames.* ACM Transactions on Graphics **27**, article 2, (2008).

[7] R. Kunkli, M. Hoffmann: *Skinning of circles and spheres.* Comput. Aided Geom. Design **27**, 611–621 (2010).

[8] C.K. Shene: *Blending two cones with Dupin cyclides.* Comput. Aided Geom. Design **15**, 643–673 (1998).

[9] G. Slabaugh, B. Whited, J. Rossignac, T. Fang, G.B. Unal: *3D ball skinning using PDEs for generation of smooth tubular surfaces.* Computer-Aided Design **42**, 18–26 (2010).

[10] G. Slabaugh, G.B. Unal, T. Fang, J. Rossignac, B. Whited: *Variational Skinning of an Ordered Set of Discrete 2D Balls.* Proc. of GMP 2008, 450–461 (2008).

[11] M. Paluszny, W. Boehm: *General cyclides.* Comput. Aided Geom. Design **15**, 699–710 (1998).

[12] H.I. Choi, S.-H. Kwon, N.-S. Wee: *Almost rotation-minimizing rational parametrization of canal surfaces.* Comput. Aided Geom. Design **21**, 859–881 (2004).

[13] M. Bizzarri, M. Lávička: *On modeling with rational ringed surfaces.* Computer-Aided Design **58**, 151–161 (2015).

[14] M. Bizzarri, M. Lávička, J. Vršek: *Canal surfaces with rational contour curves and blends bypassing the obstacle.* Computer-Aided Design **64**, 55–67 (2015).

[15] K. Maleček, Z. Šibrava: *Blending Circular Pipes with a Cyclic Surface.* J. Geometry Graphics **10**, 99–107 (2006).

[16] S. Foufou, L. Garnier: *Dupin Cyclide Blends Between Quadric Surfaces for Shape Modeling.* Computer Graphics Forum **23**, 321–330 (2004).

Received April 7, 2014; final form October 14, 2015