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Density of Optimal Packings of Three Ellipses in a Square

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Abstract. We prove that the most dense packing of three non-overlapping congruent ellipses of aspect ratio $E \in]0,1]$ in a square is obtained for $E = 1/3$, with density equal to $\pi/4$. This result was already known for two ellipses (for $E = 1/2$), but is no longer true for an arbitrary number of non-overlapping congruent ellipses.

Key Words: packing, ellipse, density

MSC 2010: 52C15, 05B40, 51N20

1. Introduction

The problem of packing circles in a given domain has been extensively studied (see [1], [6], or [7]) and has many applications in practice. Packings of ellipses are also used in many technological domains (molecules in crystals, elliptical particules,...) and can be viewed as a generalization of the previous problem. Some authors worked on the density of these packings, see for instance [8] for an upper bound for the density of a packing of ellipses of given areas, or [2, 4] for an algorithmic approach. The question of finding the optimal packing of n congruent non-overlapping ellipses in a square was completely solved for $n = 2$ in [3], by use of the notion on "unavoidable point" (see [1]). It turns out that this method does not work for $n = 3$. So we do not know the best packings of three congruent non-overlapping ellipses in a square, except using an algorithmic method for approaching them (see Figure 8). Nevertheless we are able to prove that the best density is reached for three "vertical" ellipses of aspect ratio $E = 1/3$ (cf. Theorem 1 below).

2. Statement of the results

Let $n \in \mathbb{N}^*$ and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be n non-overlapping congruent ellipses of the same aspect ratio $E \in]0,1]$. We can deal without loss of generality with *unit ellipses* i.e., we suppose that the semi-major axis is equal to 1 (thus the common semi-minor axis is E). For each such

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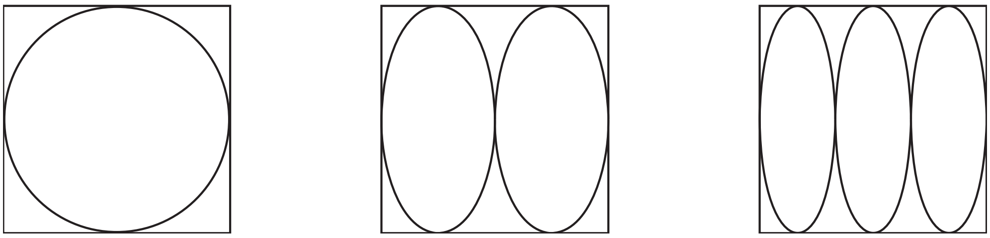

Figure 1: Configurations of optimal density for one, two or three unit ellipses

configuration, there is a square of minimal side length $S_n(E)$ containing these ellipses, and we denote by $s_n(E) = \inf S_n(E)$ the smallest value of $S_n(E)$ under all such possible configurations. In this paper, we are interested in the *density* of these optimal packings. We will denote it by $d_n(E) = \dfrac{n\pi E}{s_n^2(E)}$ and we focus on its maximum $d_n = Max\,\{d_n(E) \,|\, E \in \,]0,1]\}$. It is well known that $d_1(E) = \pi E/2(1+E^2)$, thus $d_1 = \pi/4$ is attained for a circle, and we proved in [3] that $d_2 = \pi/4$ again, the maximum being obtained for two "vertical" ellipses of aspect ratio $E = 1/2$.

It turns out that this density result remains true for three ellipses.

Theorem 1. *The optimal density of three non-overlapping congruent ellipses of the same aspect ratio is* $d_3 = \pi/4$*. Moreover, this maximum is attained for three "vertical" ellipses of aspect ratio* $E = 1/3$*.*

Unfortunately, this cannot be generalized to all values of n. An easy way to see this is to consider hexagonal packings of unit circles in a rectangle. Indeed, let n, m be two integers. If we pack m lines of n tangent unit circles as in Figure 2, we obtain a rectangle of size $(2n+1)$ times $(2+(m-1)\sqrt{3})$. We apply a vertical stretching with factor $E = (2n+1)/\left(2+(m-1)\sqrt{3}\right)$ in order to have a square. The condition $E < 1$ will be realized if $m > (2n+\sqrt{3}-1)/\sqrt{3}$. The previous circles are transformed into ellipses of semi-axis 1 and

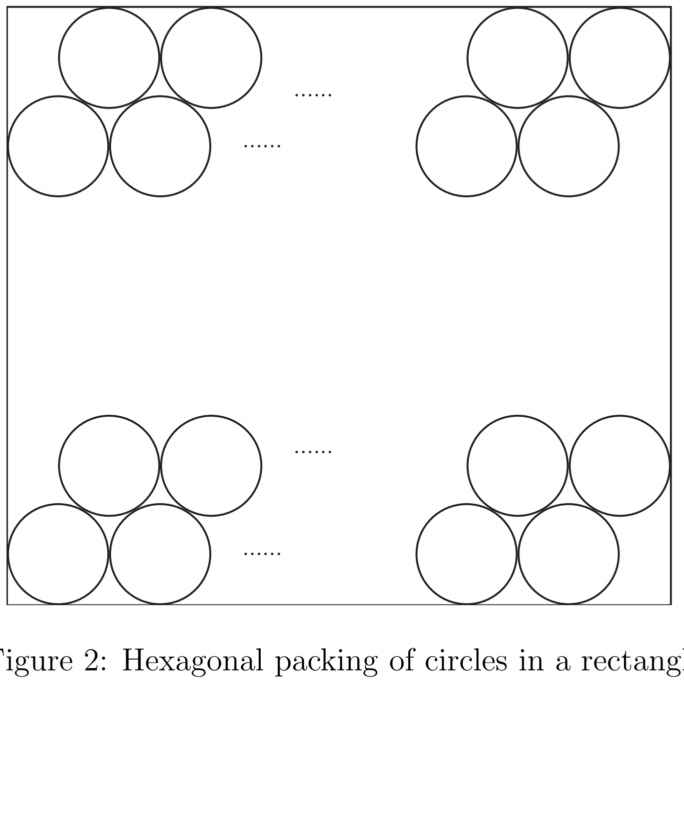

Figure 2: Hexagonal packing of circles in a rectangle

E, and the density of the new packing becomes

$$\frac{m \cdot n \cdot \pi E}{(2n+1)^2} = \frac{n\pi\left(2n+1-(2-\sqrt{3})E\right)}{\sqrt{3}(2n+1)^2}.$$

which has the limit $\pi/2\sqrt{3}$ as n tends to infinity. More precisely, it is easy to see that, for each fixed E in $]0,1]$, this density is greater than $\pi/4$ for $n \geq 5$ and $m \geq (2n+1-E(2-\sqrt{3}))/E\sqrt{3}$. Consequently, the density of the packing will also be greater than $\pi/4$.

3. Proof of the theorem

Recall that we consider three non-overlapping congruent unit ellipses $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ of the same aspect ratio E contained in a square of length $S_3(E)$. We must show that, for each configuration and each $E \in]0,1]$, we have $\dfrac{3\pi E}{S_3^2(E)} \leq \dfrac{\pi}{4}$, or equivalently

$$S_3(E) \geq \sqrt{12E}. \tag{1}$$

Suppose that there is an ellipse which is not tangent to the others. Then one can move it just a little bit without changing the sides of the square. Hence we can assume that each of these ellipses is tangent to another one. Moreover, one can see that the tangent lines cut the square in four polygons in general (see Figure 3), three of these polygons containing $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$. We divide the proof in two parts, depending on whether these polygons are quadrilateral (or even triangles) or pentagons.

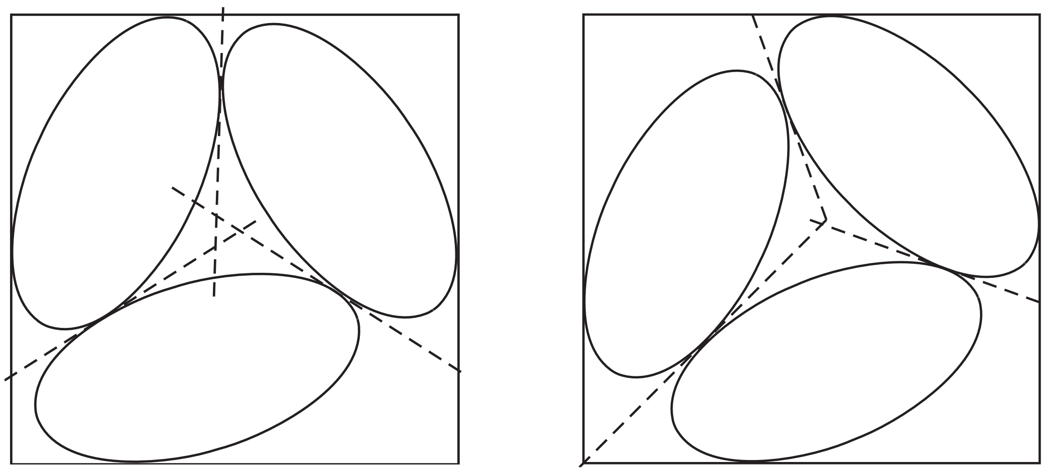

Figure 3: Pentagonal and quadrilateral cases

3.1. The quadrilateral case

This case is easy, for if a quadrilateral (or a triangle) contains a unit ellipse $\mathcal{E}$ of aspect ratio E, then its area is bigger than $4E$. One can see this for instance by stretching $\mathcal{E}$ into a unit circle and observing that the unit square is the quadrilateral of smallest area containing it. Note that a one-dimensional stretching preserves the ratio of areas. Thus we conclude that the area $S_3(E)$ of the square containing $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ is greater or equal $3 \times 4E = 12E$, and (1) is verified. $\square$

3.2. The pentagonal case

Before starting the proof, let us notice that it is possible to have a configuration of two pentagons and one triangle, but this case is also easy. In fact, the minimal area of a pentagon containing a unit circle is $5\tan(\pi/5)$ whereas the minimal area of a triangle containing a unit circle is $3\sqrt{3}$. Thus, if the common tangent lines of the three ellipses cut a square into two pentagons and one triangle, its area is greater than $\left(10\tan(\pi/5)+3\sqrt{3}\right)E \geq 12E$.

The difficult case is the one of a single pentagon, because a pentagon containing $\mathcal{E}$ may have an area less than $4E$. So, we have to look at things more closely. In order to see in detail what happens in each polygon (see Figure 4), we shall use the following lemmas which we are not going to prove (similar calculations can be found in [3] and [4]).

Lemma 1. *Let $\mathcal{E}_\alpha$ be the ellipse of semi-axes 1 and $E \in\,]0,1]$, tangent to the Cartesian segments $[O,x)$ and $[O,y)$, forming the angle $\alpha \in [-\pi/2,\pi/2]$ with the horizontal direction. Then $\mathcal{E}_\alpha$ is parametrized by:*

$$t \in [0,2\pi] \longmapsto \left\{ \begin{array}{rcl} x(t) & = & \cos(\alpha)\cos(t) - E\sin(\alpha)\sin(t) + \lambda_\alpha \\ y(t) & = & \sin(\alpha)\cos(t) + E\cos(\alpha)\sin(t) + \mu_\alpha \end{array} \right. \in \mathbb{R}^2$$

where $\lambda_\alpha = \sqrt{\cos^2(\alpha)+E^2\sin^2(\alpha)}$, $\mu_\alpha = \sqrt{\sin^2(\alpha)+E^2\cos^2(\alpha)}$ are the coordinates of the center Ω_α of $\mathcal{E}_\alpha$.

Lemma 2. *The ellipse $\mathcal{E}_\alpha$ is tangent to the x-axis at a point $(x_\alpha,0)$ with*

$$x_\alpha = \lambda_\alpha - \sin(2\alpha)(1-E^2)/2\mu_\alpha$$

and to the y-axis at a point $(0,y_\alpha)$ with

$$y_\alpha = \mu_\alpha - \sin(2\alpha)(1-E^2)/2\lambda_\alpha.$$

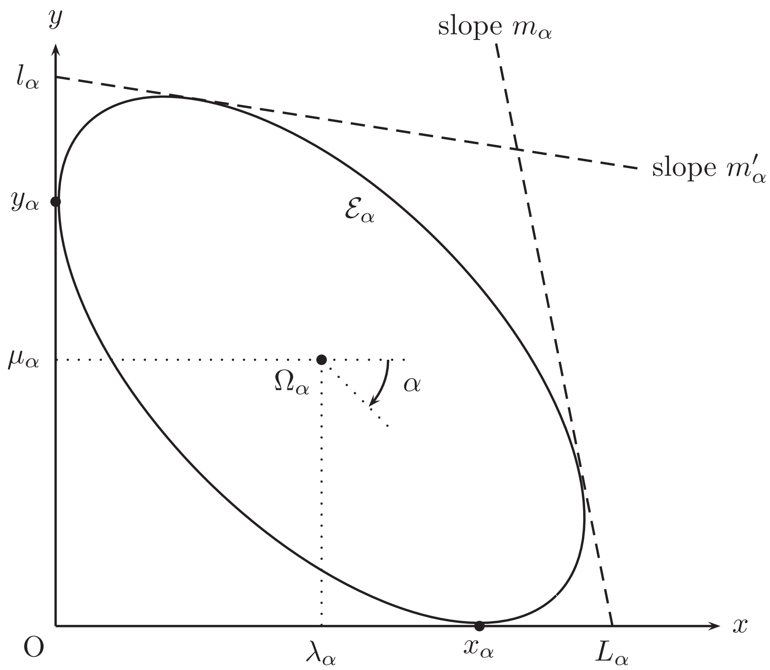

Figure 4: Tangent lines to the ellipse $\mathcal{E}_\alpha$

Moreover, the tangent line to $\mathcal{E}_\alpha$ passing through the point $(L_\alpha, 0)$, $L_\alpha \geq x_\alpha$, has slope

$$m_\alpha = \frac{2\mu_\alpha(L_\alpha - \lambda_\alpha) + \sin(2\alpha)(1-E^2)}{L_\alpha(2\lambda_\alpha - L_\alpha)},$$

and the tangent line to $\mathcal{E}_\alpha$ through the point $(0, l_\alpha)$, $l_\alpha \geq y_\alpha$, has slope

$$m'_\alpha = \frac{l_\alpha(2\mu_\alpha - l_\alpha)}{\sin(2\alpha)(1-E^2) - 2\lambda_\alpha(\mu_\alpha - l_\alpha)}.$$

The same result holds for a unit ellipse $\mathcal{E}_\beta$ of aspect ratio E forming the angle $-\beta \in [0, \pi/2]$ with the x-axis (see Figure 5).

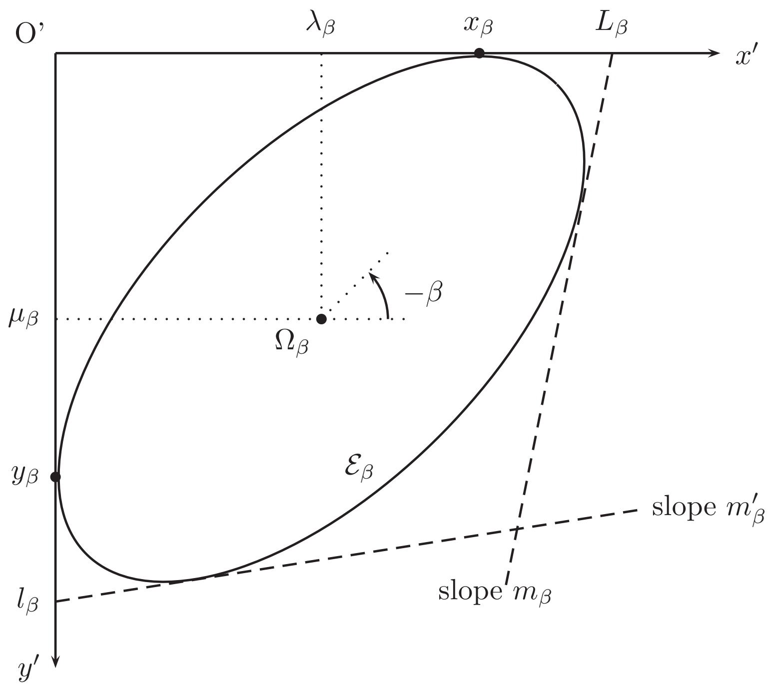

Figure 5: Tangent lines to the ellipse $\mathcal{E}_\beta$

This time, the slopes of the tangent lines are

$$m_\beta = -\frac{2\mu_\beta(L_\beta - \lambda_\beta) + \sin(2\beta)(1-E^2)}{L_\beta(2\lambda_\beta - L_\beta)} \tag{2}$$

$$m'_\beta = -\frac{l_\beta(2\mu_\beta - l_\beta)}{\sin(2\beta)(1-E^2) - 2\lambda_\beta(\mu_\beta - l_\beta)}. \tag{3}$$

Now our goal is first to put together the two previous ellipses and secondly to see if there is still some room available for a third ellipse $\mathcal{E}_\gamma$ (see Figure 6).

(i) The slopes m'_α and m'_β must be equal. This leads to:

$$l_\beta = \lambda_\beta m'_\alpha + \mu_\beta + \sqrt{\lambda_\beta^2 m'^2_\alpha + \mu_\beta^2 + m'_\alpha \sin(2\beta)(1-E^2)}.$$

Thus we know the side of the square (denoted by c):

$$c = l_\alpha + l_\beta = l_\alpha + \lambda_\beta m'_\alpha + \mu_\beta + \sqrt{\lambda_\beta^2 m'^2_\alpha + \mu_\beta^2 + m'_\alpha \sin(2\beta)(1-E^2)}.$$

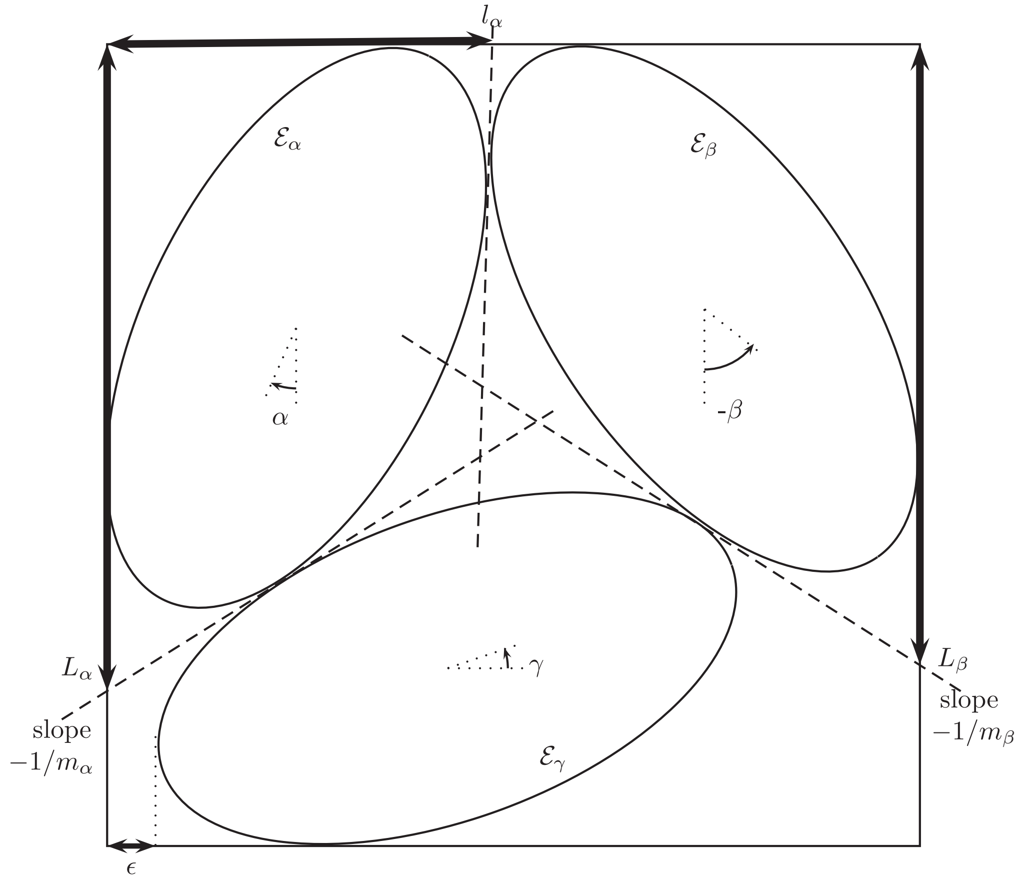

Figure 6: Pentagonal case in details

(ii) Let us search for a non-negative ϵ and an angle γ such that we can put the ellipse $\mathcal{E}_\gamma$ in the pentagon at the bottom of the square (see Figure 6). By the use of Lemma 2, we have the following system of slopes for this pentagon:

$$(\Sigma) \quad \left\{ \begin{array}{lcl} \dfrac{2\mu_\gamma(c-\epsilon+m_\beta(c-L_\beta)-\lambda_\gamma)+\sin(2\gamma)(1-E^2)}{\lambda_\gamma^2-(c-\epsilon+m_\beta(c-L_\beta)-\lambda_\gamma)^2} & = & \dfrac{-1}{m_\beta} \\ \dfrac{\mu_\gamma^2-(c-L_\alpha-\epsilon/m_\alpha-\mu_\gamma)^2}{\sin(2\alpha)(1-E^2)-2\lambda_\gamma(\mu_\gamma-(c-L_\alpha-\epsilon/m_\alpha))} & = & \dfrac{-1}{m_\alpha}. \end{array} \right.$$

Due to (1), the following proposition gives us a proof of the theorem.

Proposition 1. *If we suppose $c < \sqrt{12E}$, then the system (Σ) does not admit a solution.*

Proof: The system (Σ) is equivalent to the following one:

$$\left\{ \begin{array}{l} \epsilon = c + m_\beta(c-L_\beta) - (\lambda_\gamma + m_\beta\mu_\gamma) - \varepsilon_\beta \cdot R \\ \epsilon = -\lambda_\alpha + m_\alpha(c - L_\alpha - \mu_\gamma) - \varepsilon_\alpha \cdot R\,. \end{array} \right.$$

where $\varepsilon_\alpha = \pm 1$, $\varepsilon_\beta = \pm 1$, and $R = \sqrt{m_\beta^2\mu_\gamma^2 + \lambda_\gamma^2 + m_\beta \sin(2\gamma)(1-E^2)}$.

Moreover, $\varepsilon_\beta = 1$ because $c - \epsilon + m_\beta(c - L_\beta) \geq \lambda_\gamma + m_\beta\mu_\gamma$ (one can see that by drawing a straight line parallel to the one of slope $-1/m_\beta$ passing through the point $(\lambda_\gamma, \mu_\gamma)$), and

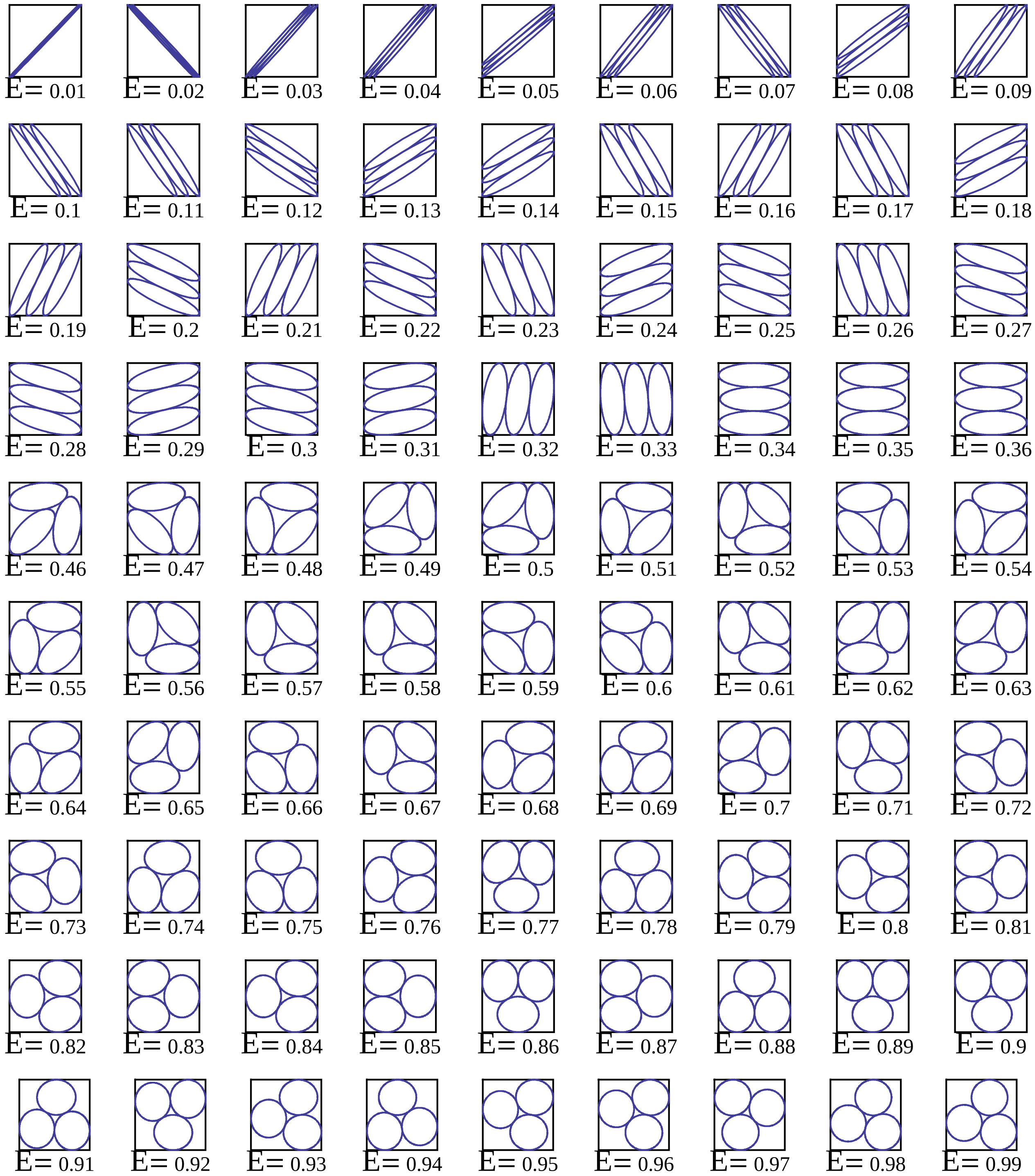

Figure 7: Best packings found by the stochastic algorithm

$\varepsilon_\alpha = -1$ because $c - L_\alpha - \epsilon/m_\alpha \geq \mu_\gamma + \lambda_\gamma/m_\alpha$ (draw the straight line parallel to the one of slope $-1/m_\alpha$ passing through the point $(\lambda_\gamma, \mu_\gamma)$).

For the sake of simplicity, we will note $M = m_\beta$, $\lambda = \lambda_\gamma$ and $\mu = \mu_\gamma$. By a symmetry argument, we can restrict our attention to $\gamma \in [0, \pi/2]$. It could be shown that the function

$$\varphi(\gamma) = c + M(c - L_\beta) - \sqrt{M^2\mu^2 + \lambda^2 + M\sin(2\gamma)(1 - E^2)} - (M\mu + \lambda)$$

is negative, which contradicts with $\varphi(\gamma) = \epsilon \geq 0$ due to the previous system. (It can easily be seen by using a computer, so we do not prove this strictly.) □

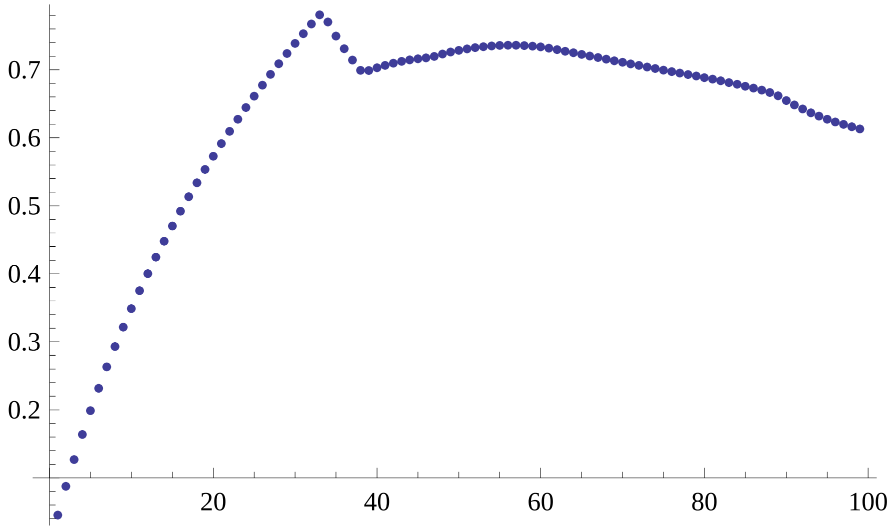

Figure 8: Density graph for three ellipses

For the convenience of the reader, we join drawings of the best packings found by Thierry GENSANE[1] for three unit ellipses with aspect ratio E varying from 0.01 to 1 with step 0.01, and the corresponding density graph (see Figures 7 and 8). I would like to thank him for his contribution and valuable discussions on the subject. He used a stochastic algorithm based on an *inflation formula* (see for instance [5]) which was already implemented in [3] to verify our theoretical results for two ellipses. The idea is to inflate the given ellipses until they contact, then to shrink them just a bit in order to move by an arbitrary little rotation-translation and so on, as if we had shaken them.

Even if there is some discontinuity on these drawings between 0.38 and 0.39, the density function is continuous (but not differentiable!).

Finally, this algorithm motivates the conjecture that the maximal density for the best packing of four ellipses in a square is again $\pi/4$, and it would be interesting to know for which values of n it is still true.

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