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Cevian Cousins of a Triangle Centroid

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Abstract. According to Seebach's theorem there exist six points inside a triangle with Cevian triangles similar to the reference triangle. Besides the centroid, other five points M, M', M_A, M_B, M_C are generally not constructable with ruler and compass. We present an access to these five points using an additional tool: a possibility to draw a conic through five given points. We provide information on barycentric coordinates of these five points and prove that $M_A M_B M_C$ is a central triangle of type 2 and that points M and M' are Brocardians of each other.

Key Words: Cevian triangle, Seebach's theorem, constructability with ruler and compass, conics, central triangle, Brocardian

MSC 2010: 51M15, 51N20, 51M04

1. Introduction

K. Seebach's theorem [6] states that if a triple of angles (A_1, B_1, C_1), $A_1 + B_1 + C_1 = 180°$ is chosen, we find exactly one point P inside a triangle ABC such that it's Cevian triangle $A_P B_P C_P$ has angles: $(A_P, B_P, C_P) = (A_1, B_1, C_1)$. We will call ordered triples (A_1, B_1, C_1) of positive numbers satisfying $A_1 + B_1 + C_1 = 180°$ *angle triples*, and the related point P *the point, corresponding to a certain angle triple.*

In this article we will consider points, corresponding to angle triples, consisting of permutations of angles A, B, C of the reference triangle ABC. The point, corresponding to the triple (A, B, C) is the centroid G. Points M and M', corresponding to the 3-cycles, i.e., the triples (C, A, B) and (B, C, A), were considered in [1] and were called *Cevian Brocard points* due to their Brocard-like property. The authors in [1] also raised the question regarding points M_A, M_B, M_C, corresponding to transpositions, i.e., to the triples (A, C, B), (C, B, A) and (B, A, C).

Since the centroid G and the above mentioned five points have the same angle set $\{A, B, C\}$ of their Cevian triangles and they have therefore a similar 'Cevian part of their genome', we will call M_A, M_B, M_C, M, and M' *Cevian cousins* of the centroid G.

In Section 2 we are going to prove that points M, M', M_A, M_B and M_C are generally not constructible with ruler and compass. (See [5] for a comprehensive review of many different

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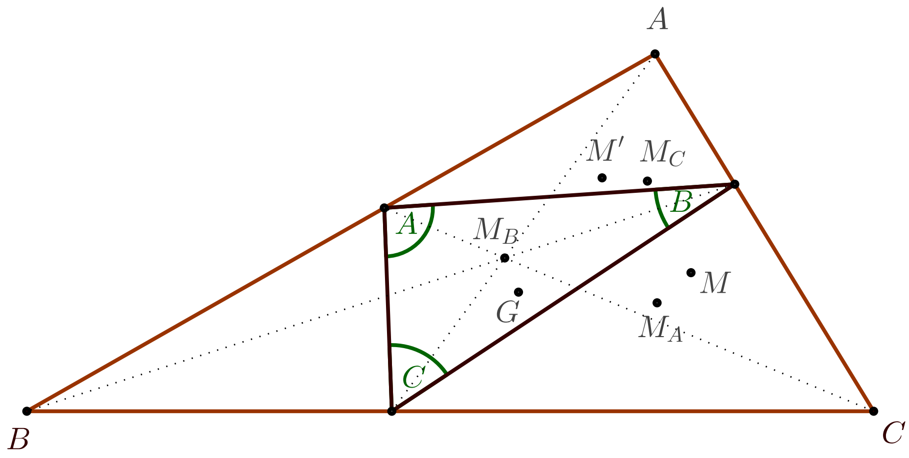

Figure 1: Points G, M, M', M_A, M_B, M_C having Cevian triangles similar to the reference triangle. Example: Cevian triangle of M_B.

aspects of constructability.) In order to investigate these points, we want to draw them using some additional tools. In Section 3 we will describe the construction of these points using a tool for drawing a conic through five given points $P_1, \ldots, P_5$. We will denote this conic by $Con_5(P_1, \ldots, P_5)$. Constructions of this kind were already considered, for instance, in [2, 7]. The effective realization of these constructions is possible in many computer programs for dynamic geometry, like Cabri-Geometry, GeoGebra etc., which integrate the conics as the base objects. Moreover, using Pascal's theorem, the conic through five given points can be drawn as soon as a program for dynamic geometry provides a possibility to draw the trace of a point. In Section 4 we will consider the barycentric coordinates of the centroid's cousins, and on this basis we derive some further relations among them.

2. Five points that are generally not constructable with ruler and compass

The basis of our considerations regarding the constructibility of the points M, M', M_A, M_B, and M_C with ruler and compass is the following result, which follows from the main theorem in [3]:

Theorem 1. *Let ABC be positively oriented triangle and (A_1, B_1, C_1) an angle triple. For $\{X, Y, Z\} = \{A, B, C\}$ define:*

$$m_X = (\cot Y_1 + \cot Z_1)\sin X, \quad n_{x,Y} = (\cot X_1 \sin Y - \cos Y) \tag{1}$$

and

$$p(\alpha) = m_A\alpha^3 + (n_{c,B} - m_A\, n_{a,C} + n_{b,C}\, n_{c,A})\alpha^2 + (n_{b,C}\, m_B - n_{a,C}\, n_{c,B} - n_{c,A})\alpha - m_B. \tag{2}$$

Since $(A + A_1) + (B + B_1) + (C + C_1) = 360°$, at most one of the sums in brackets can be equal to $180°$. Suppose $A + A_1 \neq 180°$ and $C + C_1 \neq 180°$. Then the point P corresponding to an angle triple (A_1, B_1, C_1) has trilinear coordinates $P = (\alpha : 1 : \gamma)$, where α is the only positive root of the polynomial $p(\alpha)$ in (2) such that also $\gamma = \dfrac{n_{c,A}\alpha + m_B}{\alpha - n_{a,C}}$ is positive.

In case of a point M_A we have $(A_1, B_1, C_1) = (A, C, B)$. Compute the coefficients:

$$\begin{aligned} m_A = \frac{a^2}{bc}, \quad m_B = \frac{c}{a}, \quad m_C = \frac{b}{a}, \quad n_{c,B} = n_{b,C} = 0, \\ n_{a,C} = \frac{c^2 - a^2}{ab}, \quad n_{c,A} = \frac{a^2 - b^2}{bc}, \quad n_{a,B} = \frac{b^2 - a^2}{ac}, \quad n_{b,A} = \frac{a^2 - c^2}{bc}. \end{aligned} \tag{3}$$

E.g., for the triangle with the sides $a = b = 2, c = 3$, the polynomial p has the form

$$p(\alpha) = 4\alpha^3 - 5\alpha^2 - 9.$$

This polynomial has exactly one real root α_0 which is not rational. Therefore the number α_0 is not constructible with ruler and compass. If point M_A would be constructible with ruler and compass, we could construct actual trilinear coordinates $M_A = (\alpha_1 : \beta_1 : \gamma_1)$, and since $\frac{\alpha_1}{\beta_1} = \frac{\alpha_0}{1}$, also the root α_0 would be constructible. The contradiction proves that in this triangle M_A is not constructible with ruler and compass.

In a similar way we can also prove that M_B, M_C, M and M' are generally not constructable with ruler and compass. In the case of the Cevian Brocard points M and M' we knew this before, since the authors have noticed this in [1].

3. The constructions of the points M_A, M_B, M_C, M, and M'

The only information we have about the points M_A, M_B, M_C, M, and M' are the angles of their Cevian triangles. In this situation the following result [3, section 2] is useful.

Theorem 2. *Let (A_1, B_1, C_1) be an angle triple and P the corresponding point inside the triangle ABC. Define the coefficients m_X and $n_{x,Y}$ as given in (1). Then the point P lies on three conics with the following trilinear equations:*

$$\begin{aligned} \beta\gamma &= \alpha\,(m_A\,\alpha + n_{c,B}\,\beta + n_{b,C}\,\gamma), \\ \alpha\gamma &= \beta\,(n_{c,A}\,\alpha + m_B\,\beta + n_{a,C}\,\gamma), \\ \alpha\beta &= \gamma\,(n_{b,A}\,\alpha + n_{a,B}\,\beta + m_C\,\gamma). \end{aligned} \tag{4}$$

Each of the listed conics passes through two vertices of the triangle. Moreover, each of them also passes through one of the points of the anticomplementary triangle $\tilde{A}\tilde{B}\tilde{C}$. This follows from the relations of the type $a + c\,n_{c,B} + b\,n_{b,C} = \frac{bc}{a}\,m_A$, which can be verified directly. However, in special cases as listed below this will be evident even without this last relation. We therefore already have three points on each of the conics. In order to provide a 'conic' construction of a point P we just need to detect two more constructible points on at least two of the conics in (4). Point P will then be the only intersection of these two conics inside the triangle.

In the case $P = M_A$ the related coefficients are already prepared in (3). We write the equations (4) of the associated conics and recognize, that their equations are much shorter in barycentric coordinates. Therefore, from now on, we switch to barycentrics. Point M_A thus lies on conics $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ with the respective equations

$$\begin{aligned} x^2 &= yz, \\ c^2y^2 &= (b^2 - a^2)xy + b^2xz + (a^2 - c^2)yz, \\ b^2z^2 &= c^2xy + (c^2 - a^2)xz + (a^2 - b^2)yz. \end{aligned} \tag{5}$$

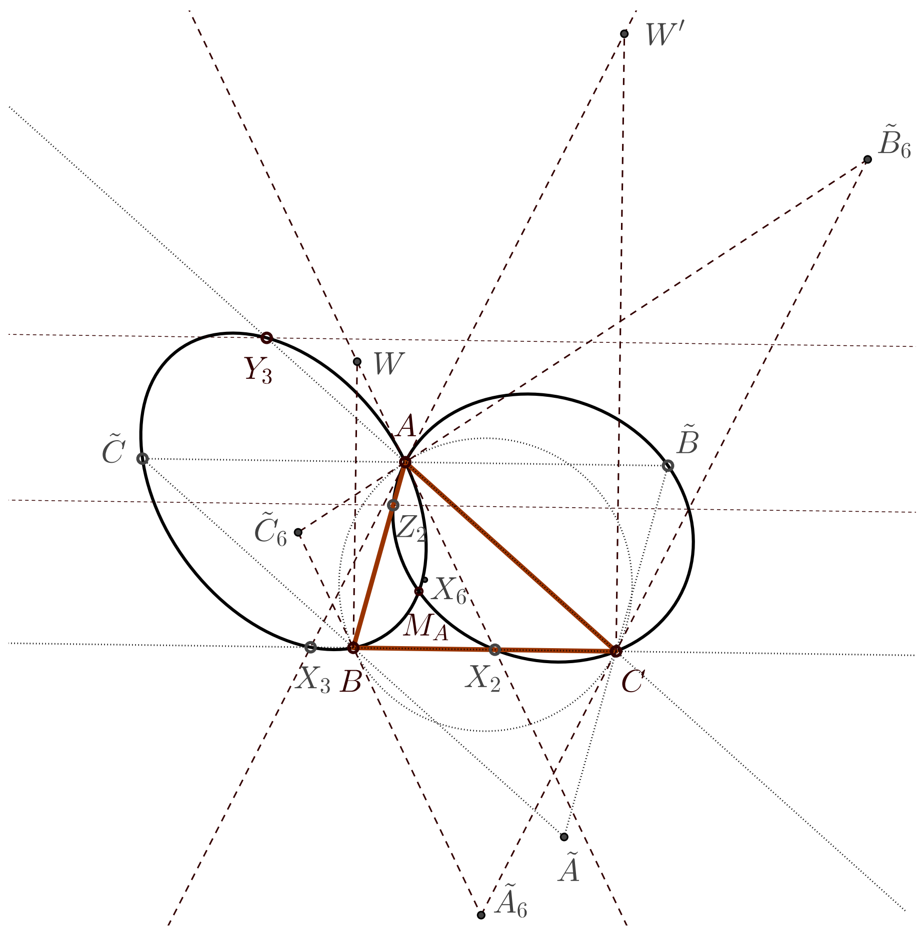

Figure 2: Conics $\mathcal{A}_2$ and $\mathcal{A}_3$

The conics $\mathcal{A}_2$ and $\mathcal{A}_3$ seem to appear somehow symmetrically. Due to this symmetry, and in spite of the simplicity of $\mathcal{A}_1$, perhaps the easiest way to construct M_A is as an intersection of $\mathcal{A}_2$ and $\mathcal{A}_3$. This is the background for the following construction (see Figure 2).

Theorem 3. *Let $\tilde{A}_6\tilde{B}_6\tilde{C}_6$ be the tangential triangle (i.e., the anticevian triangle of the symmedian point X_6).*
Let X_2 be the intersection of the sideline BC with the parallel to the line $B\tilde{A}_6$ through A and W the intersection of the line AX_2 with the perpendicular to the sideline BC at B. The bisector of the segment BW meets sideline AB at point Z_2.
Next we interchange the roles of B and C: the intersection of the sideline BC with the parallel to the line $C\tilde{A}_6$ through A is X_3 and W' is the intersection of the line AX_3 with the perpendicular to the sideline BC at C. The bisector of the segment CW' meets sideline AC at Y_3.
Then the only intersection of the conics $Con_5(A, C, \tilde{B}, X_2, Z_2)$ and $Con_5(A, B, \tilde{C}, X_3, Y_3)$ inside the triangle ABC is M_A.

This theorem follows from an easily verified fact that $\mathcal{A}_2 = Con_5(A, C, \tilde{B}, X_2, Z_2)$ and $\mathcal{A}_3 = Con_5(A, B, \tilde{C}, X_3, Y_3)$.

Similarly, a convenient way to construct M_B and M_C is to start with an adequate vertex of the anticevian triangle $\tilde{A}_6\tilde{B}_6\tilde{C}_6$ of the symmedian point X_6 and then construct the analogues of $\mathcal{A}_2$ and $\mathcal{A}_3$.

Let us now move our attention to the point M. Its corresponding angle triple is $(A_1, B_1, C_1) = (C, A, B)$. Computing the coefficients

$$m_A = \frac{c}{b}, \quad m_B = \frac{a}{c}, \quad m_C = \frac{b}{a}, \quad n_{a,C} = n_{c,B} = n_{b,A} = 0,$$
$$n_{b,C} = \frac{c^2 - a^2}{ab}, \quad n_{c,A} = \frac{a^2 - b^2}{bc}, \quad n_{a,B} = \frac{b^2 - c^2}{ac},$$

applying Theorem 2 and switching to barycentrics, we find that M lies on the following conics $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$:

$$\begin{aligned} c^2x^2 &= (a^2 - c^2)xz + a^2yz, \\ a^2y^2 &= (b^2 - a^2)xy + b^2xz, \\ b^2z^2 &= (c^2 - b^2)yz + c^2xy. \end{aligned}$$

Here we notice the following: the only intersection of $\mathcal{M}_1$ with the sideline AB is B, thus AB is a tangent of $\mathcal{M}_1$ at B.

Having five points of a conic, we can construct a tangent of a conic at one of these points [8, section 12.5.1]. Reversing this construction, having four points of a conic and a tangent to it at one of the points, we can construct a fifth point on the conic as follows:

Lemma 4. *Let T, U, V, W be four points on a conic $\mathcal{C}$ an let t be a tangent to $\mathcal{C}$ at T. Choose point R on line t and find the intersections: P of lines TU and VW, Q of lines TW and PR and E of lines VR and UQ. Then point E lies on a conic $\mathcal{C}$.*

Having four points on $\mathcal{M}_1$ and knowing that the sideline AB is tangent to $\mathcal{M}_1$ at B, we apply the Lemma, find a fifth point on the conic, and draw it. To accelerate our work in constructing conics, using the programs for dynamic geometry, we can prepare a macro, based on Lemma 4, to construct a conic through given four points $P_1, \ldots, P_4$ and a given point T on the tangent to the conic at P_1. We will denote such a conic by $Con_{41}(P_1, P_2, P_3, P_4, T)$.

Now we can describe the construction of point M, which is somehow similar to that of M_A, with the difference, that the role of X_6 is now played by the first Brocard point $\Omega = (b^{-2} : c^{-2} : a^{-2})$.

Theorem 5. *Let Ω be the first Brocard point of ABC and $\tilde{A}_\Omega\tilde{B}_\Omega\tilde{C}_\Omega$ the anticevian triangle of Ω. Let Y_M be the intersection of the sideline CA with the parallel to the line $A\tilde{B}_\Omega$ through B. Similarly, let Z_M be the intersection of AB with the parallel to $B\tilde{C}_\Omega$ through C, and X_M the intersection of BC with the parallel of $C\tilde{A}_\Omega$ through A.*

Then M is the intersection of the conics $Con_{41}(B, C, \tilde{A}, Y_M, A)$, $Con_{41}(C, A, \tilde{B}, Z_M, B)$, and $Con_{41}(A, B, \tilde{C}, X_M, C)$.

Again, this theorem follows from the easily verified fact that the quoted conics are $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$. Figure 3 shows the construction of M as the intersection of these three conics, each of them tangent to one of the triangle sides at one of the vertices of a triangle.

In case of the point M', the situation is similar, with two differences: in the constructions of the conics $\mathcal{M}'_1$, $\mathcal{M}'_2$, and $\mathcal{M}'_3$, the role of the first Brocard point is played by the second Brocard point $\Omega' = (c^{-2} : a^{-2} : b^{-2})$; and the direction of circulation is reversed (Y'_M is the intersection of the sideline AC with the parallel to the line $C\tilde{B}'_\Omega$ through B, etc.). Again, each of the appearing three conics is tangent to one of the triangle sides at one of the vertices of a triangle.

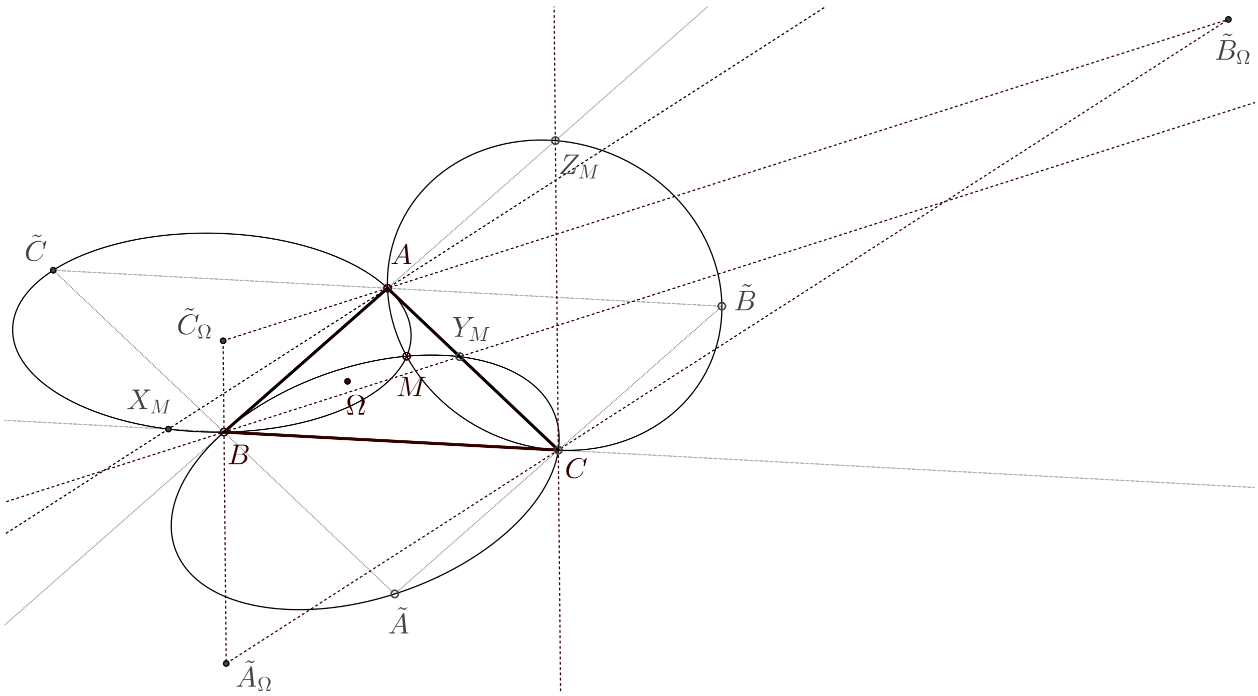

Figure 3: Conics $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$

Both configurations, the one in Figure 3 and the similar one, leading to the point M', remind us of similar configurations of three circles, each tangent to one of the triangle sides, which appear in the constructions of Brocard points Ω and Ω'.

As we will see in the next section, once we have one of the points M or M', there is no need to apply 'conic' constructions to construct the other one. We can draw it with a simple compass and ruler construction (Figure 4).

4. On barycentric coordinates of the considered five points

As a side result of our considerations up to now we can derive some results on the barycentric coordinates of the considered five points, which will bring to light some additional connections between them.

It follows from Seebach's theorem and from considerations in [3] that the triple of the barycentrics of M_A is the only solution of the system (5) with all three coordinates being positive. This fact provides an opportunity to find the barycentric coordinates of M_A and similarly of M_B and M_C. In case of M_A, choose $y = 1$ and apply $z = x^2$ to the second and third equation of (5). This leads us to the following result:

Theorem 6. *Let $\tau(a, b, c)$ be the only positive root of the equation*

$$b^2x^3 + (a^2 - c^2)x^2 + (b^2 - a^2)x - c^2 = 0.$$

Then the respective barycentric coordinates of M_A, M_B, and M_C are

$$M_A = (\tau(a, b, c) : 1 : \tau^2(a, b, c)), \quad M_B = (\tau^2(b, c, a) : \tau(b, c, a) : 1),$$
$$M_C = (1 : \tau^2(c, a, b) : \tau(c, a, b)).$$

Note that due to the symmetry of the cubic equation above we have $\tau(a,c,b) = \tau^{-1}(a,b,c)$ and therefore $M_A = (1 : \tau(a,c,b) : \tau(a,b,c))$. Switching for a moment to trilinears and defining $f(a,b,c) = \frac{1}{a}$ and $g(a,b,c) = \frac{1}{a}\tau(c,b,a)$, we notice that $M_A M_B M_C$ is a (f,g)-central triangle of type 2 as defined in [4].

In a similar way we can derive some information on barycentric coordinates of the points M and M':

Theorem 7. *Let $\eta(a,b,c)$ (or shortly η_a) be the only positive root of the equation*

$$b^2c^2x^3 - (a^2-b^2)(a^2-c^2)x^2 + a^2(-2a^2+b^2+c^2)x - a^4 = 0$$

such that also the numbers $\left[\left(\frac{a^2}{b^2}-1\right)\eta_a + \frac{a^2}{b^2}\right]$ and $\left[\left(\frac{a^2}{c^2}-1\right)\eta_a + \frac{a^2}{c^2}\right]$ are positive. Then the barycentric coordinates of M and M' are

$$M = \left(\eta_a^2 : \eta_a : \left[\left(\frac{a^2}{b^2}-1\right)\eta_a + \frac{a^2}{b^2}\right]\right) \text{ and } M' = \left(\eta_a^2 : \left[\left(\frac{a^2}{c^2}-1\right)\eta_a + \frac{a^2}{c^2}\right] : \eta_a\right).$$

This result can be improved using the following observation. Triangles ABC and BCA have the same Cevian Brocard point M. Analogue to $\eta_a = \eta(a,b,c)$ we denote $\eta_b = \eta(b,c,a)$. Applying Theorem 7, we compare the barycentrics of M with respect to the triangles ABC and BCA, namely

$$\left(\eta_a^2 : \eta_a : \left[\left(\frac{a^2}{b^2}-1\right)\eta_a + \frac{a^2}{b^2}\right]\right) \text{ and } \left(\eta_b^2 : \eta_b : \left[\left(\frac{b^2}{c^2}-1\right)\eta_b + \frac{b^2}{c^2}\right]\right).$$

Since M is the same point, the ratio of the coordinates regarding sides of triangle of lenghts b and c are the same:

$$\eta_a : \left[\left(\frac{a^2}{b^2}-1\right)\eta_a + \frac{a^2}{b^2}\right] = \eta_b^2 : \eta_b,$$

which implies

$$\left[\left(\frac{a^2}{b^2}-1\right)\eta_a + \frac{a^2}{b^2}\right] = \frac{\eta_a}{\eta_b}.$$

Moreover, using the same argument for the triangle CAB, we derive $\eta_a\eta_b\eta_c = 1$. After repeating the whole procedure also for the point M', we can state the following improved result:

Theorem 8. *Under the assumptions of Theorem 7 and denoting $\eta_a = \eta(a,b,c)$ and $\eta_b = \eta(b,c,a)$, the barycentric coordinates of the points M and M' are*

$$M = (\eta_a\eta_b : \eta_b : 1) \text{ and } M' = (\eta_a : \eta_a\eta_b : 1).$$

Remember that, given a point $P = (u : v : w)$, we define the *Brocardians* of the point P as

$$P_{\leftarrow} = \left(v^{-1} : w^{-1} : u^{-1}\right) \text{ and } P_{\rightarrow} = \left(w^{-1} : u^{-1} : v^{-1}\right)$$

(see [8]). According to Theorem 8, the points M' and M are Brocardians of each other: $M' = M_{\leftarrow}$ and $M = M'_{\rightarrow}$. Given one of the points M or M', the other one can therefore be simply constructed applying the construction [8, section 8.4].

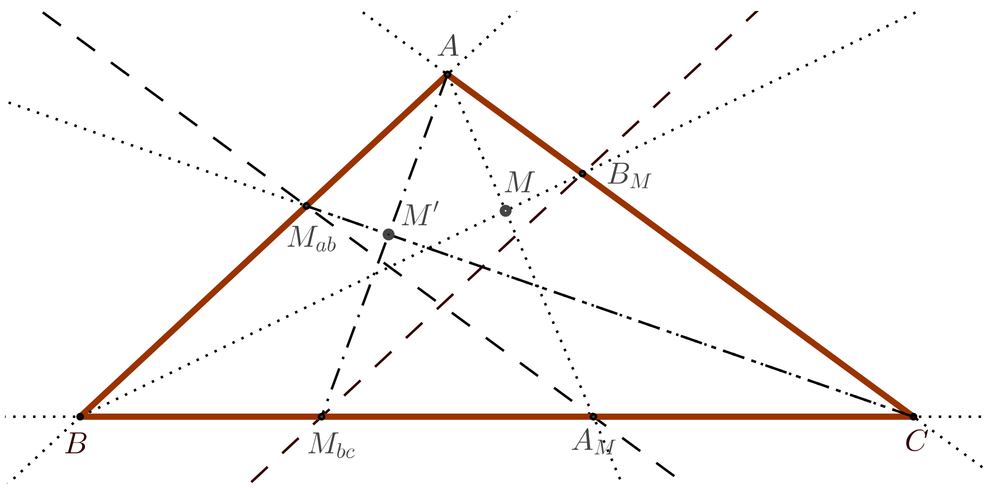

Figure 4: The construction of the point M' if point M is given

5. Some concluding remarks

The situation considered in this paper could be generalized. As well as the centroid, also every other triangle center P has five Cevian cousins P_A, P_B, P_C, M_P and M'_P, i.e., points with the same Cevian triangle angle set. The conjecture is that $P_AP_BP_C$ is always a central triangle of type 2 and that M_P and M'_P are Brocardians of each other.

When looking at the Figure 1, the first thought is that the six points might lie on an ellipse. Computer experiments do not confirm it: a considerable enlargement shows that the conic through five of the points fails to include the sixth one. The same conclusion can be reached by computing the 6×6 determinant with rows of the type $(x_i^2, y_i^2, z_i^2, x_iy_i, x_iz_i, y_iz_i)$ for $i = 1, \ldots, 6$, where $(x_i : y_i : z_i)$ are the barycentrics of the considered six points. The determinant is generally not vanishing.

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