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The Parabola in Universal Hyperbolic Geometry II: Canonical Points and the $\mathcal{Y}$-conic

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Abstract. We introduce canonical points and lines for the hyperbolic parabola in Universal Hyperbolic Geometry, and explicit formulas for them in standard coordinates. Quite a few remarkable collinearities result, with the duality of the twin parabola playing a major role. We also introduce the curious $\mathcal{Y}$-conic which is homologous to the parabola, and contains many interesting meets.

Key Words: hyperbolic geometry, projective geometry, universal geometry, hyperbolic parabola

MSC 2010: 51M10, 51N15, 14N99

1. Introduction and review of the hyperbolic parabola

We begin by reviewing the basic set-up for *Universal hyperbolic geometry* (UHG), (see [6], [7], [8], [9]), the definition of a (hyperbolic) parabola in this context which was introduced in [1], and the use of *standard coordinates* which allow a significant simplification of many formulas for such a parabola. In this algebraic version of hyperbolic geometry, we use a Cayley-Klein projective framework with metrical structure determined by an invertible symmetric projective matrix C and its adjugate D. Since the theory is independent of the particular form of C, we may employ projective (linear) transformations to simplify situations. Everything holds over a general field $\mathbb{F}$ not of characteristic two—which may for simplicity be taken to be the rational numbers. Prior classical discussions of such curves in hyperbolic geometry include [2], [3], [4], and [5].

In Figure 1 we see the null circle or absolute $\mathcal{C}$, the parabola $\mathcal{P}_0$, and some canonical points associated to it generating the homologous $\mathcal{Y}$-conic.

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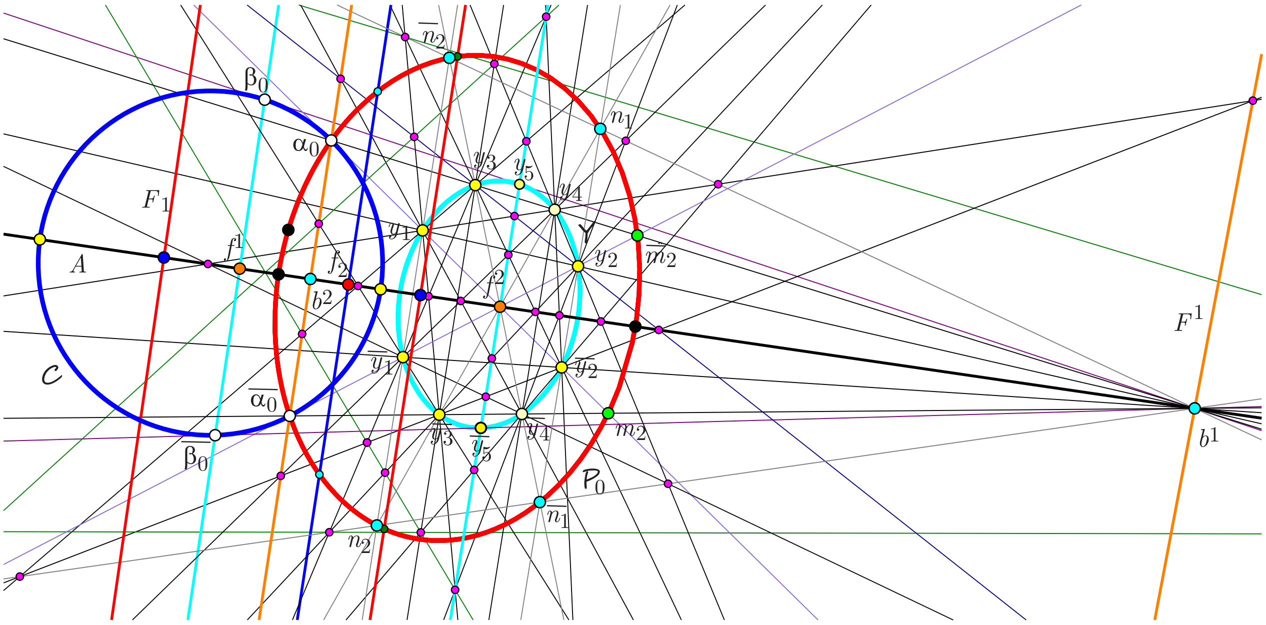

Figure 1: The hyperbolic parabola $\mathcal{P}_0$ and its $\mathcal{Y}$-conic

Here is briefly how we set up Universal hyperbolic geometry using (projective) linear algebra. A *(projective) point* is a proportion $a = [x : y : z]$ in square brackets, or equivalently a projective row vector $a = \begin{bmatrix} x & y & z \end{bmatrix}$. A *(projective) line* is a proportion $L = \langle l : m : n \rangle$ in pointed brackets, or equivalently a projective column vector

$$L = \begin{bmatrix} l \\ m \\ n \end{bmatrix}.$$

Incidence between the point $a = [x : y : z]$ and the line $L = \langle l : m : n \rangle$ is defined by

$$aL = lx + my + nz = 0.$$

The *join* a_1a_2 of distinct points $a_1 \equiv [x_1 : y_1 : z_1]$ and $a_2 \equiv [x_2 : y_2 : z_2]$ is the unique line passing through (i.e. incident with) a_1 and a_2, namely

$$a_1a_2 \equiv [x_1 : y_1 : z_1] \times [x_2 : y_2 : z_2] \equiv \langle y_1z_2 - y_2z_1 : z_1x_2 - z_2x_1 : x_1y_2 - x_2y_1 \rangle . \qquad (1)$$

The *meet* L_1L_2 of distinct lines $L_1 \equiv \langle l_1 : m_1 : n_1 \rangle$ and $L_2 \equiv \langle l_2 : m_2 : n_2 \rangle$ is the unique point lying on (i.e. incident with) L_1 and L_2, namely

$$L_1L_2 \equiv \langle l_1 : m_1 : n_1 \rangle \times \langle l_2 : m_2 : n_2 \rangle \equiv [m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : l_1m_2 - l_2m_1] . \qquad (2)$$

Three points a_1, a_2, a_3 are *collinear* precisely when they lie on a line L; in this case we also write $[[a_1a_2a_3]]$. Similarly three lines L_1, L_2, L_3 are *concurrent* precisely when they pass through a point a; in this case we will also write $[[L_1L_2L_3]]$. These conditions may be directly reduced to checking that the determinant of the matrix formed by the three points or lines is zero.

1.1. Projective quadrance and spread

If C is a symmetric invertible 3×3 matrix, with entries in $\mathbb{F}$, and D is its adjugate matrix (the inverse, up to a multiple), denote by $\mathbf{C}$ and $\mathbf{D}$ the corresponding projective matrices, each defined up to a non-zero multiple. From these we get a metrical structure: the (projective) points a_1 and a_2 are *perpendicular* precisely when $a_1\mathbf{C}a_2^T=0$, written $a_1\perp a_2$, and the (projective) lines L_1 and L_2 are *perpendicular* precisely when $L_1^T\mathbf{D}L_2=0$, written $L_1\perp L_2$. The point a and the line L are *dual* precisely when

$$L=a^{\perp}\equiv\mathbf{C}a^T\quad\text{or equivalently}\quad a=L^{\perp}\equiv L^T\mathbf{D}. \tag{3}$$

Then two points are perpendicular precisely when *one is incident with the dual of the other,* and similarly for two lines. So $a_1\perp a_2$ precisely when $a_1^{\perp}\perp a_2^{\perp}$.

A point a is *null* precisely when it is perpendicular to itself, that is, when $a\mathbf{C}a^T=0$, and a line L is *null* precisely when it is perpendicular to itself, that is, when $L^T\mathbf{D}L=0$. The null points determine the *null conic*, sometimes also called the *absolute*. *Hyperbolic* and *elliptic geometries* arise respectively from the special cases

$$C=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}=D\quad\text{and}\quad C=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=D. \tag{4}$$

In the hyperbolic case, $a=[x:y:z]$ is null precisely when $x^2+y^2-z^2=0$ and dually the line $L=(l:m:n)$ is null precisely when $l^2+m^2-n^2=0$. So the null circle $\mathcal{C}$ in affine coordinates $X\equiv x/z$ and $Y\equiv y/z$ is the circle $X^2+Y^2=1$, which is shown in blue in our diagrams.

In the general setting, the dual notions of *(projective) quadrance* $q(a_1,a_2)$ between points a_1 and a_2, and *(projective) spread* $S(L_1,L_2)$ between lines L_1 and L_2 are

$$q(a_1,a_2)\equiv1-\frac{\left(a_1\mathbf{C}a_2^T\right)^2}{\left(a_1\mathbf{C}a_1^T\right)\left(a_2\mathbf{C}a_2^T\right)}\quad\text{and}\quad S(L_1,L_2)\equiv1-\frac{\left(L_1^T\mathbf{D}L_2\right)^2}{\left(L_1^T\mathbf{D}L_1\right)\left(L_2^T\mathbf{D}L_2\right)}. \tag{5}$$

Clearly $q(a,a)=0$ and $S(L,L)=0$ for any point a and any line L, while $q(a_1,a_2)=1$ precisely when $a_1\perp a_2$, and dually $S(L_1,L_2)=1$ precisely when $L_1\perp L_2$. Then $S\left(a_1^{\perp},a_2^{\perp}\right)=q(a_1,a_2)$.

In [6], Wildberger showed that for hyperbolic geometry these metrical notions agree with a purely projective formulation using suitable cross ratios, and relate to the classical hyperbolic *distance* $d(a_1,a_2)$ and *angle* $\theta(L_1,L_2)$ between points and lines, inside the null circle $\mathcal{C}$, via $q(a_1,a_2)=-\sinh^2(d(a_1,a_2))$ and $S(L_1,L_2)=\sin^2(\theta(L_1,L_2))$. Note however that (5) are defined for all non-null points and lines in the projective plane.

Recall also that a *midpoint* of the non-null side $\overline{ab}$ is a point m lying on the line ab which satisfies $q(a,m)=q(m,s)$. There are generally zero or two midpoints of a given side. More novel is the following closely related concept, which was introduced in our paper [9]: a *sydpoint* of the non-null side $\overline{ab}$ is a point s lying on the line ab which satisfies $q(a,s)=-q(b,s)$. There are also generally zero or two sydpoints of a given side, and these are intimately related to the theory of the hyperbolic parabola.

1.2. The parabola and standard coordinates

We now introduce some basic facts from [1]. The *hyperbolic parabola* $\mathcal{P}_0$ is defined in terms of two non-null, non-perpendicular points f_1 and f_2 (called the *foci*), as the locus of a point

p_0 satisfying

$$q(p_0, f_1) + q(p_0, f_2) = 1.$$

After introducing the *directrices* $F_1 \equiv f_1^\perp$ and $F_2 \equiv f_2^\perp$ respectively, this defining equation is equivalent to either $q(p_0, f_1) = q(p_0, F_2)$ or $q(p_0, f_2) = q(p_0, F_1)$, showing that the above definition gives a hyperbolic version of the Euclidean parabola. Note also that there are *two* focus/directrix pairs.

The parabola $\mathcal{P}_0$ is indeed a conic. Define its *axis line* $A \equiv f_1 f_2$, the *vertices* v_1 and v_2 where $\mathcal{P}_0$ meets the axis, the dual *vertex lines* $V_1 \equiv v_1^\perp$ and $V_2 \equiv v_2^\perp$ which are tangents to the parabola at the vertices, and the *base points* $b_1 \equiv F_1 A$ and $b_2 \equiv F_2 A$, with dual *base lines* $B_1 \equiv b_1^\perp$ and $B_2 \equiv b_2^\perp$.

For a point c its reflection in the axis A, called the *opposite* of c, is denoted $\overline{c}$. This is a fundamental symmetry for the parabola.

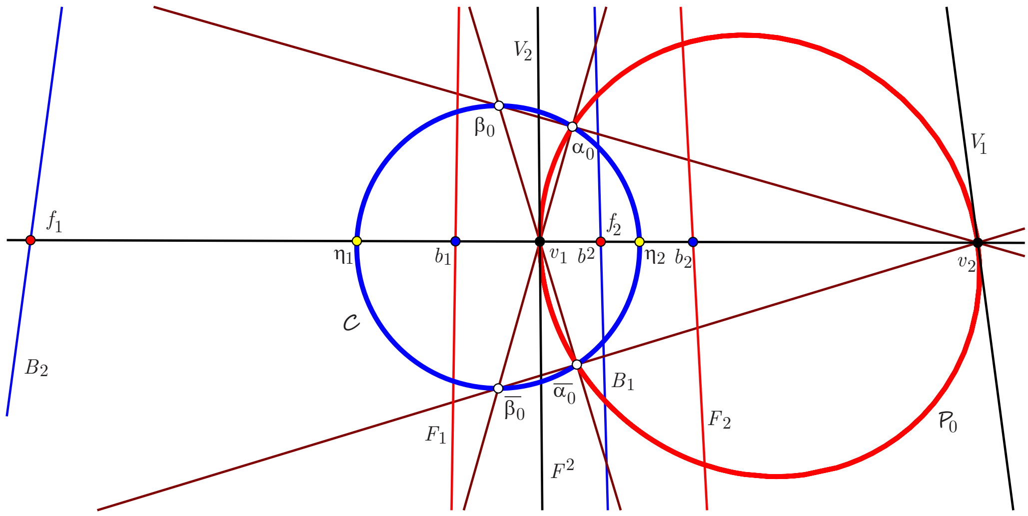

Figure 2: A parabola $\mathcal{P}_0$ and some basic points and lines

The main idea to study the parabola is to allow flexibility in our field and to *carefully choose an optimum coordinate framework*; for this we utilize four important points associated to the parabola: a pair of opposite null points $\alpha_0, \overline{\alpha_0}$ lying on $\mathcal{P}_0$, and the vertices v_1, v_2. The existence of $\alpha_0, \overline{\alpha_0}$ may well require a quadratic field extension, which we assume we have made.

We may now invoke the Fundamental theorem of projective geometry to projectively transform these four points to

$$\alpha_0 = [1:1:1], \quad \overline{\alpha_0} = [1:-1:1], \quad v_1 = [0:0:1], \quad v_2 = [1:0:0].$$

This choice is called *standard coordinates* for the parabola. It is then a pleasant fact that $\beta_0 \equiv (v_2\alpha_0)(v_1\overline{\alpha_0})$ and $\overline{\beta_0} \equiv (v_2\overline{\alpha_0})(v_1\alpha_0)$ are the null points

$$\beta_0 = [-1:1:1] \quad \text{and} \quad \overline{\beta_0} = [-1:-1:1].$$

The opposite of $c = [x:y:z]$ is then $\overline{c} = [x:-y:z]$.

The main *Parabola standard coordinates theorem* then shows that the original hyperbolic bilinear form of $C = D = J$ from (4) is transformed to one with new matrices

$$C = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & 1-\alpha^2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad D = \operatorname{adj}(C) = \begin{pmatrix} \alpha^2 - 1 & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & \alpha^2(1-\alpha^2) \end{pmatrix} \qquad (6)$$

for some parameter α. While the metrical structure has now changed, the quadrance and spread depend only on the corresponding projective matrices $\mathbf{C}$ and $\mathbf{D}$, so all the definitions of the previous section apply. Crucially, in standard coordinates the parabola $\mathcal{P}_0$ now has equation

$$y^2 = xz \tag{7}$$

and so can easily be parametrized by $p_0 = p(t) \equiv [t^2 : t : 1]$. The equation of the axis in standard coordinates is $A = \langle 0 : 0 : 1 \rangle$, while the null circle $\mathcal{C}$ is

$$\alpha^2 x^2 + \left(1 - \alpha^2\right) y^2 - z^2 = 0.$$

Almost all subsequent formulas for points, lines and related curves will involve the parameter α.

1.3. Dual conics and the connection with sydpoints

Here are the coordinates of the points and lines already defined in standard coordinates:

$$\begin{aligned}
f_1 &= [\alpha + 1 : 0 : \alpha(\alpha - 1)], & f_2 &= [1 - \alpha : 0 : \alpha(\alpha + 1)], \\
F_1 &\equiv f_1^{\perp} = \langle \alpha(\alpha + 1) : 0 : 1 - \alpha \rangle, & F_2 &= f_2^{\perp} = \langle \alpha(\alpha - 1) : 0 : 1 + \alpha \rangle, \\
b_1 &\equiv F_1 A = [\alpha - 1 : 0 : \alpha(\alpha + 1)], & b_2 &\equiv F_2 A = [\alpha + 1 : 0 : \alpha(1 - \alpha)], \\
B_1 &\equiv b_1^{\perp} = \langle -\alpha(\alpha - 1) : 0 : \alpha + 1 \rangle, & B_2 &\equiv b_2^{\perp} = \langle \alpha(\alpha + 1) : 0 : \alpha - 1 \rangle.
\end{aligned}$$

Define the *axis null points* to be the meets of the axis A and the null conic C:

$$\eta_1 \equiv A\mathcal{C} = [-1 : 0 : \alpha] \quad \text{and} \quad \eta_2 = A\mathcal{C} = [1 : 0 : \alpha];$$

note that this is a switch from the convention in [1]. We also have dual lines

$$\alpha_0^{\perp} = \mathbf{C}\,[1 : 1 : 1]^T = \left\langle \alpha^2 : 1 - \alpha^2 : -1 \right\rangle \quad \text{and} \quad \overline{\alpha_0}^{\perp} = \mathbf{C}\,[1 : -1 : 1]^T = \left\langle \alpha^2 : \alpha^2 - 1 : -1 \right\rangle.$$

The *tangent line* to $\mathcal{P}_0$ at a point $p_0 = p(t) \equiv [t^2 : t : 1]$ on it is $P^0 = \langle 1 : -2t : t^2 \rangle$, and the dual point of this tangent line is the *twin point* p^0 of p_0. The locus of p^0 as p_0 varies along $\mathcal{P}_0$ turns out, remarkably, to be another parabola $\mathcal{P}^0$ with foci which are the *sydpoints* f^1, f^2 of the side $\overline{f_1 f_2}$, as in Figure 3.

To understand this, we first introduce the lines and points

$$\begin{aligned}
F^2 &\equiv \alpha_0 \overline{\alpha_0} = \langle 1 : 0 : -1 \rangle, & B^1 &\equiv \beta_0 \overline{\beta_0} = \langle 1 : 0 : 1 \rangle, \\
b^2 &\equiv F^2 A = [1 : 0 : 1], & f^1 &= B^1 A = [-1 : 0 : 1].
\end{aligned}$$

The duals are

$$\begin{aligned}
f^2 &\equiv \left(F^2\right)^{\perp} = [1 : 0 : \alpha^2], & b^1 &\equiv \left(B^1\right)^{\perp} = [1 : 0 : -\alpha^2], \\
B^2 &\equiv \left(b^2\right)^{\perp} = \langle -\alpha^2 : 0 : 1 \rangle, & F^1 &\equiv \left(f^1\right)^{\perp} = \langle \alpha^2 : 0 : 1 \rangle.
\end{aligned}$$

The points f^1 and f^2 are the *t-foci* of the parabola $\mathcal{P}_0$, while the respective dual lines F^1 and F^2 are the *t-directrices of* $\mathcal{P}_0$. The meets of the t-directrices and the axis A are the *t-base points* $b^1 \equiv F^1 A$ and $b^2 \equiv F^2 A$, with respective dual lines B^1 and B^2.

We also introduce the points d_0 and $\overline{d_0}$ to be the meets of the directrix F_2 with the parabola $\mathcal{P}_0$, should they exist, and the corresponding twin null points $d^0 \equiv \delta_0$ and $\overline{\delta_0}$ lying on the directrix F_1.

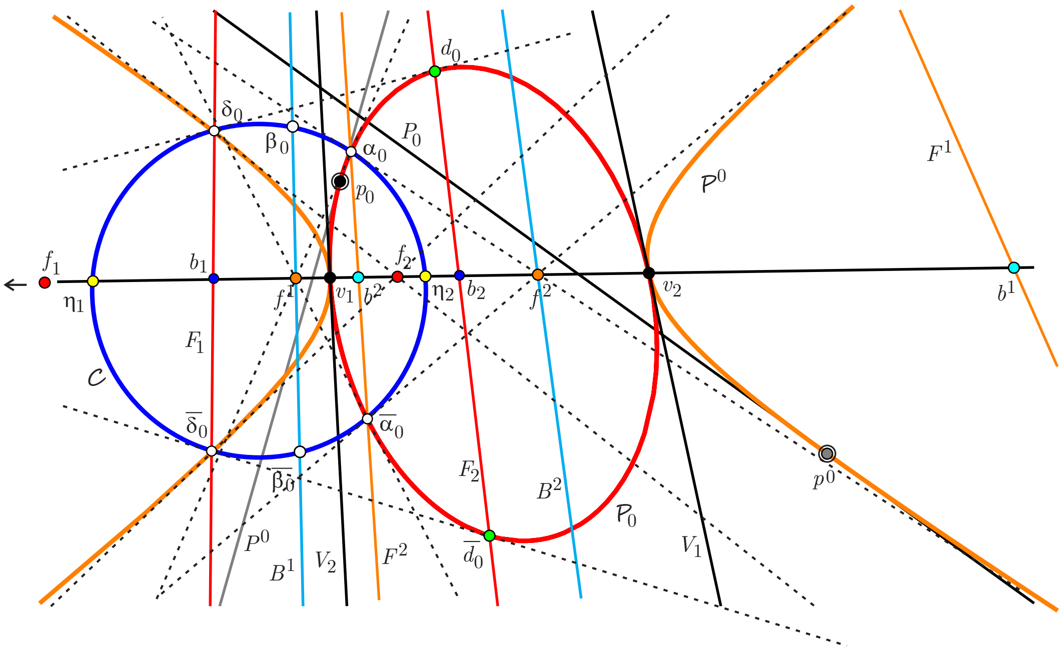

Figure 3: A point p_0 and the twin point p^0 of $\mathcal{P}_0$

The *Parabola sydpoints theorem* then asserts that the points f^1 and f^2 are in fact the *sydpoints of the original side* $\overline{f_1 f_2}$. The parabola $\mathcal{P}^0$ with foci f^1 and f^2, called the *twin parabola* of $\mathcal{P}_0$, is the dual conic of $\mathcal{P}_0$ with respect to the null circle $\mathcal{C}$; namely the locus of p^0 as p_0 varies. The equation of $\mathcal{P}^0$ in standard coordinates is

$$y^2 = \frac{-4\alpha^2}{(\alpha^2-1)^2} xz.$$

Note in Figure 3 that the tangents to both the parabola $\mathcal{P}_0$ and the null circle $\mathcal{C}$ at their common meets, namely the null points α_0 and $\overline{\alpha_0}$, pass through the foci of the twin parabola $\mathcal{P}^0$. Dually the tangents to both the parabola $\mathcal{P}^0$ and the null circle $\mathcal{C}$ at their common meets, namely the null points δ_0 and $\overline{\delta_0}$ on F_1, pass through the focus f_1 of $\mathcal{P}_0$.

2. Canonical structures on the hyperbolic parabola

In the paper [1] we mostly concentrated on properties of the hyperbolic parabola that were analogous to the classical theory for a Euclidean parabola. We now derive some interesting results that have no classical parallel: while a Euclidean parabola has relatively few canonical points and lines associated to it, the situation is dramatically different here, due to the existence of the null points α_0 and $\overline{\alpha_0}$. Here we sketch the beginnings of this theory, up to the discussion of the $\mathcal{Y}$*-conic* of a hyperbolic parabola. As usual, obtaining explicit formulae in standard coordinates is a main aim.

The proofs of most of the results are straightforward–we compute joins and meets, or duals, and occasionally verify that a point lies on the parabola $\mathcal{P}_0$ using (7). Many of these facts have *generalizations that are also valid when* α_0 *is replaced by a general point* p_0 *on the parabola*; we will discuss this in a future paper.

Define the *e-points:*

$$e_0 \equiv (\eta_1\alpha_0)(\eta_2\overline{\alpha_0}) = \left[1:\alpha:\alpha^2\right] \quad \text{and} \quad \overline{e_0} \equiv (\eta_1\overline{\alpha_0})(\eta_2\alpha_0) = \left[1:-\alpha:\alpha^2\right]$$

the *m-points:*

$$m \equiv \left(f^2\alpha_0\right)\left(b^1\overline{\alpha_0}\right) = \left[1 : -\alpha^2 : \alpha^4\right] \quad \text{and} \quad \overline{m} \equiv \left(f^2\overline{\alpha_0}\right)\left(b^1\alpha_0\right) = \left[1 : \alpha^2 : \alpha^4\right]$$

and the *n-points:*

$$\begin{aligned}
n_1 &\equiv (f_1\alpha_0)\,(b_2\overline{\alpha_0}) = \left[(\alpha+1)^2 : -\alpha\left(\alpha^2-1\right) : \alpha^2(\alpha-1)^2\right], \\
n_2 &\equiv (f_2\alpha_0)\,(b_1\overline{\alpha_0}) = \left[(\alpha-1)^2 : \alpha\left(\alpha^2-1\right) : \alpha^2(\alpha+1)^2\right], \\
\overline{n_1} &\equiv (f_1\overline{\alpha_0})\,(b_2\alpha_0) = \left[(\alpha+1)^2 : \alpha\left(\alpha^2-1\right) : \alpha^2(\alpha-1)^2\right], \\
\overline{n_2} &\equiv (f_2\overline{\alpha_0})\,(b_1\alpha_0) = \left[(\alpha-1)^2 : -\alpha\left(\alpha^2-1\right) : \alpha^2(\alpha+1)^2\right].
\end{aligned}$$

Theorem 1 (Canonical parabola points)**.** *The points $e_0, \overline{e_0}, m, \overline{m}, n_1, n_2, \overline{n_1}$ and $\overline{n_2}$ all lie on the parabola $\mathcal{P}_0$.*

Proof. This can be checked easily from the above forms of these points and the equation (7) for $\mathcal{P}_0$. $\square$

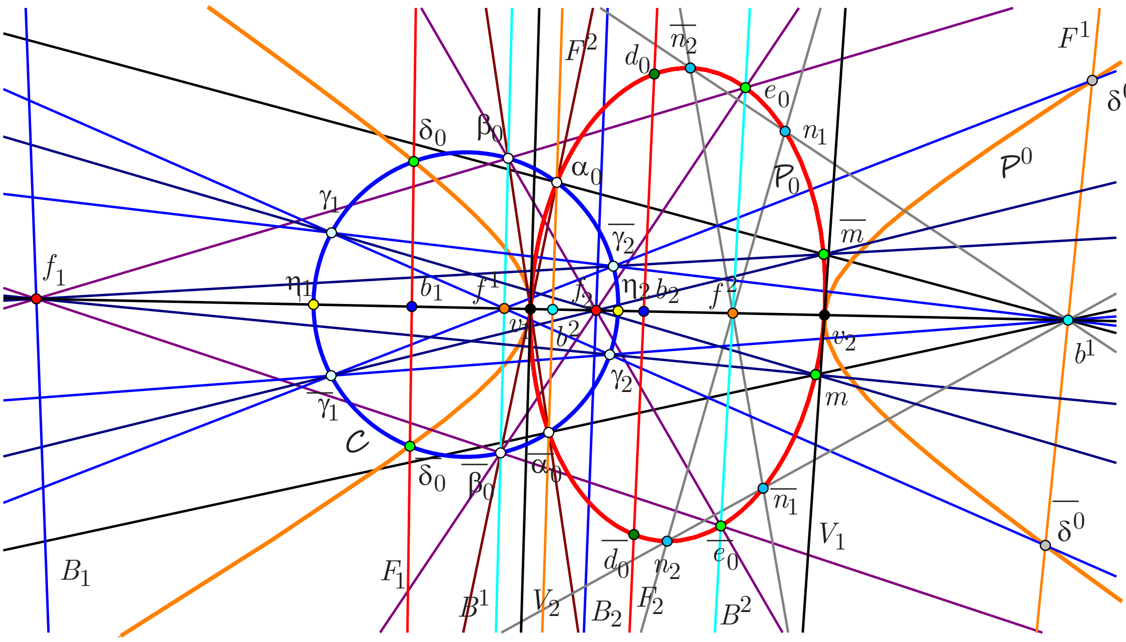

Figure 4: Additional collinearities with canonical points

We call these *canonical points* for the parabola $\mathcal{P}_0$. The dual lines of α_0 and $\overline{\alpha_0}$ are respectively $\alpha_0^{\perp} = \alpha_0 f^2 = \alpha_0 m$ and $\left(\overline{\alpha_0}\right)^{\perp} = \overline{\alpha_0} f^2 = \overline{\alpha_0 m}$, so the points m and $\overline{m}$ are also characterized by being the respective meets of these duals with $\mathcal{P}_0$.

Now we introduce the *γ-points:*

$$\begin{aligned}
\gamma_1 &\equiv (f_1\alpha_0)\,(b_1\overline{\alpha_0}) = \left[\alpha^3-\alpha^2+\alpha+1 : -2\alpha^2 : \alpha\left(\alpha^3-\alpha^2-\alpha-1\right)\right] \\
\overline{\gamma_1} &\equiv (f_1\overline{\alpha_0})\,(b_1\alpha_0) = \left[\alpha^3-\alpha^2+\alpha+1 : 2\alpha^2 : \alpha\left(\alpha^3-\alpha^2-\alpha-1\right)\right] \\
\gamma_2 &\equiv (f_2\alpha_0)\,(b_2\overline{\alpha_0}) = \left[\alpha^3+\alpha^2+\alpha-1 : 2\alpha^2 : -\alpha\left(\alpha^3+\alpha^2-\alpha+1\right)\right] \\
\overline{\gamma_2} &\equiv (f_2\overline{\alpha_0})\,(b_2\alpha_0) = \left[\alpha^3+\alpha^2+\alpha-1 : -2\alpha^2 : -\alpha\left(\alpha^3+\alpha^2-\alpha+1\right)\right].
\end{aligned}$$

Theorem 2 (Canonical null γ-points)**.** *The γ-points are all null points.*

Proof. We can check that each of $\gamma_1, \overline{\gamma_1}, \gamma_2$ and $\overline{\gamma_2}$ satisfy the equation $\alpha^2 x^2 + (1-\alpha^2) y^2 - z^2 = 0$ of the null circle $\mathcal{C}$ in standard coordinates. □

Now we introduce the δ-*points*

$$\begin{aligned}
\delta_1 &= (\beta_0 f_1)\left(\overline{\beta_0} b_1\right) = \left[\alpha^3+\alpha^2+\alpha-1 : 2\alpha^2 : \alpha\left(\alpha^3+\alpha^2-\alpha+1\right)\right], \\
\overline{\delta_1} &= \left(\overline{\beta_0} f_1\right)(\beta_0 b_1) = \left[\alpha^3+\alpha^2+\alpha-1 : -2\alpha^2 : \alpha\left(\alpha^3+\alpha^2-\alpha+1\right)\right], \\
\delta_2 &= (\beta_0 f_2)\left(\overline{\beta_0} b_2\right) = \left[\alpha^3-\alpha^2+\alpha+1 : -2\alpha^2 : \alpha\left(-\alpha^3+\alpha^2+\alpha+1\right)\right], \\
\overline{\delta_2} &= \left(\overline{\beta_0} f_2\right)(\beta_0 b_2) = \left[\alpha^3-\alpha^2+\alpha+1 : 2\alpha^2 : \alpha\left(-\alpha^3+\alpha^2+\alpha+1\right)\right].
\end{aligned}$$

Theorem 3 (Null δ points). *The δ-points are all null points.*

Proof. We can check that each of $\delta_1, \overline{\delta_1}, \delta_2$ and $\overline{\delta_2}$ satisfy the equation $\alpha^2 x^2 + (1-\alpha^2) y^2 - z^2 = 0$ of the null circle $\mathcal{C}$ in standard coordinates. □

In addition to the collinearities that define the canonical points, the following theorems bring together some remarkable relations between the points we have defined. In each case we have corresponding collinearities by considering opposite points.

Theorem 4 (e-point collinearities).
We have the collinearities $[[f_2 e_0 \overline{e_0}]], [[f_1 \beta_0 e_0]]$ *and* $\left[\left[f_2 \overline{\beta_0} e_0\right]\right]$.

Theorem 5 (n-point collinearities). *We have the collinearities* $[[f^2 n_1 n_2]]$ *and* $[[b^1 n_1 \overline{n_2}]]$.

Theorem 6 (γ-foci collinearities). *We have the collinearities* $[[f^1 \gamma_1 \gamma_2]]$ *and* $[[b^1 \gamma_1 \overline{\gamma_2}]]$.

Theorem 7 (γ, m collinearities). *We have the collinearities* $[[f_1 \overline{\gamma_2 m}]]$ *and* $[[f_2 \overline{\gamma_1 m}]]$.

Theorem 8 (α-m collinearities). *We have the collinearity* $[[b^1 \alpha_0 \overline{m}]]$.

Theorem 9 (γ-δ collinearities).
We have collinearities $[[\delta_1 \gamma_2 v_1]], [[\delta_1 \overline{\gamma_2} v_2]], [[\delta_2 \gamma_1 v_1]]$ *and* $[[\delta_2 \overline{\gamma_1} v_2]]$ *(Figure 5).*

Proof. Since we have the coordinates of all the points, these theorems can all be checked using the determinant condition for collinearity. For example to check $[[f_1 \beta_0 e_0]]$ we compute

$$\det \begin{pmatrix} \alpha+1 & 0 & \alpha(\alpha-1) \\ -1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \end{pmatrix} = 0$$

and to check $[[\delta_2 \overline{\gamma_1} v_2]]$ we compute

$$\det \begin{pmatrix} \alpha^3-\alpha^2+\alpha+1 & 2\alpha^2 & \alpha\left(-\alpha^3+\alpha^2+\alpha+1\right) \\ \alpha^3-\alpha^2+\alpha+1 & 2\alpha^2 & \alpha\left(\alpha^3-\alpha^2-\alpha-1\right) \\ 0 & 0 & 1 \end{pmatrix} = 0.$$ □

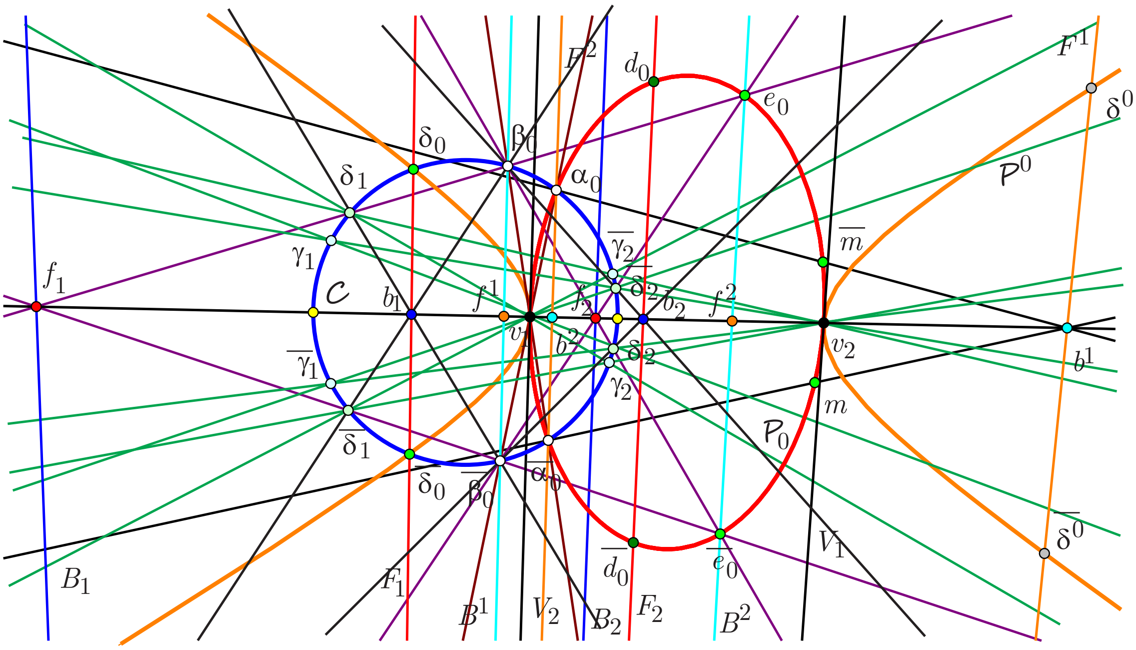

Figure 5: γ-δ collinearities

2.1. The y-points and the $\mathcal{Y}$-conic

Using the points of the previous section, we now introduce some secondary meets which determine an interesting conic. Define

$$\begin{aligned}
y_1 &\equiv (n_2\overline{n_2})(\gamma_1\overline{\gamma_2}) = \left\langle \alpha^2(\alpha+1)^2 : 0 : -(\alpha-1)^2\right\rangle \times \langle 2\alpha^3 : \alpha^4-1 : 2\alpha\rangle \\
&= \left[(\alpha-1)^3(\alpha+1) : -4\alpha^3 : \alpha^2(\alpha-1)(\alpha+1)^3\right], \\
y_2 &\equiv (n_1\overline{n_1})(\gamma_1\overline{\gamma_2}) = \left\langle -\alpha^2(\alpha-1)^2 : 0 : (\alpha+1)^2\right\rangle \times \langle 2\alpha^3 : \alpha^4-1 : 2\alpha\rangle \\
&= \left[(\alpha-1)(\alpha+1)^3 : -4\alpha^3 : \alpha^2(\alpha-1)^3(\alpha+1)\right], \\
y_3 &\equiv (\overline{n_1n_2})(\overline{\gamma_1\gamma_2}) = \langle -\alpha^2(\alpha^2-1) : 4\alpha^2 : \alpha^2-1\rangle \times \left\langle 2\alpha^2 : -(\alpha^2-1)^2 : 2\alpha^2\right\rangle \\
&= [4\alpha^2+\alpha^4-1 : 2\alpha^2(\alpha^2-1) : \alpha^2(-4\alpha^2+\alpha^4-1)], \\
y_4 &\equiv (n_1n_2)(\alpha_0b^1) = [\alpha^2(\alpha^2-1) : 4\alpha^2 : -(\alpha^2-1)] \times [\alpha^2 : -(\alpha^2+1) : 1] \\
&= [-4\alpha^2+\alpha^4-1 : 2\alpha^2(\alpha^2-1) : \alpha^2(4\alpha^2+\alpha^4-1)], \quad \text{and} \\
y_5 &\equiv (\beta_0b^1)B^2 = [\alpha^2 : \alpha^2-1 : 1] \times [-\alpha^2 : 0 : 1] \\
&= [\alpha^2-1 : -2\alpha^2 : \alpha^2(\alpha^2-1)].
\end{aligned}$$

Theorem 10 (The $\mathcal{Y}$-conic)**.** *The points $y_1, \overline{y_1}, y_2, \overline{y_2}, y_3, \overline{y_3}, y_4, \overline{y_4}, y_5$ and $\overline{y_5}$ lie on a conic, which we call the $\mathcal{Y}$-**conic**, whose equation is*

$$\alpha^4\left(\alpha^4-6\alpha^2+1\right)x^2+4\alpha^2\left(\alpha^2-1\right)^2y^2+\left(\alpha^4-6\alpha^2+1\right)z^2-2\alpha^2\left(\alpha^2+1\right)^2xz=0.$$

Proof. Since the forms of all the points involved are known, it is a lengthy but straightforward exercise (made much simpler with a computer package) to verify that the corresponding points satisfies the $\mathcal{Y}$-conic equation. □

The relationship between the $\mathcal{Y}$-conic and the parabola $\mathcal{P}_0$ is interesting; empirical investigations with GSP suggest that in some sense the $\mathcal{Y}$-conic is *symmetrically* placed both with

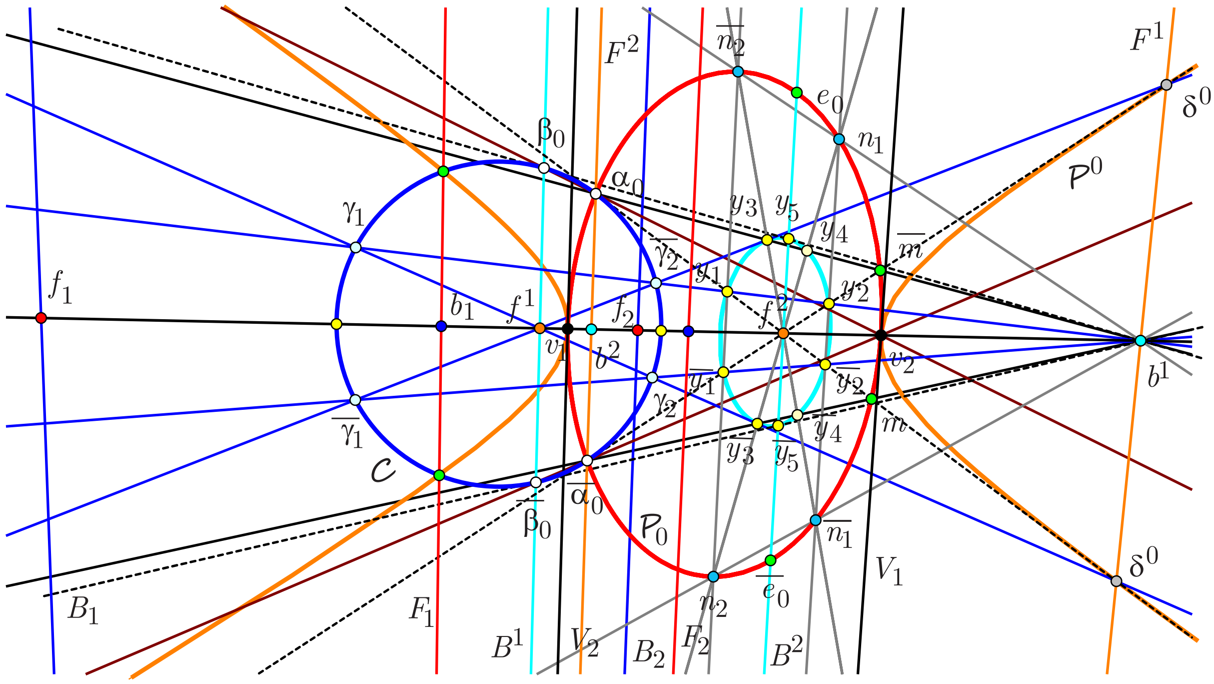

Figure 6: The parabola and its $\mathcal{Y}$-conic

respect to $\mathcal{P}_0$ and C, as Figures 6 and 7 suggest. We will give two theorems that suggest this. First recall that in projective geometry a *homology* with axis the line L and center the point a is defined in terms of two additional points c and d satisfying $[[acd]]$. In this case the homology sends the general point x to $(((cx)\,L)\,d)\,(ax)$.

Theorem 11 (Parabola null circle homology)**.** *The homology ϕ with axis F^1 and center f_1 which sends b_2 to b_1 sends $\mathcal{P}_0$ to $\mathcal{C}$.*

Proof. Using the known coordinates of the points and lines involved, the homology may be computed to be

$$\phi\left([x:y:z]\right)=\left[x\alpha^3-x\alpha^2+z\alpha+z:-2y\alpha^2:-\alpha\left(-x\alpha^3+x\alpha^2+z\alpha+z\right)\right].$$

After substitution, we find that this lies on the null conic: $\alpha^2x^2+\left(1-\alpha^2\right)y^2-z^2=0$ precisely when $4\alpha^4\left(xz-y^2\right)\left(\alpha^2-1\right)=0$. So this homology sends $\mathcal{P}_0$ to $\mathcal{C}$. $\square$

Theorem 12 ($\mathcal{Y}$-conic homology)**.** *If the meet of the tangents to $\mathcal{P}_0$ at n_2 and $\overline{n_2}$ is denoted q_2, then the homology φ with axis F^1 and center f^2 which sends q_2 to f^1 sends $\mathcal{P}_0$ to $\mathcal{Y}$.*

Proof. We first compute that $q_2=\left[(\alpha+1)^2:0:-\alpha^2\left(\alpha-1\right)^2\right]$, and then determine that

$$\begin{aligned}\varphi\left([x:y:z]\right)\;=\;&\left[z+2z\alpha+x\alpha^2-2x\alpha^3-x\alpha^4-z\alpha^2:-4y\alpha^3\right.\\&\left.:\;\alpha^2\left(z-2z\alpha+x\alpha^2+2x\alpha^3-x\alpha^4-z\alpha^2\right)\right].\end{aligned}$$

After substitution, we find that this point lies on the $\mathcal{Y}$-conic above precisely when $64\alpha^8\left(xz-y^2\right)\left(\alpha^2-1\right)^2=0$. So this homology sends $\mathcal{P}_0$ to $\mathcal{Y}$. $\square$

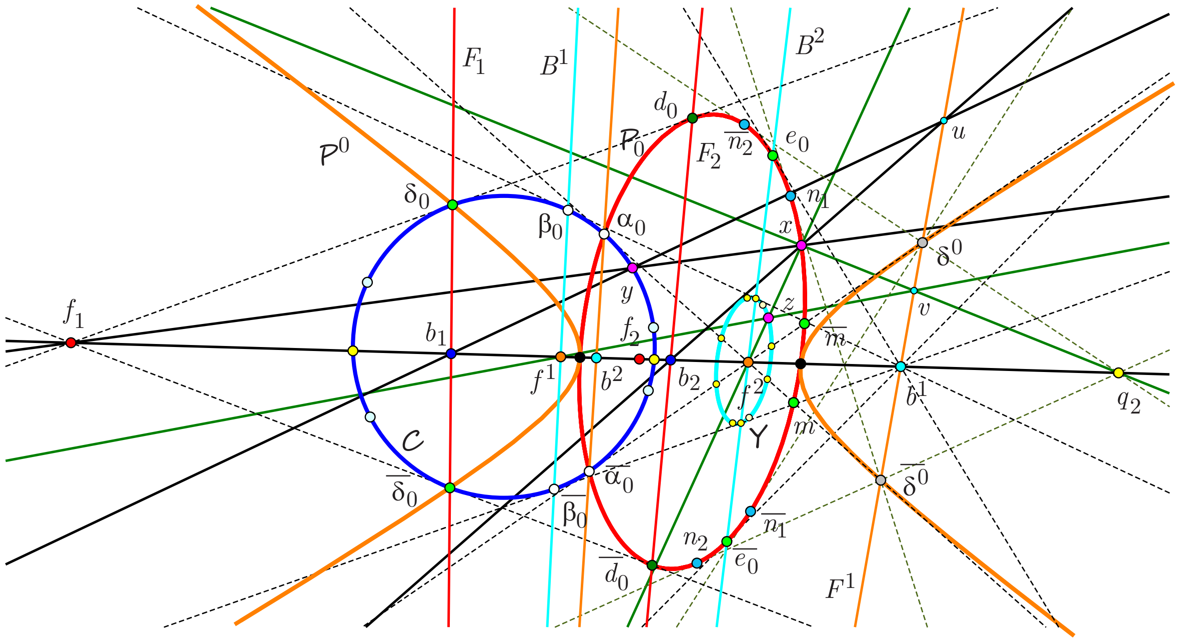

Figure 7: Another view of the $\mathcal{Y}$-conic

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