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Finding the Middle Ground Bisectors in p-Geometry

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Abstract. Bisectors of line segments are quite simple geometrical objects. Despite their simplicity, they have many surprising and useful properties. As metric objects, the shape of bisectors depends upon the metric considered. This article discusses geometric properties of bisectors of line segments in the plane, when the bisectors are taken with respect to the usual p-norms. Although the shape of bisectors changes as their defining p-norm varies, it is shown that the bisectors share exactly three points (or infinitely many points in exceptional cases determined by the orientation of the base line segment).

Key Words: taxicab norm, p-bisector, p-norm

MSC 2010: 51M05, 51B20

1. Introduction

When one ventures into the plane and decides to change the familiar Euclidean norm by another one, the new way of measuring distances between points yields a radical change in the geometry of the plane. Immediately, one falls into the category of non-Euclidean geometries, were many basic geometric principles no longer hold. For instance, principles of congruence of triangles or non distance-preserving rotations, to mention a few. As new geometric shapes are created with a new norm (by using the very same metric definition of Euclidean objects such as circles, ellipses, bisectors of a line segment and so forth), one discovers the amusing fact that our geometric way of thinking is completely Euclidean. Take, for instance, the excellent book of Krause [5]. There, he takes the reader by the hand and shows us step by step, the wonderful world of the taxicab plane with no few surprises among the taxicab versions of familiar Euclidean geometric objects. However, renorming finite dimensional spaces is not just an interesting topic in its own accord. It is related to a vast area of research known as *Minkowski Geometry* where many more interesting problems arise and even more unanswered

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questions remain. We refer the interested reader to [10] or to the survey articles [6], [7], and [8], to get a glimpse of the subject.

As we said, our setting will be the real plane, namely, the two dimensional real vector space of ordered pairs $\vec{v} = (x, y) \in \mathbb{R}^2$. We will refer to $\vec{v} = (x, y)$ as a *point* or *vector* in the plane. We will equip the plane with a *norm*, that is, a non-negative function, $\|\cdot\|$, that assigns to a vector $\vec{v}$ its "length". Recall that this "length" function must satisfy that the only vector with length zero is the origin and that for any vectors in the plane, $\vec{v}$ and $\vec{w}$, and any real number r, $\|r\vec{v}\| = |r|\,\|\vec{v}\|$ and $\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$ hold. We will only consider the so called *p-norms*: if $1 \le p \le \infty$, these norms are defined as

$$\|(x, y)\|_p = (|x|^p + |y|^p)^{1/p} \quad \text{for} \quad 1 \le p < \infty,$$

and for $p = \infty$

$$\|(x, y)\|_\infty = \max\{|x|, |y|\}.$$

The pair $(\mathbb{R}^2, \|\cdot\|_p)$ is denoted by ℓ_2^p. Recall that any norm in the plane induces a metric $\mathrm{dist}(\vec{v}_1, \vec{v}_2) = \|\vec{v}_1 - \vec{v}_2\|$, and thus a notion of distance between points in the plane.

For the cases $p = 1, 2$, and, ∞ the corresponding norms are usually known as the *taxicab norm*, the *Euclidean norm* and the *max* or *Chebyshev norm*, respectively. We denote by $S_p(\vec{v}, r)$ the p-circle with center $\vec{v}$ and radius r, that is,

$$S_p(\vec{v}, r) = \{\vec{w} \in \ell_2^p : \|\vec{w} - \vec{v}\|_p = r\}$$

(see Figure 3).[1]

In this article we focus on bisectors of a line segment with respect to the p-norm. If $\vec{v}_1$ and $\vec{v}_2$ are two distinct points in ℓ_2^p, the *p-bisector* of the line segment determined by $\vec{v}_1$ and $\vec{v}_2$ is

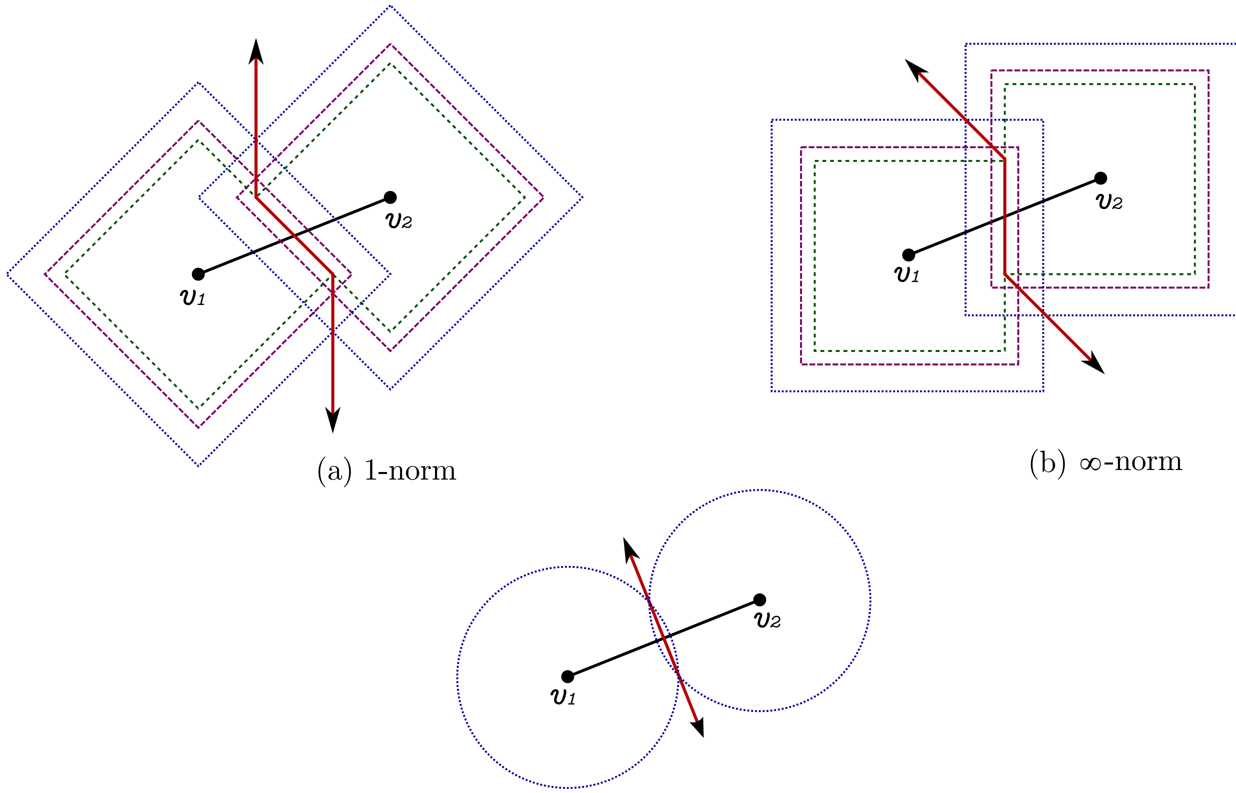
$$\mathrm{M}_p(\vec{v}_1, \vec{v}_2) = \{\vec{w} \in \ell_2^p : \|\vec{w} - \vec{v}_1\|_p = \|\vec{w} - \vec{v}_2\|_p\}.$$

Note that the p-bisector of the line segment determined by $\vec{v}_1$ and $\vec{v}_2$ is just the collection of points obtained by intersecting the circles $S_p(\vec{v}_1, r)$ and $S_p(\vec{v}_1, r)$ with r at least half the p-distance between $\vec{v}_1$ and $\vec{v}_2$ (see Figure 1). Also note that, regardless of the value of p, $\mathrm{M}_p(\vec{v}_1, \vec{v}_2)$ is nonempty, as it always contains at least the midpoint $(\vec{v}_1 + \vec{v}_2)/2$. That is,

$$\frac{\vec{v}_1 + \vec{v}_2}{2} \in \bigcap_{1 \le p \le \infty} \mathrm{M}_p(\vec{v}_1, \vec{v}_2). \tag{1}$$

We will prove that in fact the intersection above consists of exactly three points as long as the line segment $\vec{v}_1\vec{v}_2$ is not parallel to a coordinate axis or to a side of the unit taxicab circle. When the segment $\vec{v}_1\vec{v}_2$ is parallel to a coordinate axis, all p-bisectors, $1 \le p < \infty$, coincide with the Euclidean bisector: the orthogonal line to the line segment that goes through the midpoint, and the Euclidean bisector is contained in the max-bisector which surprisingly has nonempty interior (see Figure 2). When the line segment is parallel to a side of the taxicab unit circle, all p-bisectors, $1 < p \le \infty$, are the same; but now they are contained in the 1-bisector, which, in this case, is the one with nonempty interior (see Theorems 3.1 and 3.2). Figures 1 and 2 below show bisectors in different norms. The bisectors are drawn the same way as in the Euclidean case: take p-circles $S_p(\vec{v}_1, r)$ and $S_p(\vec{v}_2, r)$ with $r \ge \|(\vec{v}_1 - \vec{v}_2)/2\|_p$. The intersection points of the circles lie on the bisector, and it is easy to show that there are no more points on a bisector.

[1] Please note that all figures in this article are drawn in the cartesian plane with respect to the standard basis $\{(1, 0), (0, 1)\}$. The coordinate axes, although not shown in the figures, are important since the p-norms are analytically defined via a fixed (affine) coordinate frame.



(a) 1-norm

(b) ∞-norm

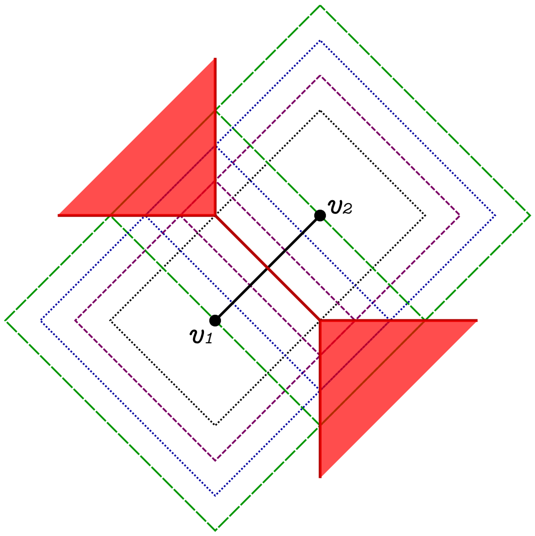
(c) 2-norm

Figure 1: Bisectors for three different norms

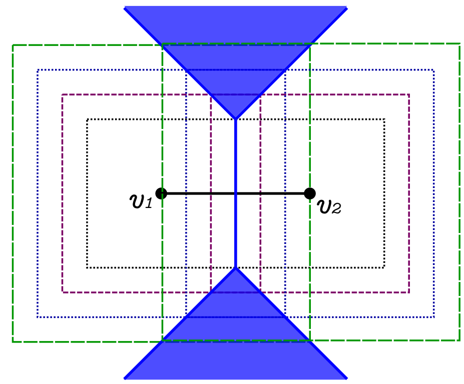
Bisectors have been a subject of intense study among mathematicians. In fact, some of the ideas presented here have been discussed under more general settings. Nevertheless, to the best of our knowledge, Theorem 3.1 has not been previously explicitly stated or proven (see [3] and [4] for related results). Choosing one's favorite norm and discovering its geometric properties is just plain instructional fun. Who does not remember a first course in geometry where after a series of geometrical constructions three points appear miraculously aligned? Or, take the case where more than four points (nine to be exact, as in the nine-point circle theorem) are in the same circle?

Bisectors have played a significant role in other contexts. For instance, in [9] Schattschneider gave a beautiful and simple geometric argument to describe the group of isometries for the taxicab norm (recall, an *isometry* is a function that preserves distances). First, she determined which Euclidean isometries preserve taxicab distance (translations plus the symmetry group of the taxicab unit circle, that is, translations plus isometries that take the unit circle onto itself) and then, by considering the taxicab metric midpoints, that is, the points $\vec{w}$ of the taxicab bisector of $\vec{v}_1$ and $\vec{v}_2$ such that $\|\vec{w}-\vec{m}\|_1 = \|\vec{v}_1-\vec{m}\|_1$ where $\vec{m}$ is the midpoint of the segment $\vec{v}_1$, $\vec{v}_2$, she proved that any taxicab isometry must be a Euclidean isometry. It is well known, and a relatively easy exercise, to show that for $p \neq 2$, all p-norms have the same group of isometries, they correspond to the group of unitary permutation matrices plus translations. This sets apart the Euclidean norm as the p-norm with the richest isometry group (translations plus the group of orthogonal matrices, which is the symmetry group of the circle) and hence our natural bias to think in an Euclidean way.

Another instance, where one can find bisectors, is when trying to find the regions, say, in the plane, that contain the points that are closer to a point than to other points in a given set.



(a) 1-norm



(b) ∞-norm

Figure 2: Bisectors with nonempty interior

One can obtain these regions by considering the bisectors among the points in the given set. The regions obtained are called *Voronoi sets*. Voronoi sets have a wide range of applications, from computational geometry to anthropology and archeology. See, for instance, the excellent survey [1].

2. A few good lemmas

The following lemma is a classical exercise from real analysis (see, for instance, [2]). The second part of the lemma can be proved by elementary calculus.

Lemma 2.1. *For a fixed point* $(x, y) \neq (0, 0)$ *in* $\mathbb{R}^2$ *we have that*

$$\lim_{p\to\infty} \|(x, y)\|_p = \|(x, y)\|_\infty .$$

Moreover, the limit is strictly decreasing whenever (x, y) *is not on a coordinate axis. If one of* x *or* y *is equal to zero then all* p*-norms have the same value,* $\max\{|x|, |y|\}$*. In particular, we obtain that for any point* (x, y) *in the plane*

$$\|(x, y)\|_\infty \leq \|(x, y)\|_p \leq \|(x, y)\|_1 \quad \textit{for any} \quad 1 < p < \infty$$

or, to put it in geometric terms,

$$B_1(\vec{0}, 1) \subset B_p(\vec{0}, 1) \subset B_\infty(\vec{0}, 1),$$

where $B_p(\vec{0}, 1)$ *denotes the closed unit ball in the corresponding norm.*

The p-norms are symmetric with respect to the origin and invariant under the group of rotations with respect to right angles. As a result, most of the time we will consider points in the plane in the first quadrant only. Moreover, given the symmetric nature of the p-norms with respect to the coordinates of a point, we will consider points of the form $(x, \alpha x)$ with

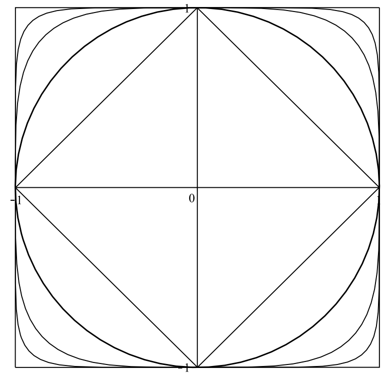


Figure 3: Unit spheres for the 1, 2, 4, 8 and ∞-norm

$x \geq 0$ and $0 \leq \alpha \leq 1$ only. That is, points in the plane that lie in the cone bounded by the non-negative x-axis and the line $y = x$, which we will refer to as the *first octant.*

A word about angles. Recall that angles in the plane are defined via the usual inner product that induces the 2-norm

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 y_1 + x_2 y_2.$$

Here you have an amusing fact to think about it: if you think about the notion of radian (a measure of angle), this notion is defined in terms of length (arclength), thus, if one measures length using different norms the corresponding arclength should change and hence the magnitude of the angle (in "radians"). For instance, with the 1-norm, the perimeter of the unit 1-circle is 8, thus the value of π is 4. Angles are not well defined when one moves away from the Euclidean plane, that is, by considering norms that are not induced by inner products (recall that all p-norms, $p \neq 2$, are nor induced by an inner product, that is, the norm cannot be obtained as $\| \cdot \| = \sqrt{\langle \cdot \, , \, \cdot \rangle}$ for some inner product $\langle \cdot \, , \, \cdot \rangle$). See for instance [11] to get a glimpse on angle discussion. In what follows we avoid these subtleties and use the notion of angle in the good Euclidean way. We use the term *angle* or *polar angle* to emphasize that we measure angles in the polar plane.

Lemma 2.2. *Let $(x, \alpha x)$ be a point in the first quadrant, with $x \geq 0$ and $0 \leq \alpha \leq 1$. Suppose furthermore, that for some $p \neq q$, $\|(x, \alpha x)\|_p = r$ and $\|(x, \alpha x)\|_q = t$ with $r \neq t$. Then, there does not exist another point in the first octant with p-norm r and q-norm t.*

Proof. First, observe that $(x, \alpha x) \neq (0, 0)$. Moreover, given that $r \neq t$, Lemma 2.1 implies $\alpha \neq 0$. Now suppose that such a second point $(y, \beta y)$ exists. By Lemma 2.1 this point cannot lie on the coordinate axes, therefore $y > 0$ and $0 < \beta \leq 1$. By our assumptions $\|(y, \beta y)\|_p = r$, whereas $\|(y, \beta y)\|_q = t$. Without loss of generality we may take $p < q$, and we first consider $q < \infty$. It follows that

$$\|(y, \beta y)\|_p \|(y, \beta y)\|_q^{-1} = \|(x, \alpha x)\|_p \|(x, \alpha x)\|_q^{-1} = r \, t^{-1}, \tag{2}$$

which is equivalent to

$$(1 + \beta^p)^{1/p} (1 + \beta^q)^{-1/q} = (1 + \alpha^p)^{1/p} (1 + \alpha^q)^{-1/q} = r \, t^{-1}. \tag{3}$$

By defining $g(\beta) = (1 + \beta^p)^{1/p} (1 + \beta^q)^{-1/q}$, it is easy to prove that $g'(\beta) > 0$ whenever $0 < \beta < 1$ (consider $\log g$). By the Mean Value Theorem, there do not exist two distinct constants $0 \leq \alpha$ and $\beta \leq 1$ that satisfy (3), and from the equivalence to (2) neither two points in the first octant with p-norm r and q-norm t if $r \neq t$.

For $p < q = \infty$, $g(\beta) = \|(y, \beta y)\|_p \|(y, \beta y)\|_\infty^{-1} = (1 + \beta^p)^{1/p}$ which also satisfies $g'(\beta) > 0$ whenever $0 < \beta < 1$. Thus, the result also extends to this case. □

An immediate consequence of Lemma 2.1 is that a p-circle and a q-circle, both centered at the origin and of the same radius, intersect at exactly four points, on the coordinate axes:

$$S_p(\vec{0}, r) \cap S_q(\vec{0}, r) = \{(\pm r, 0), (0, \pm r)\}.$$

At the same time, Lemma 2.2 gives us the points of intersection of a p and a q-circle of different radii. We summarize these geometric consequences in the following proposition:

Proposition 2.3. *If the intersection of a p-circle and a q-circle, both centered at the origin, i.e.,* $S_p(\vec{0}, r) \cap S_q(\vec{0}, t)$, *is nonempty and* $p \neq q$, *then exactly one of the following happens:*

1. $r = t$ *and the circles intersect at exactly four points on the coordinate axes,* $(\pm r, 0)$ *and* $(0, \pm r)$.
2. $r \neq t$ *and the circles intersect at exactly eight points, one in each octant:* $(\pm x, \pm y)$ *and* $(\pm y, \pm x)$, *with* $x \neq y$ *and* $x, y \neq 0$.
3. $r \neq t$ *and the circles intersect at exactly four points* $(\pm x, \pm x)$ *with* $2^{1/p}|x| = r$ *and* $2^{1/q}|x| = t$.

Proof. Case 1) is the observation following the proof of Lemma 2.2. To prove 2), first note that if $p \neq q$, $r \neq t$ and the circles intersect at a point (x, y) that does not belong to the lines $y = x$, $y = -x$, then it must be in one of the open octants. By Lemma 2.2 and the symmetry of the p-norms we know that there does not exist another point in that octant where they intersect. However, it is easy to see that $\|(\pm x, \pm y)\|_p = \|(\pm y, \pm x)\|_p = r$ and the same is true for their q-norms. So, by applying Lemma 2.2 once again, these eight points are the only intersections of the circles.

Case 3) occurs when one of the intersection points lies on $y = x$ or $y = -x$. Given that the four points $(\pm x, \pm x)$ all have the same p-norm, they are the only ones on the p-circle with that norm in each octant by Lemma 2.2, and therefore in each quadrant. □

Remark 2.4. Proposition 2.3 illustrates that the intersection of a p-circle with a q-circle at a point (x, y) with polar angle ϕ, determines the angle of all other possible intersections of the circles. Given the symmetric nature of the p-norms with respect to reflections in the lines $y = x$ and $y = -x$, we conclude that the set of polar angles where the two circles intersect is $\left\{\frac{\pi}{4} \pm \phi,\ \frac{3\pi}{4} \pm \phi,\ \frac{-\pi}{4} \pm \phi,\ \frac{-3\pi}{4} \pm \phi\right\}$ for some $0 \leq \phi \leq \frac{\pi}{4}$. Equivalently, this set of angles can be represented as $\left\{\pm\phi,\ \frac{\pi}{2} \pm \phi,\ \pi \pm \phi,\ \frac{3\pi}{2} \pm \phi\right\}$ for some $0 \leq \phi \leq \frac{\pi}{4}$.

3. Main results, or the part where one has to do the math

We start this section with a "proof by picture". Figure 4 shows a line segment (AB) with five bisectors corresponding to the p-norms, 1, 3/2, 2, 5/2, and ∞. Observe that the bundle of bisectors lies in between the taxicab bisector and the max bisector. Also observe, that all bisectors have exactly three points in common, one of them the midpoint of the segment. The details of this nice geometric property are the contents of Theorems 3.1 and 3.2 in this section.

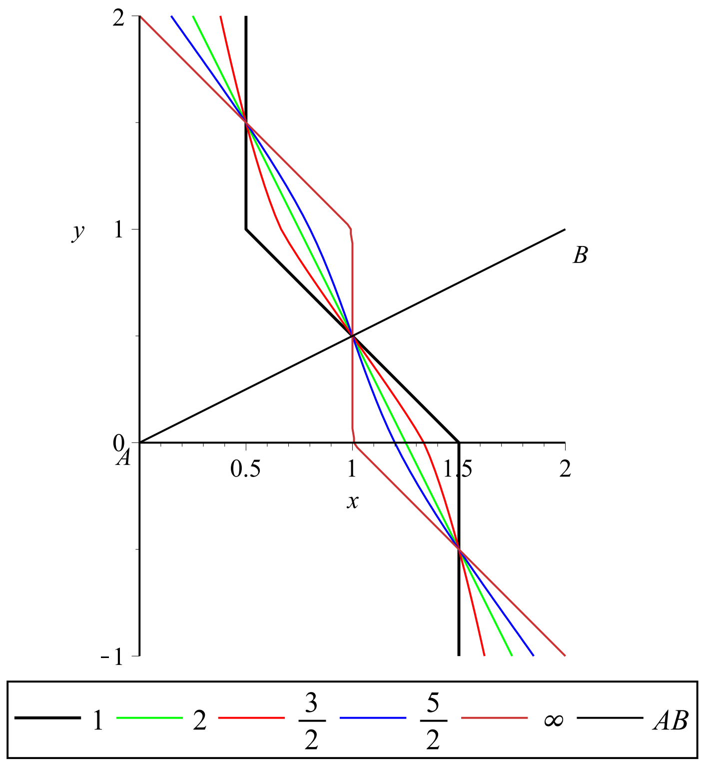


Figure 4: Bisectors of AB in five different norms

We start with a few simplifications. By the invariance of the p-norms under translation, we will assume that the midpoint of the given segment is the origin. Thus, the endpoints of the line segment are now symmetric points with respect to the origin. We will take one of the end points in the first quadrant.

Given $\vec{v} = (x, y)$, we define its *normal vector* as $\vec{v}^{\perp} = (-y, x)$. It can be easily confirmed that both $\vec{v}^{\perp}$ and $-\vec{v}^{\perp}$ belong to $\mathrm{M}_p(\vec{v}, -\vec{v})$ for all p. We will demonstrate that whenever the bisectors do not coincide, then their only points of intersection besides the origin are $\vec{v}^{\perp}$ and $-\vec{v}^{\perp}$.

Theorem 3.1. *Suppose that $1 < p < \infty$ and $\vec{v} \neq \vec{0}$. Then exactly one of the following may occur:*

1. $\mathrm{M}_p(\vec{v}, -\vec{v}) = \mathrm{M}_2(\vec{v}, -\vec{v})$, *or*
2. $\mathrm{M}_p(\vec{v}, -\vec{v}) \cap \mathrm{M}_2(\vec{v}, -\vec{v}) = \{\vec{0}, \vec{v}^{\perp}, -\vec{v}^{\perp}\}$.

Proof. It suffices to consider the case $p \neq 2$. Whether the two bisectors $\mathrm{M}_p(\vec{v}, -\vec{v})$ and $\mathrm{M}_2(\vec{v}, -\vec{v})$ coincide or not, depends solely on the position of the vector $\vec{v}$. Let ϕ be the polar angle of the vector $\vec{v}$ with respect to the x-axis. Again due to the symmetric nature of the p-norms we may assume without loss of generality that $0 \leq \phi \leq \frac{\pi}{4}$.

Let $\vec{w} \in \mathrm{M}_2(\vec{v}, -\vec{v})$, and consider the triangle $\triangle ABC$ with $A = \vec{w}$, $B = -\vec{v}$ and $C = \vec{v}$. Let $r = \|\vec{v} - \vec{w}\|_2 = \|-\vec{v} - \vec{w}\|_2$ and θ be the angle $\angle CBA$ (see Figure 5). Clearly, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\angle BCA = \theta$. The midpoint $\vec{0}$ corresponds to $\theta = 0$ whereas $\vec{v}^{\perp}$ and $-\vec{v}^{\perp}$ correspond to $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$, respectively. Thus each point $\vec{w}$ in $\mathrm{M}_2(-\vec{v}, \vec{v})$ is associated to a unique angle $\theta = \theta(\vec{w})$.

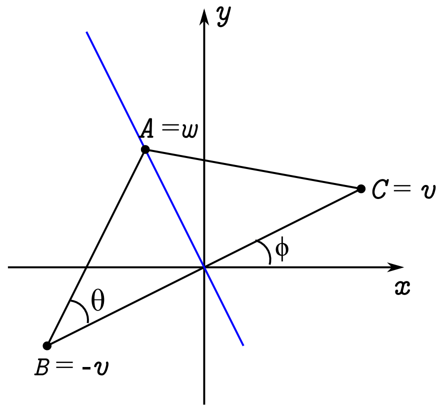


Figure 5: Polar angles ϕ and θ associated to the points $\vec{v}$ and $\vec{w}$, respectively

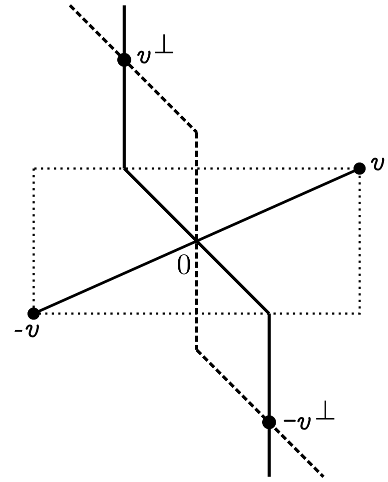


Figure 6: Taxicab (bold) and max (dotted) bisectors

The point $\vec{w}$ lies on $S_2(-\vec{v}, r)$ at polar angle $\phi + \theta$ with respect to the x-axis, and on $S_2(\vec{v}, r)$ at polar angle $\pi + \phi - \theta$. Given the invariance of the p-norms under translation, we see that $\vec{w}$ corresponds to two points on $S_2(\vec{0}, r)$, one at the angle $\phi + \theta$ and another one at the angle $\pi + \phi - \theta$.

Suppose now that the same point $\vec{w}$ is in $\mathrm{M}_p(\vec{v}, -\vec{v})$ with $p \neq 2$, and let $t = \|\vec{v} - \vec{w}\|_p = \| - \vec{v} - \vec{w}\|_p$. Then, as in the case when $\vec{w}$ is in the Euclidean bisector, $\vec{w}$ corresponds to two points on $S_p(\vec{0}, t)$, one at the angle $\phi + \theta$ and another at the angle $\pi + \phi - \theta$.

Note that $\vec{w}$ is in $S_2(\vec{v}, r) \cap S_p(\vec{v}, t)$ and $\vec{w}$ is in $S_2(-\vec{v}, r) \cap S_p(-\vec{v}, t)$ if and only if $S_2(\vec{0}, r)$ and $S_p(\vec{0}, t)$ intersect at the angles $\phi+\theta$ and $\pi+\phi-\theta$. This is the crucial geometric observation needed to prove the theorem. Indeed, since the value of $\phi + \theta$ determines, as we have seen, the set of angles where the intersection of the two circles may occur, the point $\vec{w}$ belongs to both bisectors if and only if the angles $\phi + \theta$ and $\pi + \phi - \theta$ are related as in Remark 2.4.

To begin with, consider the case $\phi = 0$. Then $\phi + \theta = \theta$, whereas $\pi + \phi - \theta = \pi - \theta$. We know from Remark 2.4 that whenever $S_2(\vec{0}, r)$ intersects $S_p(\vec{0}, t)$ at an angle $0 < \theta < \frac{\pi}{2}$ then it must also intersect it at the angle $\pi - \theta$. As a consequence, every point $\vec{w}$ in $\mathrm{M}_2(\vec{v}, -\vec{v})$ must also belong to $\mathrm{M}_p(\vec{v}, -\vec{v})$ for all p.

In the case $\phi = \frac{\pi}{4}$ holds $\phi+\theta = \frac{\pi}{4}+\theta$, whereas $\pi+\phi-\theta = \frac{3\pi}{4}-\theta$. Again by Remark 2.4, whenever $S_2(\vec{0}, r)$ intersects $S_p(\vec{0}, t)$ at an angle $\frac{\pi}{4} + \theta$, then it must also intersect it at the angle $\frac{3\pi}{4} - \theta$.

For the case $0 < \phi < \frac{\pi}{4}$ consider the points on $\mathrm{M}_2(\vec{v}, -\vec{v})$ with $0 \leq \theta \leq \frac{\pi}{4}$. Note that $0 < \phi + \theta < \frac{\pi}{2}$, whereas $\frac{3\pi}{4} < \pi + \phi - \theta < \frac{5\pi}{4}$. Again by Remark 2.4 we have that, whenever $S_2(\vec{0}, r)$ intersects $S_p(\vec{0}, t)$ at an angle $\phi + \theta$, then it also intersects it at an angle $\pi + \phi - \theta$ if and only if there exists $0 \leq \psi \leq \frac{\pi}{4}$ such that one of the following happens:

(i) $\phi + \theta = \frac{\pi}{4} - \psi$ and $\pi + \phi - \theta = \frac{3\pi}{4} + \psi$

(ii) $\phi + \theta = \frac{\pi}{4} - \psi$ and $\pi + \phi - \theta = \frac{5\pi}{4} - \psi$

(iii) $\phi + \theta = \frac{\pi}{4} + \psi$ and $\pi + \phi - \theta = \frac{3\pi}{4} + \psi$

(iv) $\phi + \theta = \frac{\pi}{4} + \psi$ and $\pi + \phi - \theta = \frac{5\pi}{4} - \psi$.

Case (i) implies that $\phi = 0$, and case (iv) implies that $\phi = \frac{\pi}{4}$, which is not possible. Case (ii) implies that $\theta = 0$, whereas case (iii) implies that $\theta = \frac{\pi}{4}$.

A similar analysis shows that there are no intersection points for $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, and by symmetry we get that the only intersection point for negative θ occurs at $-\frac{\pi}{4}$.

We conclude that $\mathrm{M}_2(\vec{v}, -\vec{v}) \subset \mathrm{M}_p(\vec{v}, -\vec{v})$ if and only if $\vec{v}$ lies on one of the coordinate axes, or on the lines $y = \pm x$, and that in the remaining of the cases $\mathrm{M}_2(\vec{v}, -\vec{v}) \cap \mathrm{M}_p(\vec{v}, -\vec{v})$ consists of exactly three points, the ones corresponding to the angles $\theta = \pm\frac{\pi}{4}$ and 0.

That $\mathrm{M}_2(\vec{v}, -\vec{v}) = \mathrm{M}_p(\vec{v}, -\vec{v})$ in the case $\phi = 0, \frac{\pi}{4}$, follows by the fact for $1 < p < \infty$, $\mathrm{M}_p(\vec{v}, -\vec{v})$ intersects the ray at angle $\theta + \phi$ from $-\vec{v}$ at exactly one point. This is a consequence of the triangle inequality. □

We can now generalize the above result to any p and q-norms.

Theorem 3.2. *Suppose that* $1 < p < q < \infty$ *and* $\vec{v} \neq \vec{0}$. *Then exactly one of the following occurs:*

1. $\mathrm{M}_p(\vec{v}, -\vec{v}) = \mathrm{M}_q(\vec{v}, -\vec{v})$, *or*
2. $\mathrm{M}_p(\vec{v}, -\vec{v}) \cap \mathrm{M}_q(\vec{v}, -\vec{v}) = \{\vec{0}, \vec{v}^\perp, -\vec{v}^\perp\}$.

Proof. Again, we let $\vec{w} \in \mathrm{M}_p(\vec{v}, -\vec{v})$ with $r = \|\vec{v} - \vec{w}\|_p = \| - \vec{v} - \vec{w}\|_p$ and define the angles ϕ and $\theta = \angle CBA$ as in the proof of Theorem 3.1. Note that in this case it is not true in general that $\angle CBA = \angle BCA$. So we define the function $f_p(\theta) = \angle BCA$, which is uniquely defined since $\mathrm{M}_p(\vec{v}, -\vec{v})$, $1 < p < \infty$, intersects the ray at angle $\theta + \phi$ from $-\vec{v}$ at exactly one point.

If $\vec{w} \in \mathrm{M}_p(\vec{v}, -\vec{v}) \cap \mathrm{M}_q(\vec{v}, -\vec{v})$, then $f_p(\theta) = f_q(\theta)$. Let $t = \|\vec{v} - \vec{w}\|_q = \| - \vec{v} - \vec{w}\|_q$. In the terms of the proof of Theorem 3.1, $\phi + \theta$, and $\pi + \phi - f_p(\theta) = \pi + \phi - f_q(\theta)$ are two angles where the circle $S_p(\vec{0}, r)$ intersects $S_q(\vec{0}, t)$. Let $s = \| - \vec{v} - \vec{w}\|_2$. Since $\vec{w}$ is in $S_2(-\vec{v}, s)$, it follows from Proposition 2.3 that $S_2(\vec{0}, s)$ also intersects these circles at these same angles. In other words, $\vec{w}$ must also belong to $\mathrm{M}_2(\vec{v}, -\vec{v})$. The theorem now follows from Theorem 3.1. □

To include the taxicab and max bisectors in the statement of Theorem 3.2, consider again the segment determined by the points $\vec{v}$ and $-\vec{v}$, and let ϕ be the polar angle of the vector $\vec{v}$ (see Figure 5). We may assume, as in the proof of Theorem 3.1, that $0 \leq \phi \leq \frac{\pi}{4}$. When $\phi = 0$, the segment lies on the x-axis and, in this case, one can easily check that all bisectors $\mathrm{M}_p(\vec{v}, -\vec{v})$, $1 \leq p < \infty$, coincide. Moreover, all of them are contained in the bisector $\mathrm{M}_\infty(\vec{v}, -\vec{v})$ (see Figure 2, (b)). The intersection of all p-bisectors, $1 \leq p \leq \infty$, is then the set of all max metric midpoints of the segment determined by $\vec{v}$ and $-\vec{v}$. The case $\phi = \frac{\pi}{4}$ is analogous, but now all bisectors $\mathrm{M}_p(\vec{v}, -\vec{v})$, $1 < p \leq \infty$, coincide and are contained in $\mathrm{M}_1(\vec{v}, -\vec{v})$. Their intersection is the set of metric midpoints of the taxicab bisector of $\vec{v}$ and $-\vec{v}$.

Finally, if $0 < \phi < \frac{\pi}{4}$, we know from Theorem 3.2 that $\mathrm{M}_p(\vec{v}, -\vec{v}) \cap \mathrm{M}_q(\vec{v}, -\vec{v}) = \{\vec{0}, \vec{v}^\perp, -\vec{v}^\perp\}$ for all $1 < p < q < \infty$. It is easy to check that all three of $\vec{0}, \vec{v}^\perp$, and $-\vec{v}^\perp$ lie on the bisectors $\mathrm{M}_1(\vec{v}, -\vec{v})$ and $\mathrm{M}_\infty(\vec{v}, -\vec{v})$. To see that these are the only points common to both bisectors, we show that both $\mathrm{M}_1(\vec{v}, -\vec{v})$ and $\mathrm{M}_\infty(\vec{v}, -\vec{v})$ are piecewise linear curves. Indeed, if we write $\vec{v} = (x, y)$ and $\vec{v}^\perp = (-y, x)$ it follows that $\mathrm{M}_1(\vec{v}, -\vec{v})$ consists of the following three lines (see Figure 6), the line segment of taxicab metric midpoints, which is the the segment with endpoints $(-y, y)$ and $(y, -y)$, and the vertical rays starting at the points $(-y, y)$ and $(y, -y)$. On the other hand, $\mathrm{M}_\infty(\vec{v}, -\vec{v})$ consists of the line segment of max metric midpoints, which is the segment on the y-axis with endpoints $(0, (\alpha-1)x)$ and $(0, (1-\alpha)x)$ where $\alpha = \tan\phi$, and the rays with slope -1 starting at $(0, (1-\alpha)x)$ and $(0, (\alpha-1)x)$ (see Figure 6).

The segments of metric midpoints of $\mathrm{M}_1(\vec{v}, -\vec{v})$ and $\mathrm{M}_\infty(\vec{v}, -\vec{v})$ intersect at the origin, while the lines above the segment of metric midpoints intersect at $v^\perp$, and the ones below intersect at $-v^\perp$, as claimed.

4. Epilogue

The invitation is clear, take your favourite norm and wonder about geometric objects that you can get just by reading off the metric definition of them. What are their properties? Are they of any interest? Do you get common properties when norms are changed? Has anyone studied them? We ventured into this direction for the case of p-norms in the two dimensional space. Rest assured that there is still a lot more to get discovered and a lot of fun to be had.

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