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Angular Coordinates and Rational Maps

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Abstract. Associated with a triangle in the real projective plane are three standard transformations: inversion in the circumcircle, isogonal conjugation and antigonal conjugation. These are investigated in terms of angular and related coordinates, and are found to be part of a group of more general transformations. This group can be identified with a group of automorphisms of a real two-torus. The torus is in essence the surface obtained by starting with the projective plane, performing blowups on the three vertices, and then collapsing the triangle's circumcircle and the line at infinity. A conjecture concerning Hofstadter points is proved as an immediate consequence of this viewpoint.

Key Words: Trilinear coordinates, angular coordinates, birational transformation, group action, triangle center

MSC 2010: 51N20, 14E05, 20B27, 14P25

1. Introduction

Advancement in the understanding of triangle geometry has benefited greatly from the introduction of special coordinate systems. This is especially true for trilinear and barycentric coordinates; less known is the angular coordinate system associated with a triangle, though this notion has appeared in the literature for more than a century (cf. [3]). It is the intention of this article to motivate the usage of these latter coordinates in certain circumstances, and to investigate the consequences. Particular attention is paid to the singularities inherent in working with angular coordinates. Upon resolving these, a number of elegant results are easily revealed, some of which will be presented here.

For instance, our research explores a certain two-dimensional continuous group of transformations of the plane, which is shown to include some standard involutions, namely, isogonal conjugation, antigonal conjugation and inversion in the circumcircle. A discrete subgroup of this group, containing these three involutions, has an orbit consisting almost entirely of Hofstadter points and similar points. Moreover, the elimination of the singularities that naturally occur as a result of using angular coordinates produces, as a bonus, a demonstration of Dyck's famous result on the topological equivalence of two surfaces.

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A new coordinate system is also introduced in this article. These "tricyclic" coordinates arise as a simple reparameterization of angular coordinates. They have the advantage of being rationally related to trilinear and Cartesian coordinates; hence, plane transformations which are rational in Cartesian coordinates are rational in tricyclic coordinates, and algebro-geometric methods can be applied to eliminate singularities. As with trilinear coordinates, it is useful to study both an exact version and a homogeneous version of the tricyclic coordinate system.

Section 1 of this paper introduces basic concepts, notation and conventions used throughout the paper, along with some preliminary results. Section 2 explores further the concepts of angular and tricyclic coordinates. Section 3 looks at the three involutions mentioned earlier, and provides a rapid proof of a theorem that relates these, and which was previously proven by D. M. Bailey [2] and independently by J. Van Yzeren [11].

Section 4 identifies the singularities inherent in the usage of angular coordinates; in particular, points on the circumcircle cannot be distinguished by angular coordinates, and the triangle vertices have ill-defined angular coordinates. Algebro-geometric constructions meant to eliminate these issues are carried out in detail. Section 5 introduces the continuous group mentioned above as a group of automorphisms of the torus of angular coordinates; it is identified as the set of plane transformations satisfying a property naturally expressed within the framework introduced in Section 1.

1.1. Notation and conventions

Let A, B, and P be points in the plane. Define the *directed angle* $\measuredangle APB$ to be the angle through which the line $\overleftrightarrow{AP}$ can be rotated about P to coincide with the line $\overleftrightarrow{BP}$. The angle is signed, with positive values indicating counterclockwise rotation, and is only well-defined modulo π. Any equation involving directed angles should be considered modulo π.

We record some immediately observed properties of directed angles below:

Lemma 1.1. *Let A, B, C, and P be points in the plane. Then*

(i) $\measuredangle APB = -\measuredangle BPA$,

(ii) $\measuredangle BAC + \measuredangle CBA + \measuredangle ACB = 0$,

(iii) $\measuredangle APB + \measuredangle BPC + \measuredangle CPA = 0$.

The well-known theorem equating inscribed angles for a circle can be rewritten in terms of directed angles as follows:

Lemma 1.2. *Let A, B, P, and Q be points in the plane. Then A, B, P, and Q are concyclic if and only if $\measuredangle APB = \measuredangle AQB$ if and only if $\measuredangle PAQ = \measuredangle PBQ$.*

Proof. The key difference from the traditional inscribed angle theorem can be seen as follows: If P and Q are on opposite sides of a chord AB of a circle, then $\angle APB = \pi - \angle AQB$. But the directed angles $\measuredangle APB$ and $\measuredangle AQB$ must have opposite orientation in this case, so $\measuredangle APB = \pi + \measuredangle AQB = \measuredangle AQB$. □

We will fix a triangle ΔABC with circumcenter O, circumradius R, and with A, B, and C not collinear. The interior angles at A, B, and C will be denoted by θ_1, θ_2, and θ_3, respectively. We will write $L_i = R\sin\theta_i$ and $M_i = R\cos\theta_i$. Observe that L_i is half the length of its corresponding edge and M_i is the signed distance from O to the corresponding edge. This is shown in Figure 1.

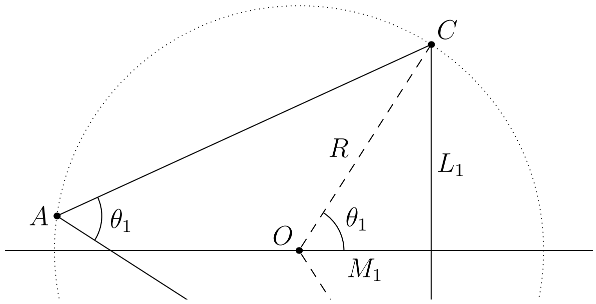

Figure 1: The quantities L_1 and M_1. For this triangle, $M_1 > 0$.

A subscript used to indicate an edge or vertex of ΔABC may be dropped when it can be understood from context. Subscripts may also be dropped in expressions that would be written the same way for each subscript. For example, $L = R\sin\theta$ means that $L_i = R\sin\theta_i$ for each i.

1.2. Angular and tricyclic coordinates

In [3, Chapter II] and [9], the *angular coordinates* for a point P inside a triangle ΔABC are given by the angles $\angle BPC$, $\angle CPA$, and $\angle APB$. In [11], directed angles are used instead. This has the advantage of providing a unique triple (ψ_1, ψ_2, ψ_3) for every point P not on the circumcircle, where

$$\psi_1 = \measuredangle BPC, \quad \psi_2 = \measuredangle CPA, \quad \psi_3 = \measuredangle APB.$$

An alternative geometric description of angular coordinates can be seen as follows: By Lemma 1.2, each circle through B and C can be uniquely identified by the directed angle $\measuredangle BPC$, where P is any point on the circle other than B and C. In other words, the set of circles through B and C is in one-to-one correspondence with the set of values for ψ_1, provided that the sideline is considered such a circle with infinite radius. The triple (ψ_1, ψ_2, ψ_3) can therefore be interpreted as specifying a configuration of three circles, one for each pair of vertices of ΔABC. That a point P has angular coordinates equal to this triple means that the three circles intersect at P.

The circles in the above geometric description will play a prominent role in what follows. For the sake of brevity, then, we will refer to any circle or line $\mathcal{C}$ through two vertices of ΔABC as a *Bailey circle*[1]. To be specific, if the two vertices are A and B, then we will say that $\mathcal{C}$ is a Bailey circle for the edge AB, and similarly for the other edges. As discussed, an individual Bailey circle can be specified using a directed angle ψ, and any configuration of three Bailey circles, one for each edge of ΔABC, can be specified by a triple (ψ_1, ψ_2, ψ_3).

Other quantities may be used in place of directed angles: Let $\mathcal{C}$ be a Bailey circle for an edge E and let X be its center. The position of X along the perpendicular bisector $E^\perp$ of E determines $\mathcal{C}$. Let c denote the directed distance from the circumcenter O to X, with the outward-pointing normal to E indicating the positive direction. This is shown in Figure 2. If X, Y, and Z are the centers of Bailey circles for edges BC, CA, and AB, respectively, then this configuration can be specified by the triple (c_1, c_2, c_3), with c_1, c_2, and c_3 the directed distances OX, OY, and OZ, respectively.

Remark 1.3. If the sidelines of ΔABC are to be considered Bailey circles of infinite radius, then the directed distance c described above should be considered as a value $[c : d]$ in $\mathbb{RP}^1$.

[1] in recognition of D. M. Bailey's investigations into these circles (cf. [2])

This will be done explicitly in Section 4, but elsewhere it will be restricted to $\mathbb{R}$.

Angular coordinates arise by specifying the three Bailey circles through P using the triple (ψ_1, ψ_2, ψ_3) of directed angles. Another coordinate system may be defined by specifying the same three Bailey circles using the triple (c_1, c_2, c_3). It will be convenient to use non-standard terminology and refer to this triple as the *exact tricyclic coordinates* of P. For any $\lambda \in \mathbb{R}^*$, we will refer to $(\lambda c_1 : \lambda c_2 : \lambda c_3)$ as the *homogeneous tricyclic coordinates* of P. We will see that, unlike angular coordinates, this coordinate system is rationally related to Cartesian, trilinear, and barycentric coordinates.

Remark 1.4. Note that

(i) Points not on the circumcircle have unique, well-defined angular and tricyclic coordinates, with the caveat that points on the sidelines have one infinite tricyclic coordinate.

(ii) Points on the circumcircle other than A, B, and C cannot be distinguished using angular or tricyclic coordinates, but the coordinate triples are well-defined.

(iii) If $P = A$, B, or C, then P does not have well-defined angular or tricyclic coordinates. There are infinitely many configurations of Bailey circles such that each Bailey circle passes through P.

We establish the relationship between c and ψ in Lemma 1.5 and Lemma 1.6.

Lemma 1.5. $\cot(\psi) = \dfrac{M - c}{L}$ *and* $\cot(\theta - \psi) = \dfrac{M - R^2\, c^{-1}}{L}$.

Proof. We prove the first assertion for the edge $E = BC$. Let $\mathcal{C}$ be a Bailey circle for BC with center X. Let P and Q be the two points at which $\mathcal{C}$ meets $E^{\perp}$, chosen so that P and X are on the same side of BC. This is shown in Figure 2.

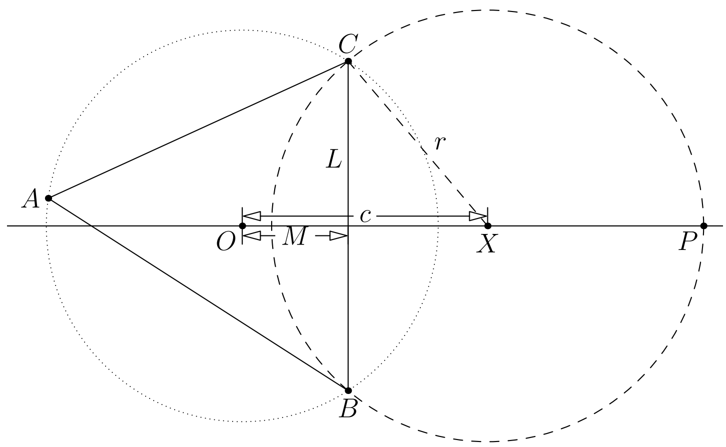

Figure 2: A Bailey circle with radius r.

By the inscribed angle theorem, $\angle BPC = \frac{1}{2}\angle BXC = \angle QXC$. Observe that the rotation of $\overleftrightarrow{PB}$ onto $\overleftrightarrow{PC}$ is a clockwise acute angle when $c > M$ and a counterclockwise acute angle when $c < M$, and similarly for the rotation of $\overleftrightarrow{XQ}$ onto $\overleftrightarrow{XC}$. Since $\angle BPC$ and $\angle QXC$ are oriented the same way,

$$\psi = \measuredangle BPC = \measuredangle QXC.$$

Now let $\sigma = \operatorname{sgn}(M - c)$ so that $\measuredangle QXC = \sigma \angle QXC$ and the absolute distance from X to BC is $\sigma(M - c)$. Then

$$\cot \measuredangle QXC = \sigma \cot \angle QXC = \sigma \frac{\sigma(M - c)}{L} = \frac{M - c}{L}.$$

For the other equality,

$$\cot(\theta - \psi) = \frac{\cot(\theta)\cot(\psi) + 1}{\cot(\psi) - \cot(\theta)} \cdot \frac{L^2}{L^2} = \frac{M(M-c) + L^2}{L(M-c) - LM} = \frac{R^2 - Mc}{-Lc}.$$

The last equality follows from the fact that $M^2 + L^2 = R^2$. ☐

Lemma 1.6. $c = R\dfrac{\sin(\psi - \theta)}{\sin(\psi)}$.

Proof. The result is obtained by replacing $L = R\sin\theta$ and $M = R\cos\theta$ in Lemma 1.5:

$$c = R\cos(\theta) - R\sin(\theta)\cot(\psi) = R\frac{\cos(\theta)\sin(\psi) - \sin(\theta)\cos(\psi)}{\sin(\psi)}.$$ ☐

2. Properties of angular and tricyclic coordinates

2.1. Exactness

Proposition 2.1. *If (ψ_1, ψ_2, ψ_3) are the angular coordinates of a point P, then*

$$\psi_1 + \psi_2 + \psi_3 = 0. \tag{2.1}$$

Proof. By definition, $\psi_1 + \psi_2 + \psi_3 = \measuredangle BPC + \measuredangle CPA + \measuredangle APB$. The right-hand side is 0 by Lemma 1.1. ☐

A partial converse is given in Proposition 2.3. Before proceeding, we establish a condition for neighboring Bailey circles to be tangent.

Lemma 2.2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be Bailey circles for edges BC and CA, and let ψ_1 and ψ_2 be their respective ψ-coordinates. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent if and only if $\psi_1 + \psi_2 = -\theta_3$.*

Proof. From Figure 2, one can deduce that

$$\measuredangle OCX = \measuredangle OCB + \measuredangle BCX = (\pi/2 - \theta_1) + (\pi/2 + \psi_1) = \psi_1 - \theta_1.$$

By a symmetrical argument, $\measuredangle OCY = -(\psi_2 - \theta_2)$. Therefore

$$\measuredangle YCX = \measuredangle YCO + \measuredangle OCX = (\psi_2 - \theta_2) + (\psi_1 - \theta_1) = \psi_1 + \psi_2 + \theta_3.$$

But $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent precisely when $\measuredangle YCX = 0$. ☐

Proposition 2.3. *Let ψ_1, ψ_2, and ψ_3 be any triple of directed angles such that*

$$\psi_1 + \psi_2 + \psi_3 = 0.$$

The configuration of Bailey circles given by (ψ_1, ψ_2, ψ_3) satisfies exactly one of the following:

(i) *All three Bailey circles are sidelines ($\psi_1 = \psi_2 = \psi_3 = 0$),*

(ii) *At least one Bailey circle is the circumcircle and the other two are tangent,*

(iii) *There is a common point of intersection P not on the circumcircle.*

The case that all three Bailey circles are the circumcircle ($\psi = \theta$) is a special case of (ii).

Proof. Clearly (i), (ii), and (iii) are each possible and are mutually exclusive. We will show they are exhaustive. L et $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ denote the Bailey circles in the configuration for the edges BC, CA, and AB, respectively.

Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at a point P not on the circumcircle. Let $\mathcal{K}$ be the circle APB and $\mathring{\psi}$ its ψ-coordinate. Then the angular coordinates of P are $(\psi_1, \psi_2, \mathring{\psi})$, so by Proposition 2.1, $\psi_1 + \psi_2 + \mathring{\psi} = 0$. Hence $\psi_3 = \mathring{\psi}$ and $\mathcal{C}_3 = \mathcal{K}$. This is case (iii).

If $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at a point on the circumcircle other than C, then at least one of $\mathcal{C}_1$ and $\mathcal{C}_2$ must be the circumcircle. Given the condition that $\psi_1 + \psi_2 + \psi_3 = 0$, Lemma 2.2 implies that two Bailey circles are tangent if and only if the other is the circumcircle. So this is case (ii).

Finally, if $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect only at C, then they are either tangent, which is case (ii), or they are both sidelines. If they are both sidelines, then $\psi_1 = \psi_2 = 0$. This implies $\psi_3 = 0$, which is case (i). □

Proposition 2.4. *If* (c_1, c_2, c_3) *are the exact tricyclic coordinates of a point* P*, then*

$$R(L_1c_1 + L_2c_2 + L_3c_3) = L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2. \tag{2.2}$$

Proof. It is sufficient to show that (2.2) is equivalent to (2.1). Let $\Theta = e^{i\theta}$, so that $R\Theta = M + iL$. Then

$$e^{2i\psi} = \frac{\cot\psi + i}{\cot\psi - i} = \frac{(M-c)+Li}{(M-c)-Li} = \frac{R\Theta - c}{R\overline{\Theta} - c}, \tag{2.3}$$

where the middle equality follows from Lemma 1.5. Now let

$$\xi = (R\Theta_1 - c_1)(R\Theta_2 - c_2)(R\Theta_3 - c_3).$$

Then

$$e^{2i(\psi_1+\psi_2+\psi_3)} = \frac{(R\Theta_1 - c_1)(R\Theta_2 - c_2)(R\Theta_3 - c_3)}{(R\overline{\Theta_1} - c_1)(R\overline{\Theta_2} - c_2)(R\overline{\Theta_3} - c_3)} = \xi/\overline{\xi}.$$

Therefore (2.1) holds if and only if $\xi/\overline{\xi} = 1$, or equivalently $\mathrm{Im}(\xi) = 0$.

Observe that $\Theta_1\Theta_2\Theta_3 = e^{i(\theta_1+\theta_2+\theta_3)} = -1$, so $\mathrm{Im}(\Theta_1\Theta_2\Theta_3) = 0$. Also, $\Theta_1\Theta_2 = -\overline{\Theta_3}$, so $\mathrm{Im}(\Theta_1\Theta_2) = \mathrm{Im}(\Theta_3) = L_3$; similarly, $\mathrm{Im}(\Theta_1\Theta_3) = L_2$ and $\mathrm{Im}(\Theta_2\Theta_3) = L_1$. Expanding ξ and extracting the imaginary part yields

$$\mathrm{Im}(\xi) = R(L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2) - R^2(L_1c_1 + L_2c_2 + L_3c_3).$$

Setting this equal to zero and rearranging yields (2.2). □

Corollary 2.5. *If* P *is not on the circumcircle and has homogeneous tricyclic coordinates* $(c_1 : c_2 : c_3)$*, then it has exact tricyclic coordinates* (Kc_1, Kc_2, Kc_3)*, where*

$$K = R\frac{L_1c_1 + L_2c_2 + L_3c_3}{L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2}. \tag{2.4}$$

2.2. Trilinear and cartesian coordinates

In order to establish the relationship between tricyclic and trilinear coordinates, it will be useful to first make an observation concerning antipedal triangles.

Lemma 2.6. *Let P be a point not on the circumcircle or sidelines of ΔABC. Let ΔXYZ be the triangle with vertices at the centers of the three Bailey circles through P. Then ΔXYZ is the antipedal triangle of ΔABC with respect to P, scaled by $\frac{1}{2}$.*

Proof. Let P be as in the statement of the lemma and let X and Y denote the centers of the circles BPC and CPA, respectively. Since PC is a chord of both of these circles, its perpendicular bisector passes through both X and Y. That is, the sideline $\overleftrightarrow{XY}$ is the perpendicular bisector of PC. The same reasoning shows that the sidelines $\overleftrightarrow{YZ}$ and $\overleftrightarrow{ZX}$ are the perpendicular bisectors of PA and PB, respectively. This is shown in Figure 3. □

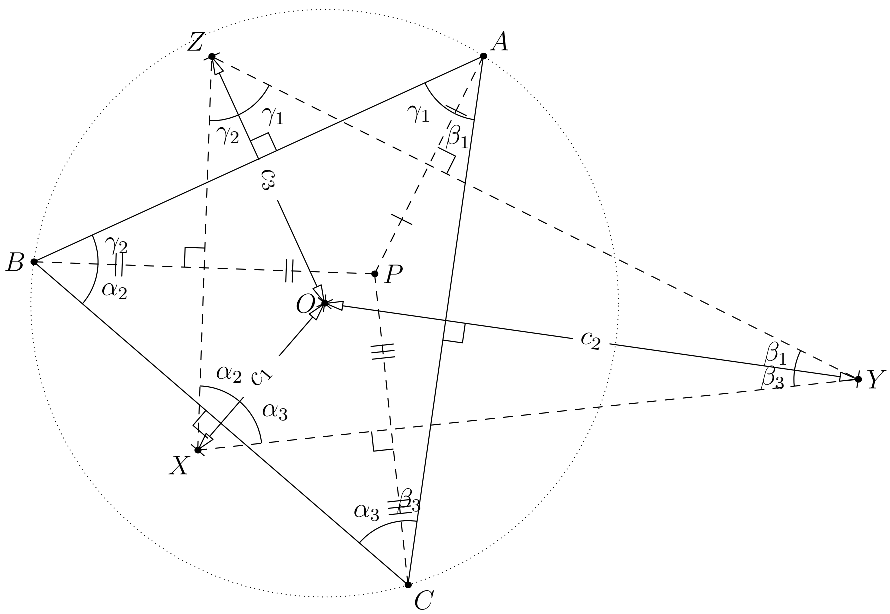

Figure 3: ΔXYZ is half the antipedal triangle of ΔABC with respect to P.

Proposition 2.7. *Let P be a point not on the sidelines of ΔABC. Let (ℓ_1, ℓ_2, ℓ_3) and (c_1, c_2, c_3) denote the exact trilinear and exact tricyclic coordinates, respectively, of P. Then for each i,*

$$2c_i\ell_i = -\mathcal{P}, \tag{2.5}$$

where $\mathcal{P} = |OP|^2 - R^2$ is the power of P for the circumcircle, and

$$c_i\ell_i = \frac{|\Delta ABC|}{L_1c_1^{-1} + L_2c_2^{-1} + L_3c_3^{-1}}. \tag{2.6}$$

Proof. If P is on the circumcircle, then (2.5) is clearly true. Now assume P is not on the circumcircle, and let ΔXYZ be the triangle formed by the centers of the Bailey circles induced by P, as in Figure 3. Observe that $\overrightarrow{PB} \cdot \overrightarrow{OX} = \ell_1 c_1$. Also, since $\overrightarrow{PB} \cdot (\overrightarrow{XP} + \overrightarrow{PB}/2) = 0$, it follows that $2\overrightarrow{PB} \cdot \overrightarrow{PX} = |PB|^2$. Therefore

$$\begin{aligned} 2c_1\ell_1 &= 2\overrightarrow{PB} \cdot (\overrightarrow{OP} + \overrightarrow{PX}) \\ &= 2\overrightarrow{PB} \cdot \overrightarrow{OP} + |PB|^2 \\ &= |\overrightarrow{OP} + \overrightarrow{PB}|^2 - |OP|^2 \\ &= R^2 - |OP|^2. \end{aligned}$$

The arguments for $c_2\ell_2$ and $c_3\ell_3$ are similar, so (2.5) holds.

Since

$$L_1\ell_1 + L_2\ell_2 + L_3\ell_3 = |\Delta ABC|,$$

it follows that

$$-\mathcal{P}L_1c_1^{-1} - \mathcal{P}L_2c_2^{-1} - \mathcal{P}L_3c_3^{-1} = 2|\Delta ABC|.$$

Solving for $\mathcal{P}$ yields (2.6). □

Applying Lemma 1.6, (2.5) can be rewritten as

$$2R\ell = -\mathcal{P}\frac{\sin(\psi)}{\sin(\psi - \theta)}.$$

This is proved in [3, Chapter II], which discusses the notion of "power", although the angular coordinates in [3] differ slightly from those defined here.

Corollary 2.8. *Let P be a point not on the circumcircle or sidelines of ΔABC. Let $(\ell_1 : \ell_2 : \ell_3)$ and $(c_1 : c_2 : c_3)$ denote the homogeneous trilinear and homogeneous tricyclic coordinates, respectively, of P. Then*

$$(\ell_1 : \ell_2 : \ell_3) = (c_1^{-1} : c_2^{-1} : c_3^{-1}) \tag{2.7}$$

Proof. This follows directly from Proposition 2.7. □

Corollary 2.9. *Let P be a point not on the circumcircle or sidelines of ΔABC. Suppose that the Cartesian coordinates of P are given by dehomogenizing $(x : y : z) \in \mathbb{RP}^2$ at $z = 1$, and that the Cartesian coordinates of A, B, and C are given by (x_1, y_1), (x_2, y_2), and (x_3, y_3), respectively. Let $(c_1 : c_2 : c_3)$ denote the homogeneous tricyclic coordinates of P. Then*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} L_1\, c_1^{-1} \\ L_2\, c_2^{-1} \\ L_3\, c_3^{-1} \end{pmatrix}. \tag{2.8}$$

Proof. From (2.7), the barycentric coordinates of P are

$$(L_1\, c_1^{-1} : L_2\, c_2^{-1} : L_3\, c_3^{-1}).$$

The result follows. □

Remark 2.10. Both Collorarie 2.8 and 2.9 can be extended to the case that P is on a sideline of ΔABC by considering each tricyclic coordinate as belonging to $\mathbb{RP}^1$ and scaling the homogeneous quantities in the typical way. This is discussed further in Section 4.

3. Transformations

Given a triangle ΔABC, we will say that a birational automorphism of the plane F *preserves Bailey circles* if, for any Bailey circle $\mathcal{C}$ for an edge of ΔABC, there is a Bailey circle $\mathcal{C}'$ for the same edge such that F restricts to a map $\mathcal{C} \dashrightarrow \mathcal{C}'$.

Each tricyclic coordinate specifies the Bailey circle for one edge of ΔABC. Hence, a map F preserves Bailey circles if and only if it is "diagonal" when written in tricyclic coordinates, in the sense that F can be expressed as

$$(c_1, c_2, c_3) \mapsto (f_1(c_1), f_2(c_2), f_3(c_3)) \tag{3.1}$$

for some functions f_i. The same principle holds for angular coordinates.

3.1. Three involutions

Antigonal conjugation, isogonal conjugation, and inversion in the circumcircle each preserve Bailey circles, and can therefore be expressed in the form of (3.1). The formulas in terms of angular coordinates are equivalent to the "characteristic equations" appearing in [11].

What we refer to here as antigonal conjugates are referred to as reflective points in [2], reflective conjugates in [4], and antigonal pairs in [11]: Let P be a point in the plane other than A, B, and C, and consider the three Bailey circles induced by P. Reflect each circle in its corresponding edge. The reflected circles will intersect in a unique point P' called the antigonal conjugate of P.

Proposition 3.1. *Antigonal conjugation is given in angular coordinates as*

$$(\psi_1, \psi_2, \psi_3) \mapsto (-\psi_1, -\psi_2, -\psi_3)$$

and in exact tricyclic coordinates as

$$(c_1, c_2, c_3) \mapsto (2M_1 - c_1, 2M_2 - c_2, 2M_3 - c_3)\,.$$

Proof. The first formula appears in [11].

Fix an edge E of ΔABC, let $\mathcal{C}$ be a Bailey circle for E, and let $\mathcal{C}'$ be its image under antigonal conjugation. Let c and c' be the coordinates of $\mathcal{C}$ and $\mathcal{C}'$, respectively. Since the midpoint of E is exactly between the centers of $\mathcal{C}$ and $\mathcal{C}'$, $\dfrac{c+c'}{2} = M$. □

Isogonal conjugation is defined as follows: Let P be a point in the plane. Reflect the lines $\overleftrightarrow{AP}$, $\overleftrightarrow{BP}$, and $\overleftrightarrow{CP}$ over the internal angle bisectors at A, B, and C, respectively. The reflected lines will intersect in a point P', called the isogonal conjugate of P.

Proposition 3.2. *Isogonal conjugation is given in angular coordinates as*

$$(\psi_1, \psi_2, \psi_3) \mapsto (-\psi_1 + \theta_1, -\psi_2 + \theta_2, -\psi_3 + \theta_3)$$

and in exact tricyclic coordinates as

$$(c_1, c_2, c_3) \mapsto \left(R^2\, c_1^{-1}, R^2\, c_2^{-1}, R^2\, c_3^{-1}\right).$$

Proof. The first formula appears in [11]. It follows by Lemma 1.6 that

$$c' = R\frac{(\psi' - \theta)}{(\psi')} = R\frac{\sin(-\psi)}{\sin(-\psi + \theta)} = \frac{\sin(\psi)}{\sin(\psi - \theta)} = \frac{R^2}{c}.$$ □

Remark 3.3. Proposition 3.2 implies that isogonal conjugation preserves Bailey circles. This can be seen directly as follows: Let P and P' be isogonal conjugates and let $\mathcal{C}$ and $\mathcal{C}'$ denote the circles ABP and ABP'. Moving P along $\mathcal{C}$ will rotate $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$ some common angle α about A and B, respectively. Hence the reflections $\overleftrightarrow{AP'}$ and $\overleftrightarrow{BP'}$ are rotated the common angle $-\alpha$, which has the effect of moving P' about the circle $\mathcal{C}'$.

Inversion scales the coordinates of a point P by $R^2/|P|^2$. It is well known that this transformation preserves the set of circles and lines in the plane, and it clearly fixes A, B, and C. Therefore it must also preserve Bailey circles.

Proposition 3.4. *Inversion in the circumcircle is given in angular coordinates as*

$$(\psi_1, \psi_2, \psi_3) \mapsto (-\psi_1 + 2\theta_1, -\psi_2 + 2\theta_2, -\psi_3 + 2\theta_3)$$

and in exact tricyclic coordinates as

$$(c_1, c_2, c_3) \mapsto \left(\frac{R^2\, c_1}{2M_1c_1 - R^2}, \frac{R^2\, c_2}{2M_2c_2 - R^2}, \frac{R^2\, c_3}{2M_3c_3 - R^2} \right).$$

Proof. The first formula appears in [11]. Since $\psi' = -\psi + 2\theta$ can be rearranged as $\theta - \psi' = -(\theta - \psi)$, it follows that $\cot(\theta - \psi') = -\cot(\theta - \psi)$. Therefore, by Lemma 1.5, $M - R^2/c' = M - R^2/c$. Solving for c' yields the second formula. □

3.2. Bailey's theorem and dihedral groups

The following theorem is not new, appearing as [2, Theorem 5] and [4, Theorem 13]. It is also proved in [11] by a technique equivalent to using angular coordinates. All subsequent results and proofs reported in this article are new.

Theorem 3.5. *Isogonal conjugation maps inverse points to antigonal conjugates.*

Proof. Let a denote antigonal conjugation, s isogonal conjugation, and v inversion. By Propositions 3.1, 3.2, and 3.4,

$$(s \circ v)(\psi) = s(-\psi + 2\theta) = -(-\psi + 2\theta) + \theta = \psi - \theta$$

and

$$(a \circ s)(\psi) = a(-\psi + \theta) = -(-\psi + \theta) = \psi - \theta.$$

Therefore $s \circ v = a \circ s$, which is equivalent to the statement of the theorem. □

The next theorem characterizes the group generated by isogonal conjugation and inversion.

Theorem 3.6. *Let s denote isogonal conjugation and v inversion. If the interior angles of ΔABC are rational multiples of π, written in lowest terms as $\theta_i = \pi k_i/n_i$, then*

(i) *The order of $v \circ s$ is $n = \operatorname{lcm}(n_1, n_2, n_3)$,*

(ii) *v and s generate the dihedral group of order $2n$.*

If the interior angles of ΔABC are not all rational multiples of π, then

(i) *$v \circ s$ has infinite order,*

(ii) *v and s generate the infinite dihedral group.*

Proof. By Propositions 3.2 and 3.4,

$$(v \circ s)(\psi_1, \psi_2, \psi_3) = (\psi_1 + \theta_1, \psi_2 + \theta_2, \psi_3 + \theta_3),$$

so in the first case,

$$(v \circ s)^n(\psi_1, \psi_2, \psi_3) = \left(\psi_1 + \pi\frac{nk_1}{n_1}, \psi_2 + \pi\frac{nk_2}{n_2}, \psi_3 + \pi\frac{nk_3}{n_3}\right).$$

The smallest n such that each $\pi nk_i/n_i$ is a multiple of π is $\operatorname{lcm}(n_1, n_2, n_3)$. The second case is clear, since if $n\theta_i$ is a multiple of π for some n, then θ_i is a rational multiple of π. □

Remark 3.7. Inversion may be replaced with antigonal conjugation in the statement of Theorem 3.6 with no significant change in the proof.

4. Resolution of singularities

Points in the plane are not in one-to-one correspondence with angular or tricyclic coordinate triples satisfying (2.1) and (2.2). In this section, we will modify the plane using algebro-geometric methods in order to obtain a surface whose points are in one-to-one correspondence with such triples. We will see that this surface is a torus, and that any birational automorphism of the plane that preserves Bailey circles is a regular automorphism of this torus, with no singularities.

As described in Subsection 1.2, triples of angular or tricyclic coordinates can be regarded as a geometric configuration of circles; the ambiguity of these coordinates for a point on the circumcircle corresponds to a degenerate configuration. Distinguishing degenerate geometric configurations using algebraic geometry has an extensive history; for example, Semple's study of the space of triangles in [10] utilizes ideas dating back to Schubert the 19th century.

The steps carried out in the rest of this section can be motivated as follows: First, points on the circumcircle other than A, B, and C all have the same representation, since the only Bailey circle through such points is the circumcircle. That is, $c_1 = c_2 = c_3 = 0$. This suggests that the circumcircle should be collapsed to a point.

Second, the vertices of ΔABC have ambiguous representation: If $P = A$, for example, then the Bailey circle for BC must be the circumcircle. But, as suggested by Proposition 2.3, any other two Bailey circles which are tangent at A will yield a configuration given by a triple (c_1, c_2, c_3) satisfying (2.2). The fact that there is one triple of exact coordinates for each line of tangency through A suggests that the plane should be blown up at A: This replaces A with its exceptional divisor $E_A \simeq \mathbb{RP}^1$, representing each direction through A.

We therefore carry out the following steps: First extend the plane to include the line at infinity and blow up each vertex of ΔABC. Then collapse the circumcircle to a point P_0 and the line at infinity to a point P_∞. Theorem 4.3 shows that the result is the desired surface.

4.1. Construction

We will consider homogeneous tricyclic coordinates explicitly as triples of values in $\mathbb{RP}^1$, each written as $[c : d]$. The value c in $\mathbb{R}$ corresponds to $[c : 1]$ in $\mathbb{RP}^1$. Denote $[1 : 0]$ by ∞ and $[0 : 1]$ by 0. Let T_c^3 be the 3-torus of all possible coordinate triples.

The surface of exact tricyclic coordinates given by (2.2) is defined on that subset $\mathbb{R}^3 \subset T_c^3$ where $d_1 = d_2 = d_3 = 1$. Its closure $\Sigma \subset T_c^3$ is given by the equation

$$R(L_1\, c_1d_2d_3 + L_2\, d_1c_2d_3 + L_3\, d_1d_2c_3) = L_1\, d_1c_2c_3 + L_2\, c_1d_2c_3 + L_3\, c_1c_2d_3.$$

The surface Σ is in fact a 2-torus. This can be seen as follows: First, the set of all triples (ψ_1, ψ_2, ψ_3) is a 3-torus T^3_ψ. The surface of angular coordinates $\Sigma_\psi \subset T^3_\psi$ is given by $\psi_1 + \psi_2 + \psi_3 = 0$, which is a 2-torus. Extending Lemma 1.6 as

$$[c : d] = [R \sin(\psi - \theta) : \sin(\psi)]$$

defines an isomorphism $T^3_c \xrightarrow{\sim} T^3_\psi$ which restricts to $\Sigma \xrightarrow{\sim} \Sigma_\psi$.

The conversion to Cartesian coordinates, (2.8), extends to a rational map $T^3_c \dashrightarrow \mathbb{RP}^2$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} L_1\, d_1\, c_2\, c_3 \\ L_2\, c_1\, d_2\, c_3 \\ L_3\, c_1\, c_2\, d_3 \end{pmatrix}. \tag{4.1}$$

This map is undefined only when $d_1c_2c_3 = c_1d_2c_3 = c_1c_2d_3 = 0$. This occurs only at (∞, ∞, ∞) and the lines $(-, 0, 0)$, $(0, -, 0)$, and $(0, 0, -)$.

Let Φ denote (4.1) restricted to Σ. Then Φ is undefined only at (∞, ∞, ∞) and $(0, 0, 0)$. Note that Φ does not extend to (∞, ∞, ∞) because, away from this point, $d_1 = 0$, $d_2 = 0$, and $d_3 = 0$ map to the sidelines BC, AC, and AB, respectively, which have no point in common. Similarly, Φ does not extend to $(0, 0, 0)$ because, away from this point, $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$ collapse to A, B, and C, respectively.

Let $\mathring{\Sigma} \subset \Sigma$ be given by removing (∞, ∞, ∞) and the curves $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Let $\mathring{\mathbb{RP}}^2 \subset \mathbb{RP}^2$ be given by removing the line at infinity and the circumcircle. The geometric definition of exact tricyclic coordinates shows that Φ restricts to an isomorphism $\mathring{\Sigma} \to \mathring{\mathbb{RP}}^2$.

Lemma 4.1. *Let $\widetilde{\mathbb{RP}^2}$ denote the blowup of $\mathbb{RP}^2$ at the vertices A, B, and C. Then Φ lifts to a rational map $\widetilde{\Phi}$ as in the following diagram:*

$$\begin{tikzcd} & \widetilde{\mathbb{RP}^2} \arrow[d] \\ \Sigma \arrow[ur, dashed, "\widetilde{\Phi}"] \arrow[r, dashed, "\Phi"'] & \mathbb{RP}^2 \end{tikzcd}$$

Like Φ, the map $\widetilde{\Phi}$ is undefined only at (∞, ∞, ∞) and $(0, 0, 0)$. Let $\mathring{Z}_1$ denote the curve $c_1 = 0$ minus the point $(0, 0, 0)$, and define $\mathring{Z}_2$ and $\mathring{Z}_3$ analogously. Let $\mathring{E}_A$ denote the exceptional divisor of A minus the direction tangent to the circumcircle, and define $\mathring{E}_B$ and $\mathring{E}_C$ analogously. Then whereas Φ collapses each $\mathring{Z}_i$ to a point, $\widetilde{\Phi}$ restricts to isomorphisms

$$\mathring{Z}_1 \xrightarrow{\sim} \mathring{E}_A, \quad \mathring{Z}_2 \xrightarrow{\sim} \mathring{E}_B, \quad \mathring{Z}_3 \xrightarrow{\sim} \mathring{E}_C.$$

Proof. There is clearly a rational map $\widetilde{\Phi} : \Sigma \dashrightarrow \widetilde{\mathbb{RP}^2}$ that is identical to Φ, excluding from the domain those points mapping to A, B, or C. From (4.1), it can be deduced that $\Phi^{-1}(A) = \mathring{Z}_1$, and similarly for B and C. It must be shown that $\widetilde{\Phi}$ extends to $\mathring{Z}_1$, mapping it isomorphically onto $\mathring{E}_A$, and similarly for $\mathring{Z}_2$ and $\mathring{Z}_3$. We will demonstrate the first assertion, the others being analogous.

To understand how $\widetilde{\Phi}$ behaves near $c_1 = 0$, consider the open subset of $\widetilde{\mathbb{RP}^2}$ which excludes E_B, E_C, and the line at infinity. This is identical to the original plane, minus B and C, blown up at $A = (x_1, y_1)$. Points in this blowup can be understood as consisting of a point

$P = (x, y)$ along with a direction through A, with the restriction that if $P \neq A$, the direction must coincide with $\overleftrightarrow{AP}$. By (4.1), the direction of $\overleftrightarrow{AP}$ is given by

$$[x - x_1 : y - y_1]^T = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} L_2\, d_2 c_3 \\ L_3\, c_2 d_3 \end{pmatrix}. \tag{4.2}$$

This clearly extends to $\mathring{Z}_1$, and we may consider (4.2) as a rational map $\mathring{Z}_1 \dashrightarrow \mathring{E}_A$. Now consider the parameterization of $c_1 = 0$ in Σ by the parameter $[u : v] \in \mathbb{RP}^1$:

$$\begin{aligned} [c_1 : d_1] &= [0 : 1], \\ [c_2 : d_2] &= [L_3^2\, u + L_2^2\, v : R^{-1} L_1 L_3\, u], \\ [c_3 : d_3] &= [L_3^2\, u + L_2^2\, v : R^{-1} L_1 L_2\, v]. \end{aligned}$$

Away from $[u : v] = [L_2^2 : -L_3^2]$, which corresponds to $(0, 0, 0)$, it is easily verified that along this parameterization, the quantity $[L_2\, d_2 c_3 : L_3\, c_2 d_3]$ appearing in (4.2) simplifies to $[u : v]$. Hence $\widetilde{\Phi}$ maps the point $[u : v]$ on $c_1 = 0$ to the direction $u\overrightarrow{AB} + v\overrightarrow{AC}$ through A. The missing direction $L_2^2\overrightarrow{AB} - L_3^2\overrightarrow{AC}$ is tangent to the circumcircle, as shown below. So (4.2) is in fact an isomorphism $\mathring{Z}_1 \xrightarrow{\sim} \mathring{E}_A$.

Let P be a point in the plane and suppose that $\overleftrightarrow{AP}$ is tangent to the circumcircle. Let u and v satisfy $\overrightarrow{AP} = u\overrightarrow{AB} + v\overrightarrow{AC}$. Then

$$0 = \overrightarrow{OA} \cdot \overrightarrow{AP} = u\, \overrightarrow{OA} \cdot \overrightarrow{AB} + v\, \overrightarrow{OA} \cdot \overrightarrow{AC},$$

so

$$[u : v] = [-\overrightarrow{OA} \cdot \overrightarrow{AC} : \overrightarrow{OA} \cdot \overrightarrow{AB}].$$

Now observe that

$$\begin{aligned} 4L_2^2 = |\overrightarrow{AC}|^2 &= (\overrightarrow{OC} - \overrightarrow{OA}) \cdot (\overrightarrow{OC} - \overrightarrow{OA}) \\ &= 2R^2 - 2\, \overrightarrow{OA} \cdot \overrightarrow{OC} \\ &= 2\, \overrightarrow{OA} \cdot (\overrightarrow{OA} - \overrightarrow{OC}) = -2\, \overrightarrow{OA} \cdot \overrightarrow{AC}. \end{aligned}$$

Similarly, $4L_3^2 = |\overrightarrow{AB}|^2 = -2\, \overrightarrow{OA} \cdot \overrightarrow{AB}$. Therefore $[u : v] = [L_2^2 : -L_3^2]$. □

Lemma 4.2. *Let $\widetilde{\Sigma}$ denote the blowup of Σ at (∞, ∞, ∞) and $(0, 0, 0)$. Then $\widetilde{\Phi}$ extends to an isomorphism $\widetilde{\Psi}$ as in the following diagram:*

$$\begin{tikzcd} \widetilde{\Sigma} \arrow[r, "\widetilde{\Psi}"] \arrow[d] & \widetilde{\mathbb{RP}^2} \\ \Sigma \arrow[ur, dashed, "\widetilde{\Phi}"'] & \end{tikzcd}$$

The exceptional divisors of (∞, ∞, ∞) and $(0, 0, 0)$ map isomorphically via $\widetilde{\Psi}$ onto the line at infinity and the proper transform of the circumcircle in $\widetilde{\mathbb{RP}^2}$, respectively.

Proof. Let E_∞ and E_0 denote the exceptional divisors of (∞,∞,∞) and $(0,0,0)$, respectively, in $\widetilde{\Sigma}$. Let $\widetilde{\Psi}:\widetilde{\Sigma} \dashrightarrow \widetilde{\mathbb{RP}^2}$ be the rational map which is undefined on E_∞ and E_0 but is otherwise identical to $\widetilde{\Phi}$. It must be shown that $\widetilde{\Psi}$ extends to E_∞ and E_0 as in the statement of the lemma.

First we determine how $\widetilde{\Psi}$ behaves near E_∞. Consider the open subset $V \subset T_c^3$ obtained by dehomogenizing at $c_i = 1$ for each i. Then $V \simeq \mathbb{R}^3_{(d_1,d_2,d_3)}$ and the point $(\infty,\infty,\infty) \in T_c^3$ is given by $\hat{0} \in V$. Let $\widetilde{V}$ denote the blowup of V at $\hat{0}$. That is, $\widetilde{V}$ is the set of

$$\big((d_1,d_2,d_3),\ [\beta_1:\beta_2:\beta_3]\big) \in V \times \mathbb{RP}^2$$

satisfying $[\beta_1:\beta_2:\beta_3] = [d_1:d_2:d_3]$ when $\hat{d} \neq \hat{0}$. The map $\widetilde{V} \dashrightarrow \mathbb{RP}^2$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} L_1\,\beta_1 \\ L_2\,\beta_2 \\ L_3\,\beta_3 \end{pmatrix} \tag{4.3}$$

agrees with (4.1) away from $\hat{d} = \hat{0}$. Now consider the intersection of Σ with V, defined by the equation

$$L_1d_1 + L_2d_2 + L_3d_3 - R(L_1d_2d_3 + L_2d_1d_3 + L_3d_1d_2) = 0.$$

The tangent plane to Σ at $\hat{0} \in V$ is given by $L_1d_1 + L_2d_2 + L_3d_3 = 0$, so E_∞ sits inside of $\{\hat{0}\} \times \mathbb{RP}^2 \subset \widetilde{V}$ as $L_1\beta_1 + L_2\beta_2 + L_3\beta_3 = 0$. Clearly (4.3) extends to E_∞, and in fact maps it isomorphically onto $z = 0$, the line at infinity. Moreover, this determines the extension of $\widetilde{\Psi}$ to E_∞, since the image of E_∞ under (4.3) does not include A, B, or C.

We now determine the behavior of $\widetilde{\Psi}$ near E_0. Consider the open subset $U \subset T_c^3$ obtained by dehomogenizing at $d_i = 1$ for each i. Then $U \simeq \mathbb{R}^3_{(c_1,c_2,c_3)}$ and the point $(0,0,0) \in T_c^3$ is given by $\hat{0} \in U$. Let $\widetilde{U}$ denote the blowup of U at $\hat{0}$, which is the set of

$$\big((c_1,c_2,c_3),\ [\alpha_1:\alpha_2:\alpha_3]\big) \in U \times \mathbb{RP}^2$$

satisfying $[\alpha_1:\alpha_2:\alpha_3] = [c_1:c_2:c_3]$ when $\hat{c} \neq \hat{0}$. The map $\widetilde{U} \dashrightarrow \mathbb{RP}^2$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} L_1\,\alpha_2\alpha_3 \\ L_2\,\alpha_1\alpha_3 \\ L_3\,\alpha_1\alpha_2 \end{pmatrix} \tag{4.4}$$

agrees with (4.1) away from $\hat{c} = \hat{0}$. The intersection of Σ with U is defined by the equation

$$L_1c_1 + L_2c_2 + L_3c_3 - R^{-1}(L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2) = 0.$$

The tangent plane to Σ at $\hat{0} \in U$ is given by $L_1c_1 + L_2c_2 + L_3c_3 = 0$, so E_0 sits inside of $\{\hat{0}\} \times \mathbb{RP}^2 \subset \widetilde{U}$ as $L_1\alpha_1 + L_2\alpha_2 + L_3\alpha_3 = 0$. Observe that (4.4) extends to E_0.

The image of E_0 under (4.4) includes A, B, and C, so to determine $\widetilde{\Psi}$, we must also determine a direction through A, B, and C. Again assuming $\hat{c} \neq \hat{0}$, (4.2) shows that the direction through A is given by

$$L_2\alpha_3\overrightarrow{AB} + L_3\alpha_2\overrightarrow{AC}. \tag{4.5}$$

This extends to E_0 as well, and directions through the other vertices can be determined similarly. We have therefore determined how $\widetilde{\Psi}$ extends to E_0, but it remains to be shown that it maps E_0 isomorphically onto the proper transform of the circumcircle.

By (2.6) and (2.5), assuming exact tricyclic coordinates,

$$R^2 - |OP|^2 = \frac{2\,|\Delta ABC|\,c_1c_2c_3}{L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2}.$$

For homogeneous tricyclic coordinates, each c_i must be scaled according to (2.4). This yields

$$R^2 - |OP|^2 = \frac{2R\,|\Delta ABC|\,c_1c_2c_3(L_1c_1 + L_2c_2 + L_3c_3)}{(L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2)^2}.$$

It follows that away from $\hat{0}$ in U, the surface $L_1c_1 + L_2c_2 + L_3c_3 = 0$ maps via (4.1) to the circumcircle. By continuity, $\widetilde{\Psi}$ must therefore map E_0 to the proper transform of the circumcircle. It remains to be seen that it is an isomorphism.

Observe that $[0 : -L_3 : L_2]$ is the only point of E_0 with $\alpha_1 = 0$, and therefore the only point of E_0 mapped by (4.4) to A. By (4.5), it is sent to the direction $L_2^2\overrightarrow{AB} - L_3^2\overrightarrow{AC}$ in the exceptional divisor of A which, as discussed in Lemma 4.1, is the direction tangent to the circumcircle. The same principle holds for the two points of E_0 with $\alpha_2 = 0$ and $\alpha_3 = 0$, respectively.

Finally, let $\mathring{E}_0$ denote E_0 minus the three points with $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$, and let $\mathring{\mathfrak{C}}$ denote the proper transform of the circumcircle minus the three points meeting the exceptional divisors of the vertices. All that remains to be shown is that the map $\mathring{E}_0 \to \mathring{\mathfrak{C}}$ induced by (4.4) is an isomorphism. This is true, since (4.4) is invertible away from $\alpha_1\alpha_2\alpha_3 = 0$. □

Theorem 4.3. *Suppose the line at infinity and the proper transform of the circumcircle in $\widetilde{\mathbb{RP}^2}$ are collapsed to points P_∞ and P_0, respectively, yielding a surface T. Then the isomorphism $\widetilde{\Psi}$ descends to an isomorphism Ψ as in the following diagram:*

$$\begin{tikzcd}
\widetilde{\Sigma} \arrow[r, "\widetilde{\Psi}"] \arrow[d] & \widetilde{\mathbb{RP}^2} \arrow[d] \\
\Sigma \arrow[r, "\Psi"] & T
\end{tikzcd}$$

The points (∞, ∞, ∞) and $(0, 0, 0)$ map via Ψ to P_∞ and P_0, respectively.

Proof. This follows immediately from Lemma 4.2. □

Remark 4.4. A slight modification of Theorem 4.3 illustrates Dyck's theorem: If E_∞ is collapsed to a point in $\widetilde{\Sigma}$, the result is the same as Σ blown up at one point; topologically, this is $T^2\#\mathbb{RP}^2$. If the line at infinity is collapsed to a point in $\widetilde{\mathbb{RP}^2}$, the result is the same as a sphere blown up at three points; topologically this is $\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$. The isomorphism $\widetilde{\Psi}$ descends to an isomorphism between these surfaces.

4.2. Features on the torus

The construction of the surface T is described topologically in Figures 4 and 5. The curves $d_i = 0$ in Σ map onto the sidelines of ΔABC. These are represented by dashed lines in Figures 4 and 5. The curves $c_i = 0$ in Σ map onto the exceptional divisors of A, B, and C. These are represented by solid lines in Figures 4 and 5.

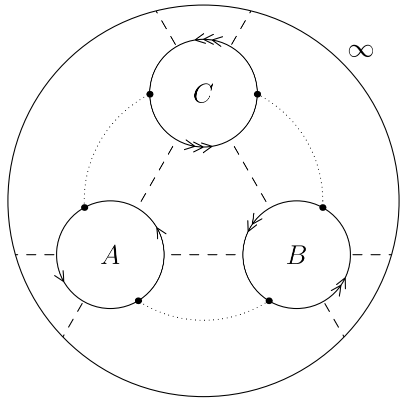

Figure 4: Blowing up at A, B, and C.

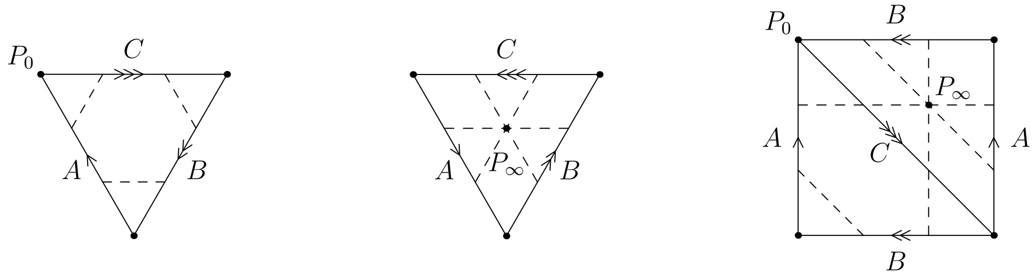

Figure 5: Inside and outside the circumcircle; the torus T.

Remark 4.5. Points in Σ are in one-to-one correspondence with configurations of Bailey circles (ψ_1, ψ_2, ψ_3) satisfying $\psi_1+\psi_2+\psi_3 = 0$. Such configurations were classified in Proposition 2.3: Case (i), in which all three Bailey circles are sidelines, corresponds to P_∞. Case (ii), in which one Bailey circle is the circumcircle and the other two are tangent, corresponds to the exceptional divisors of the vertices, shown as solid lines in Figures 4 and 5; the special case that all three Bailey circles are the circumcircle corresponds to their intersection P_0.

In Proposition 4.6 we summarize the coordinates for several points on T. The angular coordinates in this list agree with those found in [9]. Hofstadter points and related centers are investigated in Subsection 5.3. Other points, including Brocard points, isodynamic points, and isogonic centers, fit naturally into the perspective of Bailey circles and angular coordinates, but we omit these from the list.

Proposition 4.6. *Let H, O, and I denote the orthocenter, circumcenter, and incenter, respectively. The following table shows angular and exact tricyclic coordinates.*

	c	ψ
P_∞	∞	0
P_0	0	θ
I	R	$(\theta+\pi)/2$
O	$R^2/(2M)$	2θ
H	$2M$	$-\theta$

Proof. The coordinates for P_∞ and P_0 follow directly from Theorem 4.3. The angular coordinates for I, O, and H can be determined as follows:

The angular coordinate ψ_1 for I is $\measuredangle BIC = \pi - \theta_2/2 - \theta_3/2$. This is equal (modulo π, as usual) to $(\theta_1+\pi)/2$. The coordinates ψ_2 and ψ_3 can be deduced similarly.

The angular coordinate ψ_1 for O is $\measuredangle BOC$; by the inscribed angle theorem this is equal to $2\measuredangle BAC = 2\theta_1$. The other coordinates are similar. It is well known that H and O are isogonal conjugates; by Proposition 3.2 it follows that H has angular coordinates $\psi = -\theta$.

The exact tricyclic coordinates can now be determined using Lemma 1.6. □

Birational automorphisms of the plane which preserve Bailey circles can be viewed as automorphisms of the torus T. On the one hand, when viewed as acting on the plane,

(i) Antigonal conjugation is undefined at H and fixes the sidelines,

(ii) Isogonal conjugation is not defined on the circumcircle,

(iii) Inversion in the circumcircle is undefined at O and fixes the circumcircle.

On the other hand, when viewed as automorphisms of T, Propositions 3.1, 3.2, 3.4, and 4.6 show that

(i) Antigonal conjugation exchanges $H \leftrightarrow P_0$ and fixes P_∞,

(ii) Isogonal conjugation exchanges $P_0 \leftrightarrow P_\infty$,

(iii) Inversion in the circumcircle exchanges $O \leftrightarrow P_\infty$ and fixes P_0.

5. Angular translations and reflections

5.1. Characterization

Consider a point P with angular coordinates $(\alpha_1, \alpha_2, \alpha_3)$. Then the map

$$\psi \mapsto -(\psi - \alpha) + \alpha \tag{5.1}$$

fixes P. We will refer to this map as *angular reflection about* P. By Propositions 3.1, 3.2, and 3.4, we see that the following maps are angular reflections:

(i) Antigonal conjugation: Reflection about P_∞ $(\alpha = 0)$

(ii) Isogonal conjugation: Reflection about the incenter $(\alpha = (\theta + \pi)/2)$

(iii) Inversion in the circumcircle: Reflection about P_0 $(\alpha = \theta)$

Let ω_1, ω_2, and ω_3 be any triple satisfying $\omega_1 + \omega_2 + \omega_3 = 0$. The map

$$\psi \mapsto \psi + \omega \tag{5.2}$$

will be referred to as an *angular translation*. The transformations $v \circ s$ and $a \circ s$ (where a denotes antigonal conjugation, s isogonal conjugation, and v inversion) of Theorems 3.5 and 3.6 are angular translations.

The collection of angular reflections and angular translations forms a continuous group of automorphisms of the torus T described in Section 4. The dihedral group of Theorem 3.6 forms a discrete subgroup of this group. Each of these transformations can be viewed as a birational automorphism of the plane which preserves Bailey circles. In Theorem 5.1 we show that, in fact, the converse is true.

Theorem 5.1. *Any birational automorphism of the plane which preserves Bailey circles must be an angular reflection or an angular translation.*

Proof. Let F be any birational automorphism of the plane which preserves Bailey circles. Recall that such a map may be written in the form (3.1). Moreover, by (2.8), each f_i must be a birational map of one variable, hence a Möbius transformation. Without loss of generality, we may write the matrix defining f_i as $\begin{pmatrix} AR & BR^2 \\ C & DR \end{pmatrix}$ for some A_i, B_i, C_i, and D_i.

Let $z = e^{2i\psi}$ and $\Theta = e^{i\theta}$. By (2.3), z is equal to the Möbius transformation $\begin{pmatrix} -1 & R\Theta \\ -1 & R\overline{\Theta} \end{pmatrix}$ applied to c. Hence c is equal to the Möbius transformation $\begin{pmatrix} R\overline{\Theta} & -R\Theta \\ 1 & -1 \end{pmatrix}$ applied to z. It follows that f_i is given by the Möbius transformation in z with matrix

$$\begin{aligned} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \frac{1}{R^2} \begin{pmatrix} -1 & R\Theta \\ -1 & R\overline{\Theta} \end{pmatrix} \begin{pmatrix} AR & BR^2 \\ C & DR \end{pmatrix} \begin{pmatrix} R\overline{\Theta} & -R\Theta \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -(A\overline{\Theta} + B) + \Theta(C\overline{\Theta} + D) & (A\Theta + B) - \Theta(C\Theta + D) \\ -(A\overline{\Theta} + B) + \overline{\Theta}(C\overline{\Theta} + D) & (A\Theta + B) - \overline{\Theta}(C\Theta + D) \end{pmatrix}. \end{aligned} \tag{5.3}$$

Observe that $\beta = -\overline{\alpha} + \overline{\zeta}$, $\gamma = \alpha - \zeta$, and $\delta = -\overline{\alpha}$, where $\zeta = (\Theta - \overline{\Theta})(C\overline{\Theta} + D)$. Moreover, $\zeta \neq 0$ since θ is not a multiple of π.

The transformation F must map exact triples to exact triples. Exactness is equivalent to $z_1 z_2 z_3 = e^{2i(\psi_1+\psi_2+\psi_3)} = 1$. Similarly, $z'_1 z'_2 z'_3 = 1$, so

$$1 = z'_1 z'_2 z'_3 = \frac{(\alpha_1 z_1 + \beta_1)(\alpha_2 z_2 + \beta_2)(\alpha_3 z_3 + \beta_3)}{(\gamma_1 z_1 + \delta_1)(\gamma_2 z_2 + \delta_2)(\gamma_3 z_3 + \delta_3)}.$$

Therefore

$$(\alpha_1 z_1 + \beta_1)(\alpha_2 z_2 + \beta_2)(\alpha_3 z_3 + \beta_3) - (\gamma_1 z_1 + \delta_1)(\gamma_2 z_2 + \delta_2)(\gamma_3 z_3 + \delta_3) = 0.$$

Expanding, replacing $z_3 = 1/(z_1 z_2)$, and multiplying through by $z_1 z_2$ yields

$$\begin{gathered} \big[(\alpha_1\alpha_2\alpha_3 - \gamma_1\gamma_2\gamma_3) \;+\; (\beta_1\beta_2\beta_3 - \delta_1\delta_2\delta_3)\big]\, z_1 z_2 \;+ \\ [\alpha_1\alpha_2\beta_3 - \gamma_1\gamma_2\delta_3]\, z_1^2 z_2^2 \;+\; [\alpha_1\beta_2\alpha_3 - \gamma_1\delta_2\gamma_3]\, z_1 \;+ \\ [\alpha_1\beta_2\beta_3 - \gamma_1\delta_2\delta_3]\, z_1^2 z_2 \;+\; [\beta_1\alpha_2\alpha_3 - \delta_1\gamma_2\gamma_3]\, z_2 \;+ \\ [\beta_1\alpha_2\beta_3 - \delta_1\gamma_2\delta_3]\, z_1 z_2^2 \;+\; [\beta_1\beta_2\alpha_3 - \delta_1\delta_2\gamma_3] \;=\; 0. \end{gathered}$$

That is, the polynomial on the left-hand side must vanish whenever $|z_1| = |z_2| = 1$. But this implies that the polynomial is identically zero, so each quantity in square brackets must be zero. In particular,

$$\begin{aligned} (\overline{\alpha_1} - \overline{\zeta_1})\alpha_2\alpha_3 &= \overline{\alpha_1}(\alpha_2 - \zeta_2)(\alpha_3 - \zeta_3), \\ \alpha_1(\overline{\alpha_2} - \overline{\zeta_2})\alpha_3 &= (\alpha_1 - \zeta_1)\overline{\alpha_2}(\alpha_3 - \zeta_3), \\ \alpha_1\alpha_2(\overline{\alpha_3} - \overline{\zeta_3}) &= (\alpha_1 - \zeta_1)(\alpha_2 - \zeta_2)\overline{\alpha_3}. \end{aligned}$$

Observe that if $\alpha_i = 0$ for any i, then $\alpha_i = 0$ for all i. Similarly, if $\alpha_i = \zeta_i$ for any i, then $\alpha_i = \zeta_i$ for all i. In the first case, the matrix (5.3) is $\begin{pmatrix} 0 & \overline{\zeta} \\ -\zeta & 0 \end{pmatrix}$. In the second case, the matrix (5.3) is $\begin{pmatrix} \zeta & 0 \\ 0 & -\overline{\zeta} \end{pmatrix}$. These are angular reflection and angular translation, respectively.

Finally, suppose $\alpha_i \neq 0, \zeta_i$ for any i. Let $\xi = (\alpha - \zeta)/\alpha$. Then the equations above can be rewritten as

$$\overline{\xi_1} = \xi_2\xi_3, \quad \overline{\xi_2} = \xi_1\xi_3, \quad \overline{\xi_3} = \xi_1\xi_2.$$

Therefore $\xi_3 = \overline{\xi_1\xi_2} = \overline{\xi_1}\xi_1\xi_3$. Since $\xi_3 \neq 0$, it follows that $|\xi_1| = 1$. Repeating this argument yields

$$|\xi_1| = |\xi_2| = |\xi_3| = 1.$$

Hence $|\alpha| = |\alpha - \zeta|$, so there exists ρ such that $|\rho| = 1$ and $\alpha - \zeta = \rho\alpha$. The matrix (5.3) is therefore $\begin{pmatrix} \alpha & -\overline{\alpha\rho} \\ \alpha\rho & -\overline{\alpha} \end{pmatrix}$. But this is the constant map $z \mapsto \overline{\rho}$, which is impossible. □

5.2. Angular reflections

Inversion in the circumcircle exchanges the regions inside and outside the circumcircle. In [2, Theorem 6], it is proved that antigonal conjugation also exchanges two regions, with the sidelines acting as boundaries between these. These are special cases of a more general phenomenon: Inversion and antigonal conjugation are both angular reflections; we will show that every angular reflection exchanges two regions of the plane.

The torus of angular coordinates is shown in Figure 6. The vertical lines have fixed ψ_1 value, the horizontal lines have fixed ψ_2 value, and the diagonal lines have fixed ψ_3 value (since $\psi_1 + \psi_2 + \psi_3 = 0$). In terms of the isomorphism given in Theorem 4.3, the features of the diagram map to the modified plane in the following way: The dashed lines through P_∞ correspond to the sidelines of ΔABC; the dotted lines through P_0 correspond to the exceptional divisors of A, B, and C (hence these lines collapse to the vertices in the original plane); and the solid lines correspond to the Bailey circles through P.

When viewed as acting on the $\psi_1\psi_2$-plane, angular reflection about P simply reflects each point through P or, equivalently, rotates about P through an angle of π. This exchanges the shaded and unshaded regions shown in Figure 6. These two regions correspond to two regions in the original plane; angular reflection about P exchanges these.

Let $\mathcal{R}_T$ denote the shaded region of the torus shown in Figure 6, and let $\mathcal{R}$ denote the corresponding region in the plane. To understand what $\mathcal{R}$ looks like, observe that containment in $\mathcal{R}_T$ flips exactly when a solid line is crossed. Hence, containment in $\mathcal{R}$ flips exactly when a Bailey circle for P is crossed.

The dotted lines correspond to the exceptional divisors of the vertices, so each point on a dotted line can be interpreted as an infinitesimal line segment through a vertex in the plane. The points of intersection with solid lines correspond to those infinitesimal line segments tangent to a Bailey circle for P. Following along a dotted line in the torus, containment in $\mathcal{R}_T$ flips exactly when a solid line is crossed. Hence, as an infinitesimal line segment through a vertex is rotated, containment in $\mathcal{R}$ flips exactly when it passes through a direction tangent to a Bailey circle for P.

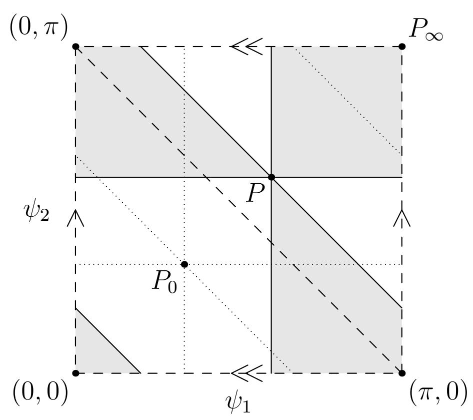

Figure 6: Two regions on the torus of angular coordinates.

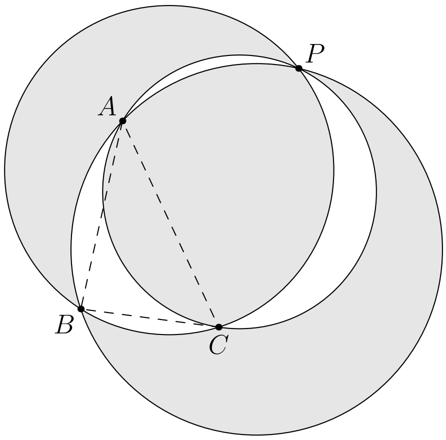

Figure 7: Angular reflection about P exchanges the shaded and unshaded regions.

This observation is summarized in Theorem 5.2, leaving reflection about P_0 and about P_∞ as special cases. These cases correspond to inversion in the circumcircle and antigonal conjugation, respectively. The fact that $\mathcal{R}$ is the region inside the circumcircle in the case $P = P_0$ is illustrated by Figures 4 and 5. A general example is shown in Figure 7.

Theorem 5.2. *Let P be a point not on the circumcircle.*
Let $\mathcal{B}$ denote the union of the three Bailey circles for P. Each Bailey circle divides the plane into two open regions; let $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ denote one of those regions for each Bailey circle. Finally, let $\mathcal{R}$ (respectively, $\mathcal{R}'$) denote the regions consisting of those points not in $\mathcal{B}$ which are contained in an even (respectively, odd) number of $\mathcal{D}_i$.
Then angular reflection about P maps $\mathcal{R}$ to $\mathcal{R}'$ and vice-versa.

5.3. Hofstadter points

In [6], the Hofstadter r-point H_r is defined as follows for $r \neq 0, 1$: Rotate the sideline $\overleftrightarrow{BC}$ counterclockwise about B through an angle of $r\theta_2$ and clockwise about C through an angle of $r\theta_3$. The resulting lines intersect at a point A'. Construct B' and C' analogously. Then $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are concurrent, and H_r is defined as their intersection. This point is shown to have homogeneous trilinear coordinates

$$\ell = \frac{\sin(r\theta)}{\sin(r\theta - \theta)}. \tag{5.4}$$

The construction of H_r can be modified slightly: Rotate the sidelines through an angle of $r\theta + \pi/2$ instead of $r\theta$, then continue the construction as before. It is easily verified that the proof appearing in [6], that $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are concurrent and intersect in a point with trilinear coordinates given by (5.4), applies virtually unchanged to this modified construction. Thus the resulting point, which we denote by $H_r^\perp$, exists and has homogeneous trilinear coordinates

$$\ell = \frac{\sin(r\theta + \pi/2)}{\sin((r\theta + \pi/2) - \theta)} = \frac{\cos(r\theta)}{\cos(r\theta - \theta)}. \tag{5.5}$$

We now determine the angular coordinates of H_r and $H_r^\perp$.

Lemma 5.3. *H_r and $H_r^\perp$ have angular coordinates $\psi = r\theta$ and $\psi = r\theta + \pi/2$, respectively.*

Proof. Let P_r and $P_r^{\perp}$ be the points with angular coordinates as in the statement of the lemma. By Lemma 1.6, P_r has exact tricyclic coordinates

$$c = R\frac{\sin(r\theta - \theta)}{\sin(r\theta)}$$

and $P_r^{\perp}$ has exact tricyclic coordinates

$$c = R\frac{\sin((r\theta + \pi/2) - \theta)}{\sin(r\theta + \pi/2)} = R\frac{\cos(r\theta - \theta)}{\cos(r\theta)}.$$

By (2.7), it follows that P_r and $P_r^{\perp}$ have trilinear coordinates given by (5.4) and (5.5), respectively. So $H_r = P_r$ and $H_r^{\perp} = P_r^{\perp}$. □

The following theorem appears as a conjecture of Randy Hutson in entry $X(360)$ and $X(5961)$ of [8]. Part (ii) of the theorem is already known, and can be found in [6] without the restriction on r (that is, it is in fact true that H_0 and H_1 are isogonal conjugates).

Theorem 5.4. *Let H_r denote the Hofstadter r-point. Then*

(i) *The inverse-in-circumcircle of H_r is H_{2-r} when $r \neq 0, 1, 2$,*
(ii) *The isogonal conjugate of H_r is H_{1-r} when $r \neq 0, 1$,*
(iii) *The antigonal conjugate of H_r is H_{-r} when $r \neq -1, 0, 1$.*

Proof. Let $r \neq 0, 1$. By Lemma 5.3, H_r has angular coordinates $\psi = r\theta$. By Proposition 3.4, the inverse of H_r has angular coordinates

$$\psi = -r\theta + 2\theta = (2 - r)\theta.$$

By Proposition 3.2, the isogonal conjugate of H_r has angular coordinates

$$\psi = -r\theta + \theta = (1 - r)\theta.$$

By Proposition 3.1, the antigonal conjugate of H_r has angular coordinates

$$\psi = -r\theta.$$

By Lemma 5.3, the right-hand sides match the angular coordinates of H_{2-r}, H_{1-r}, and H_r, respectively, provided the listed constraints on r are observed. □

Let h denote the angular translation given by $\psi \mapsto \psi + (\pi + \theta)/2$. Let $\rho = h^2$. Observe that ρ can be taken as the rotation in the dihedral group of Theorem 3.6, and is given by isogonal conjugation followed by inversion.

Theorem 5.5. *The maps h and $\rho = h^2$ act on the points H_r and $H_{r+1/2}^{\perp}$ as in Figure 8.*

Proof. As described in Theorem 4.3, the point P_∞ has angular coordinates $\psi = 0$.

Hence $h^n(P_\infty)$ has angular coordinates $\psi = (n/2)(\pi + \theta)$. When $n = 2r$, this becomes $\psi = r\theta$. Therefore $h^{2r}(P_\infty) = H_r$, provided that $r \neq 0, 1$. Similarly, when $n = 2r + 1$, $\psi = \left(r + \frac{1}{2}\right)\theta + \frac{\pi}{2}$. So $h^{2r+1}(P_\infty) = H_{r+1/2}^{\perp}$. □

Remark 5.6. Some of the points in Figure 8 are labeled according to their designation in [8]; other points in the sequence with such a designation include:

$$H_{-4} = X(5964),\ H_{-3} = X(5962),\ H_4 = X(5961),\ H_5 = X(5963).$$

To the authors' knowedge, this exhausts the list of points in the sequence appearing in [8].

Remark 5.7. The points H_0 and H_1 are missing from the diagram in Theorem 5.5, and these are exactly the "special" Hofstadter points which are obtained as a limit of other Hofstadter points as $r \to 0$ and $r \to 1$.

$$
\begin{tikzcd}[column sep=tiny]
 & X(265) \arrow[d, equal] & & H \arrow[d, equal] & & & & & & O \arrow[d, equal] & & X(186) \arrow[d, equal] & \\
\cdots \arrow[r, dashed] & H_{-2} \arrow[rr, mapsto, "\rho"] \arrow[dr, mapsto, "h"'] & & H_{-1} \arrow[rr, mapsto, "\rho"] \arrow[dr, mapsto, "h"'] & & P_\infty \arrow[rr, mapsto, "\rho"] \arrow[dr, mapsto, "h"'] & & P_0 \arrow[rr, mapsto, "\rho"] \arrow[dr, mapsto, "h"'] & & H_2 \arrow[rr, mapsto, "\rho"] \arrow[dr, mapsto, "h"'] & & H_3 \arrow[r, mapsto] & \cdots \\
 & \cdots \arrow[r, dashed] & H^\perp_{-1.5} \arrow[rr, mapsto, "\rho"'] \arrow[ur, mapsto, "h"] & & H^\perp_{-0.5} \arrow[rr, mapsto, "\rho"'] \arrow[ur, mapsto, "h"] \arrow[d, equal] & & H^\perp_{0.5} \arrow[rr, mapsto, "\rho"'] \arrow[ur, mapsto, "h"] \arrow[d, equal] & & H^\perp_{1.5} \arrow[rr, mapsto, "\rho"'] \arrow[ur, mapsto, "h"] \arrow[d, equal] & & H^\perp_{2.5} \arrow[r, mapsto] \arrow[ur, mapsto, "h"] & \cdots & \\
 & & & & X(80) & & I & & X(36) & & & &
\end{tikzcd}
$$

Figure 8: The maps h and ρ acting on Hofstadter points.

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