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Plastic Number and Origami

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Abstract. This paper proposes a simple folding algorithm for the construction of VAN DER LAAN's plastic number using origami. Furthermore, it is shown how the algorithm can be slightly modified to produce ROSENBUSCH's "cubi ratio", a number which can be defined analogously to the plastic number.

Key Words: plastic number, cubi ratio, origami

MSC 2010: 51M04, 51M15

1. Introduction

In search for a suitable and original design tool, H. VAN DER LAAN[1] invented a spatial generalization of the *golden rectangle*, i.e., a rectangle that can be decomposed into a similar rectangle and a square. It can be interpreted as one possible solution to the following problem.

Problem 1. *Find a cuboid C which can be decomposed into a similar cuboid C' and a square cuboid.*

VAN DER LAAN's version of C is shown in Figure 1, left (the cuboid C' is shadowed out). The following equations hold for its side lengths a, b and c:

$$\frac{a}{b} = \frac{b}{c} = \psi, \tag{1}$$

where $\psi = 1.324717\ldots$, called the *plastic number* (see [6] and [7]), is the unique real solution to the cubic equation

$$x^3 = 1 + x. \tag{2}$$

From (1) and (2) follows

$$\frac{a-c}{c} = \frac{a}{c} - 1 = \frac{a}{b} \cdot \frac{b}{c} - 1 = \psi^2 - 1 = \psi^{-1},$$

which proves the similarity of C and C'.

[1] Dom Hans VAN DER LAAN (1904–1991), a Dutch architect.

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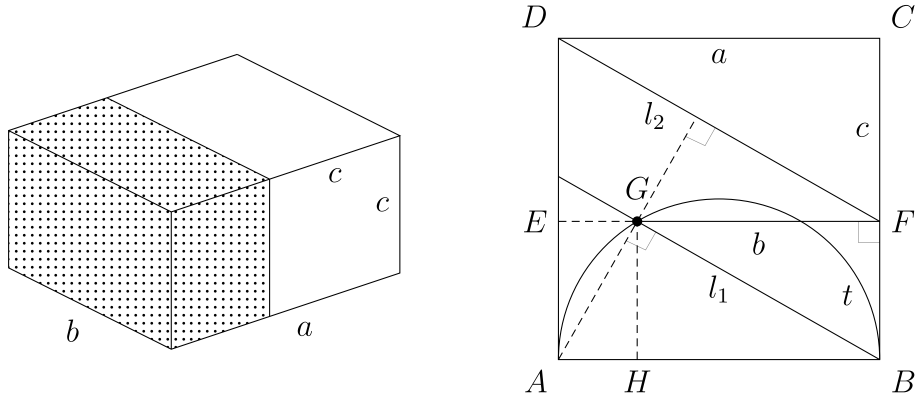


Figure 1: Van der Laan's generalization of the golden rectangle

According to [2], the side lengths of the cuboid C can be dynamically constructed from a square $ABCD$ with side length a by drawing a Thales circle t above $\overline{AB}$ and two parallel lines l_1 and l_2 passing through B and D, respectively, such that l_2 intersects $\overline{BC}$ at a point which coincides with the projection F of the intersection point G of l_1 and t on the edge $\overline{BC}$ (see Figure 1, right). Under these conditions we have $|FG| = b$ and $|CF| = c$.

Point G and its projections E, F and H define the *ψ-decomposition* of the square $ABCD$ into three similar rectangles $AHGE$, $HBFG$ and $EFCD$.

A folding algorithm for the construction of a line segment of length ψ using origami is provided in [8]. The goal of this paper is to propose a simple folding algorithm, implied in [2], for the construction of the point G, from which the ψ-decomposition and the ψ-ratio itself can be easily obtained. In the rest of this text, G will be called the *center* of the ψ-decomposition.

2. Axioms of paper folding

This section covers some preliminaries necessary for understanding the folding process which will be described in the next section. Folds required by that process are obtained using Huzita's[2] axioms (see [4]). There are six Huzita axioms in total[3], but in this paper only three axioms, as stated below, are needed.

Axiom 1 (Huzita's third axiom)**.** *Given two straight lines l_1 and l_2, there is a fold that places l_1 onto l_2.*

Axiom 2 (Huzita's fourth axiom)**.** *Given a point P and a straight line l, there is a fold perpendicular to l that passes through P.*

Axiom 3 (Huzita's sixth axiom)**.** *Given two points P_1, P_2 and two straight lines l_1, l_2, there is a fold that simultaneously places P_1 onto l_1 and P_2 onto l_2.*

The most important is Axiom 3, which is also the most complicated one. It was originally discovered by Margherita P. Beloch[4] in 1936, who used it as a tool for solving cubic equations. For an outline of her work related to paper folding see [3].

[2]Humiaki Huzita (1924–2005), a Japanese-Italian mathematician and origami artist.

[3]Actually, the complete set of origami folding axioms contains, beside Huzita's, one more axiom, discovered independently by Jacques Justin and Koshiro Hatori (see [1]).

[4]Margherita Piazzolla Beloch (1879–1986), an Italian mathematician.

2.1. Beloch fold

An interesting geometric interpretation of Axiom 3, found by Beloch, is based on the following Proposition.

Proposition 1. *Given a point P and a straight line l, a fold that places P onto l is necessary a tangent line to the parabola determined by the focus P and the directrix l.*

Proof. Without loss of generality it can be assumed that $P = (0, d)$ for some $d > 0$ and $l = \{(x, y) \mid y = 0\}$. Every other possible placement of these objects is therefore achievable by a single rotation and/or translation.

An arbitrary point (x, y) of the implied parabola, determined by the pair (P, l) of its focus and directrix, respectively, is by definition equidistant from these objects. Therefore, the implicit equation of that parabola is

$$|y| = \sqrt{x^2 + (y - d)^2}.$$

Squaring both sides of the above equation and simplifying rational expression yields its explicit form

$$f(x) = y = \frac{x^2 + d^2}{2\,d}. \tag{3}$$

A fold that places P onto l is essentially an axis of symmetry l_s that reflects point P onto some point $Q = (x_0, 0) \in l$. That line is the perpendicular bisector of the line segment $\overline{PQ}$, therefore passes through its midpoint $\left(\frac{x_0}{2}, \frac{d}{2}\right)$ (see Figure 2). The slopes of the lines PQ and l_s are negative reciprocals of each other. Since the slope of PQ is equal to $-\frac{d}{x_0}$, the slope of l_s equals $\frac{x_0}{d}$. Hence l_s is determined by the following equation:

$$g(x) = \frac{x_0}{d}\left(x - \frac{x_0}{2}\right) + \frac{d}{2}. \tag{4}$$

By subtracting the equations (3) and (4) one easily obtains

$$f(x) - g(x) = \frac{(x - x_0)^2}{2\,d},$$

which means that $f(x) = g(x)$ if and only if $x = x_0$. Since g is bijective, there exists exactly one intersection $T = (x_0, y_0)$ of the curves represented by the equations (3) and (4) (see Figure 2).

Differentiating equation (3) yields

$$f'(x) = \frac{x}{d},$$

which implies that the slope of l_s is equal to $f'(x_0)$. Therefore, l_s is the tangent line to the given parabola at point T, what completes the proof. □

Corollary 1 (Beloch fold)**.** *Given two points P_1, P_2 and two lines l_1, l_2, a fold that places P_1 onto l_1 and P_2 onto l_2 is a simultaneous tangent to two parabolas determined by the focus-directrix pairs (P_1, l_1) and (P_2, l_2).*

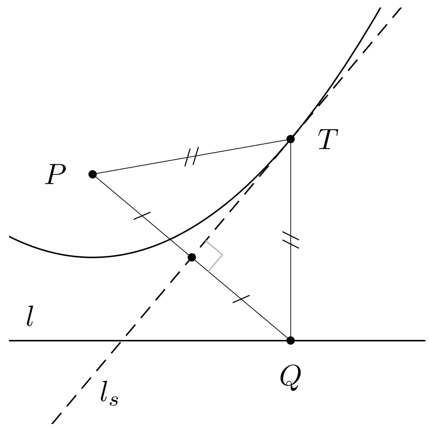


Figure 2: Beloch fold

3. A folding algorithm producing the center of a ψ-decomposition

A square sheet of paper (i.e., origami paper) contains a point being the center of a ψ-decomposition. One possible folding algorithm for constructing that point, divided into three steps each one utilizing one of Huzita's axioms, is shown below.

Step 1. Axiom 1 implies a fold that places the left edge of the paper onto the right edge. The resulting crease divides the piece of paper into two rectangular halves. Folding the longer sides of each of these rectangles on top of each other gains two more creases parallel to the first one. Finally, there is a fold that places the lower edge of the paper onto the upper edge, producing a crease perpendicular to the previous three.
Let l_0 denote the leftmost vertical crease, P_1 the intersection of l_0 with the upper edge of the paper and P_2 the vertex of the square which is nearest to P_1. Furthermore, let Q and R denote the upper and lower right vertices, respectively (see Figure 3a).

Step 2. Let l_1 and l_2 denote the rightmost vertical and the horizontal crease, respectively (see Figure 3b). Axiom 3 implies a fold that simultaneously places point P_1 onto the crease l_1 and point P_2 onto the crease l_2 (see Figure 3b).

Step 3. Let l_3 be the crease obtained in the previous step. Axiom 2 implies a fold that passes through R and is perpendicular to l_3, yielding the crease l_4. Similarly, there is a fold that passes through Q and is perpendicular to l_4, producing the final crease l_5 (see Figure 3c). That completes the algorithm.

Proposition 2. *The intersection of the creases l_4 and l_5, as shown in Figure 3c, is the center of a ψ-decomposition of the corresponding square.*

Proof. Corollary 1 implies that the line l_3 is a common tangent to the parabolas defined by the focus-directrix pairs (P_1, l_1) and (P_2, l_2). In order to obtain equations defining these curves, the creases and points obtained in the above algorithm need to be interpreted in the context of a coordinate system. Let such system be defined with the abscissa P_2Q, the ordinate l_1 and the unit length $|P_1P_2| = 1$. Under these conditions, $P_1 = (-2, 0)$, $P_2 = (-3, 0)$ and $x = 0$ resp. $y = -2$ are equations of the lines l_1 resp. l_2. Therefore, the equation of the parabola defined by the pair (P_1, l_1) is

$$|x| = \sqrt{(x+2)^2 + y^2}; \quad \text{squaring yields} \quad x = f(y) = -\frac{4+y^2}{4}. \tag{5}$$

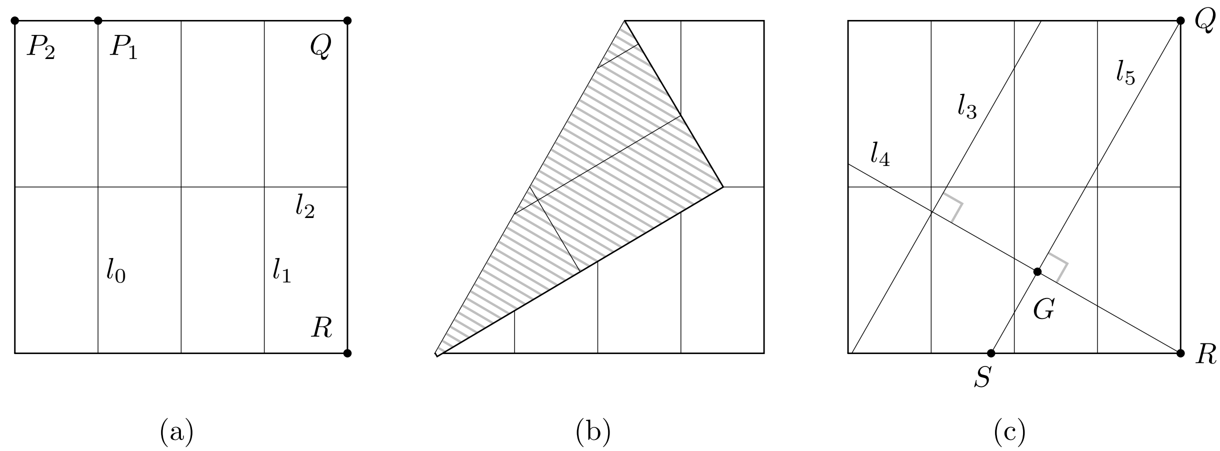


Figure 3: Construction of the center of ψ-decomposition

Analogously, the equation of the parabola defined by (P_2, l_2) is

$$|y+2| = \sqrt{(x+3)^2+y^2}; \quad \text{squaring yields} \quad y = g(x) = \frac{x^2+6\,x+5}{4}. \tag{6}$$

Let p denote the slope of the line l_3. The latter has exactly one point $Q_k = (x_k, y_k)$ in common with the parabola defined by (P_k, l_k) for $k = 1, 2$. The components of these points can be expressed in terms of p only, as given below.

Since l_3 is a tangent to f at point Q_1, the equation $\frac{\mathrm{d}f}{\mathrm{d}y}(y_1) = p^{-1}$ must hold. By differentiating f with respect to y, one obtains

$$y_1 = -\frac{2}{p}\,. \tag{7}$$

Since Q_1 lies on the graph of function f, from (5) and (7) follows

$$x_1 = f(y_1) = -1 - \frac{1}{p^2}\,. \tag{8}$$

Analogously to the above, the equation $\frac{\mathrm{d}g}{\mathrm{d}x}(x_2) = p$ must hold, which yields

$$x_2 = 2\,p - 3. \tag{9}$$

Since Q_2 lies on the graph of the function g, from (6) and (9) follows

$$y_2 = g(x_2) = p^2 - 1. \tag{10}$$

The line l_3 contains the points (x_1, y_1) and (x_2, y_2), which means that there exists a real number q such that $y_k = p\,x_k + q$ for $k = 1, 2$. Subtracting these two equations and substituting (7), (8), (9), and (10) yields

$$0 = p\,(x_2 - x_1) - (y_2 - y_1) = \frac{p^3 - 2\,p^2 + p - 1}{p}\,. \tag{11}$$

Since it would be impossible to place point P_1 on the line l_1 in the case $p = 0$, it follows $p \neq 0$. Therefore, in order to solve equation (11), it is enough to obtain the real roots of the following cubic function:

$$h(x) = p^3 - 2\,p^2 + p - 1. \tag{12}$$

The discriminant of the polynomial h is negative, as one can easily check by direct computation. Therefore, h has exactly one real root. Now, using (2), one obtains

$$h\left(\psi^2\right) = \psi^6 - 2\,\psi^4 + \psi^2 - 1 = (\psi^6 - \psi^4) - (\psi^4 - \psi^2) - 1 = \psi^3 - \psi - 1 = 0,$$

hence $p = \psi^2$ is the only real solution of the equation (11).

The creases l_3 and l_5 are parallel, therefore they have the same slope. That means $|QR|$ and $|RS|$ (see Figure 3c) are in the ratio ψ^2, i.e., the triangle QRS is similar to the triangle FCD in Figure 1. Hence the intersection G is the center of a ψ-decomposition, being the common point of l_4, l_5 and the Thales circle over $\overline{QR}$. □

4. Extending the algorithm to the Rosenbusch number

Another solution to the Problem 1, found by L. Rosenbusch[5] independently of Hans van der Laan (see [5]), is shown in Figure 4, left. For the side lengths of the Rosenbusch cuboid a, b and c the following equations hold:

$$\frac{b}{a} = \frac{c}{b} = \rho, \tag{13}$$

where $\rho = 0.682327\ldots$, called the *cubi ratio*, is the unique real solution to the cubic equation quite similar to (2):

$$x^3 = 1 - x. \tag{14}$$

As found by Rosenbusch himself (see [8]), for a given square of side length a the lengths of two shorter sides b and c of a Rosenbusch's cuboid can be dynamically constructed analogously to the center of the ψ-decomposition shown in Figure 1, the only difference being that the line l_1 resp. l_2 now passes through the other lower resp. other upper vertex of the square (see Figure 4, right). The point G, thus obtained, determines another decomposition of a square into three rectangles $EBCF$, $AEGH$ and $HGFD$, which may be called a *ρ-decomposition*. The sides of the first two rectangles are in the ratio ρ, while in the third rectangle the ratio is $1 + \rho$. To prove the latter statement, let $d = |AE|$. Now $\frac{d}{c} = \rho$, so from (13) follows

$$\rho^3 = \frac{b}{a} \cdot \frac{c}{b} \cdot \frac{d}{c} = \frac{d}{a}\,. \tag{15}$$

The ratio r of the last rectangle in the ρ-decomposition is equal to

$$r = \frac{a - c}{d} = \frac{a}{d} - \frac{c}{d} = \frac{1}{\rho^3} - \frac{1}{\rho} = \frac{1 - \rho^2}{\rho^3}.$$

But (14) implies $1 - \rho^2 = (1 - \rho)\,(1 + \rho) = \rho^3\,(1 + \rho)$, hence $r = 1 + \rho$. The point G will further be called the *center* of the ρ-decomposition.

For a given square, the center of a ρ-decomposition can be obtained using the algorithm defined in the previous section, with one small modification of Step 2: l_1 should denote the

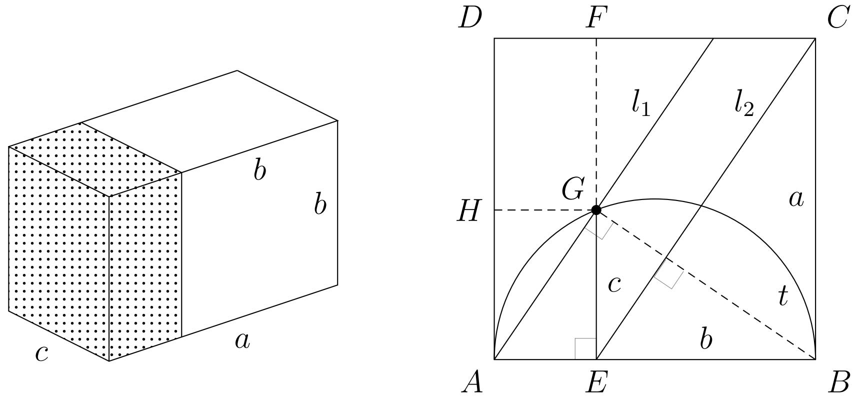
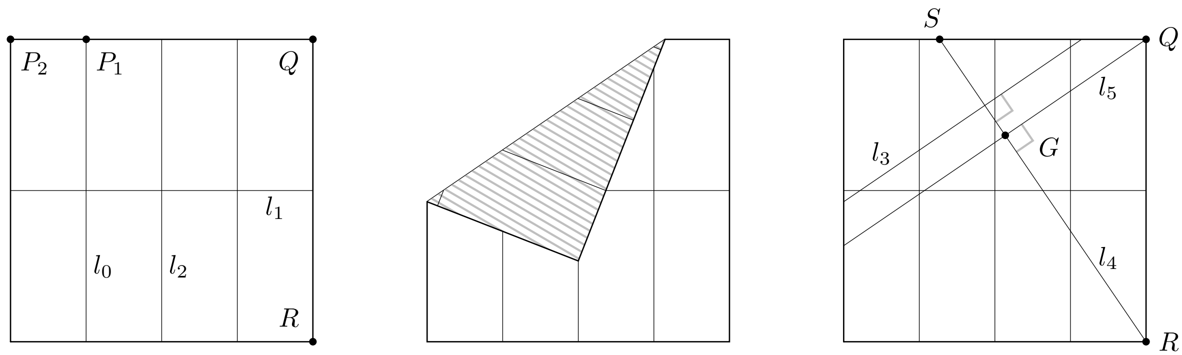


Figure 4: ROSENBUSCH's generalization of the golden rectangle



(a) (b) (c)

Figure 5: Construction of the center of a ρ-decomposition

horizontal, and l_2 the middle vertical crease. The rest of the algorithm should remain the same. The execution of the modified algorithm is shown in Figure 5.

Proposition 3. *The intersection of the creases l_4 and l_5, as shown in Figure 5c, is the center of a ρ-decomposition of the corresponding square.*

Proof. The proof is completely analogous to the proof of Proposition 2. In the coordinate system with the abscissa P_2Q, the ordinate l_2 and the unit length $|P_1P_2| = 1$ follows $P_1 = (-1, 0)$ and $P_2 = (-2, 0)$, while $y = -2$ and $x = 0$ are equations of the lines l_1 and l_2, respectively. The parabolas f and g, implied by the focus-directrix pairs (P_1, l_1) and (P_2, l_2), are defined with the equations

$$f(x) = \frac{x^2 + 2\,x - 3}{4} \quad \text{and} \quad g(y) = -\frac{y^2}{4} - 1.$$

If p denotes the slope of the line l_3, the coordinates of its common point with the parabola f are

$$x_1 = 2\,p - 1 \quad \text{and} \quad y_1 = p^2 - 1,$$

while the coordinates of its common point with the parabola g are

$$x_2 = -1 - \frac{1}{p^2} \quad \text{and} \quad y_2 = -\frac{2}{p}\,.$$

[5]Lambert ROSENBUSCH (1940–2009), a German architect.

Substituting these coordinates into the equation $p\,(x_2 - x_1) - (y_2 - y_1) = 0$ yields

$$\frac{p^3 + p - 1}{p} = 0.$$

Since $p \neq 0$, it follows $p^3 + p - 1 = 0$; so p satisfies (14), which implies $p = \rho$. Therefore, $|QS| : |QR| = \rho$ (see Figure 5c), hence the triangle SQR is similar to the triangle EBC in Figure 4. The statement now follows by the same argumentation as used in the proof of Proposition 2. □

5. Conclusion

Problem 1 has only two solutions, found by van der Laan and Rosenbusch. These solutions are based on two special constants ψ and ρ, which are interpreted as three-dimensional generalizations of the *golden ratio* $\varphi = 1.618033\ldots$, equal to the ratio of the golden rectangle. Although origami algorithms yielding line segments of these lengths are already provided in [8], they represent two quite different sets of folding operations. Focusing on the construction of the centers of ψ- and ρ-decompositions, from which the ratios ψ and ρ are easy obtained, results in two very similar folding algorithms.

References

[1] R.C. Alperin, R.J. Lang: *One-, Two-, and Multi-Fold Origami Axioms.* `http://www.math.sjsu.edu/~alperin/AlperinLang.pdf`, accessed Feb. 22, 2016.

[2] V.W. de Spinadel, A.R. Buitrago: *Towards van der Laan's Plastic Number in the Plane.* J. Geometry Graphics **13**/2, 163–175 (2009).

[3] Th.C. Hull: *Solving Cubics With Creases: The Work of Beloch and Lill.* Amer. Math. Monthly **118**/4, 307–315 2011.

[4] H. Huzita: *Axiomatic development of origami geometry.* Proc. First International Meeting of Origami Science and Technology 1989, pp. 143–158.

[5] L. Rosenbusch: *Cubi Ratio und der Goldene Schnitt.* `http://www.lambertrosenbusch.de/Theorie/cubiratio1eNetz.pdf`. accessed Feb. 22, 2016.

[6] D.H. van der Laan: *Architectonic Space: Fifteen Lessons on the Disposition of the Human Habitat.* Brill Academic Pub, Leiden 1983.

[7] C. Voet: *The poetics of order: Dom Hans van der Laan's architectonic space.* Architectural Research Quarterly **16**/2, 137–154 (2012).

[8] G. Weiss, V. de Spinadel: *Remarks to Classical Cubic Problems and the Mean Values of van der Laan and Rosenbusch.* Proc. 14th Internat. Conference on Geometry and Graphics, Kyoto/Japan 2010.

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