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Constructions in the Absolute Plane – to the Memory of Gyula Strommer (1920–1995)

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Abstract. Professor Gyula Strommer was a charismatic teacher at the Budapest University of Technology and Economics (BME) and from the beginning a leading personality in the International Society of Geometry and Graphics and also in our society in a smaller circle of surrounding countries of Hungary. On his anniversary, we like to remind on his main scientific activities and impact in the field of foundations of geometry, or in particular, of geometric constructions in the absolute plane.

Key Words: foundations of geometry, absolute geometry, geometric constructions, Mohr-Mascheroni.

MSC 2010: 01A70, 51F05, 51M15, 51M09

1. Introduction

In this obituary we would like to remember the scientific work of Professor Gyula (Julius) Strommer, mainly on the base of his academic doctor dissertation (1974), *"On the theory of geometric constructions, independent of the axiom of parallels"* [4][1].

The author became Strommer's scientific aspirant just in 1974 on the axiomatic topic of 'reflection geometry in the sense of Friedrich Bachmann'. The author's former mentor, Ferenc Kárteszi (1907–1989), suggested to apply for Strommer's supervision. The personality of János Bolyai and his wonderful discovery, the absolute geometry, attracted the author at that time and for ever to non-Euclidean geometries.

Professor Strommer, as compatriot of the two Bolyai's, was born in Nagyenyed in Transylvania (in Romania, nowadays). His talent led him to astronomy, to engineering and to geometry in the tradition of the two Bolyai's. Especially, he arrived at the absolute geometry, where the axiom of parallels of Euclid is not required. He reached a great reputation as a

[1] His further publications and a CV can be found at the home page of the Strommer Gyula Nemzetközi Geometria Alapítváy (= International Geometric Foundation Gyula (Julius) Strommer), http://www.math.bme.hu/~szirmai/strommer.html.

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researcher, excellent teacher and for his human values. He had already been a master of his topic, familiar with the vast classical literature. He also became the dean of the Faculty of Mechanical Engineering at the former Technical University of Budapest.

STROMMER's 'bible' was David HILBERT's *"Foundation of Geometry"*, and the starting point was the famous *Theorem of Mohr-Mascheroni* which states: Every Euclidean plane construction with ruler and compass can be carried out only by compass (in the usual sense).
NAPOLEON's famous task — admired, e.g., by LAPLACE in that time – was: For given three points, construct the centre of their circumcircle by compass only. NAPOLEON had learned the work of L. MASCHERONI during his stay in Italy.
The main tool for this classical problem was the circle inversion (circle reflection) and the construction of the inverse of a circle and of a straight line given by two points.

Figure 1: Prof. Gyula STROMMER, caricatured by Mária SALCA in the humorous journal of TU Budapest, "Vicinális Dugóhúzó" (Vicinalis Corkscrew).[2]

2. The dissertation

One of Gyula STROMMER's theorems states that in the absolute plane any construction using ruler and compass and finitely many steps can be realized in finitely many steps only by compass. This became an essential result in the second part of his dissertation. Moreover, he allowed to predefine lower and upper bounds $0 < r < R$ for the used circle radius c, where $r < c < R$, thus extending a result of the Euclidean case obtained 1931 by the Japanese mathematician K. YANAGIHARA.

STROMMER considered his results very modestly only as the development of initiatives originating from Johann HJELMSLEV, 'the second Euclides Danicus'; the 'first one' was G. MOHR. In the preface (see Figure 2) he expresses — very characteristically in footnotes — the history, motivations, and comments to the topic. He also emphasizes recent results, e.g., those of our colleague, Imre VERMES (1940–2002).

He cited HJELMSLEV mainly in the first introductory part of his dissertation. The absolute plane can be extended to a projective metric plane by introducing non-proper (ideal) points and lines, furthermore a line-point polarity for describing the orthogonality of lines, as usual. The main tool is a mapping, the so-called *half-rotation* ('Halbdrehung' in German).

He used this machinery in a restricted sense (keeping the former Hungarian traditional elementary intentions), so that any two proper lines have an intersection point. This is analogous to the Euclidean plane geometry, where we assign a common ideal (infinitely far) point to each pair of parallel lines. The concept of a non-proper ideal line has not been used.

[2]Our late colleague Endre PETHES (1922–2005) was always afraid that the tie, as marked above by ∞, will end in a plate of soup.

Igen valószínű, hogy *Hjelmslev* már a pusztán körzővel végezhető szerkesztéseknek ilyen értelemben való vizsgálatával is foglalkozott. Ő ugyanis egy dán nyelvű dolgozatában[†] kimutatja, hogy a derékszögű háromszögnek két befogójából, ill. az átfogójából és egyik befogójából való ama szerkesztése, melyet *Mohr* fentidézett művében adott, megfelelő módosításokkal a nem-euklidesi geometriában is elvégezhető. De p. o. egy adott távolság felezésére az ott adott megoldások a nem-euklidesi geometriában nem használhatók. "És itt azután" — mondja *Hjelmslev* — "a nem-euklidesi geometria mély problémájához érkeztünk, melyet egy későbbi alkalommal vizsgálok meg alaposabban." Azonban e tárgyra nem tért többé vissza. —

Ezeknek a vizsgálatoknak a folytatása képezi a jelen dolgozat tárgyát.

Dolgozatom I. részében néhány olyan tételnek a párhuzamosok elméletétől független levezetését adom, amelyekre tulajdonképeni tárgyalásunkban többször szükségünk lesz.[††] Ugyanitt kimutatom, hogy bármely olyan

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* Die geometrischen Konstruktionen mittels Lineals und Eichmasses, Opuscula mathematica Andreae Wiman dedicata, Upsala 1930, 175—177. lap.

** Később ugyanerre az eredményre jutott *S. Guber* (Strecken und Winkelübertragung mit Lineal und Eichmass in der absoluten Geometrie, Sitz.-Ber.d.Bayer.Acad.d.Wiss., math.-nat. Klasse 1959, 251—261. l.), ki *Hjelmslev* dolgozatát nem ismerte.

E körbe vágó szerkesztéseket közölt még *Szász Pál* következő dolgozatában: New gauge constructions of perpendiculars without assumming the parallel axiom, Archiv der Math. 13. köt. 1962, 147—150. lap.

Hjelmslev vonalzóval és alapmértékkel végzett szerkesztéseit magyar nyelven *Kürschák József* ismertette: A parallelák axiomájától független szerkesztések, Mat.Fiz. Lapok 37. köt., 1930, 1—20. l.

Figure 2: Strommer's Thesis: Detail from the preface with characteristic footnotes

Megjegyzés. E segédtétel analogonja az 1. segédtételnek.*** Ez különösen szembetűnik, ha az 1. segédtételt így fogalmazzuk:

Ha az ABC háromszög BC, CA, AB oldalait, vagy ezek meghosszabbításait egy tetszés szerinti egyenes az A_1, B_1, C_1 pontokban metszi, továbbá ha a, b, c és a_1, b_1, c_1 az A, B, C és A_1, B_1, C_1 pontokat a tőlük különböző P ponttal összekötő egyenesek, akkor az

$$(a,b) = (b_1,a_1), \qquad (b,c) = (c_1,b_1), \qquad (c,a) = (a_1,c_1)$$

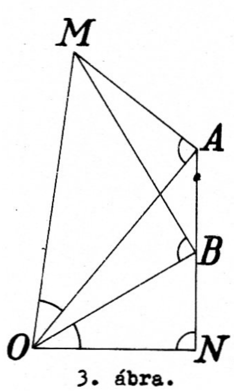
egyenlőségek egyszerre állanak fenn.

Hjelmslev-től* származik a következő

3. segédtétel. *Ha az AOB háromszög OA és OB oldalaira A-ban és B-ben merőlegest emelünk* (3. ábra),** *melyek a sík tényleges vagy nem tényleges M pontjában metszik egymást, és az O csúcsból megrajzoljuk az ON magasságot, akkor*

$$(OA,OM) = (ON,OB).$$

Amilyen egyszerűnek látszik e tétel, olyan fontosak következményei. Úgy, hogy bízvást alaptételnek tekinthető.***



3. ábra.

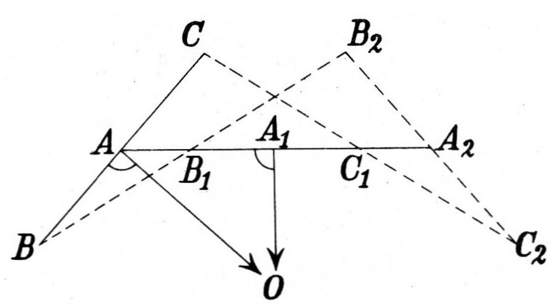
Figure 3: Description of the content and Hjelmslev's Lotensatz

Legyen $(ABC\ldots)$ egy tetszőleges idom, melynek pontjai mind tényleges pontok, továbbá legyen a és b két egyenes, melyek egy és ugyanazon tényleges vagy nem tényleges O ponton mennek át; ha az $(ABC\ldots)$ idom az a és b egyenesre vonatkozó tükrözés egymásután való alkalmazása által az $(A_2B_2C_2\ldots)$ idomba vihető át, akkor az $AA_2, BB_2, CC_2, \ldots$ közök $A_1, B_1, C_1, \ldots$ felező pontjai oly idomot alkotnak, amelyben az eredeti idom minden egyes pontjának egyetlen egy pont felel meg. Az ily leképezést az O pont körül való *félforgás*nak nevezzük;* ha az O ponton átmenő egyeneseknek egy l közös merőlegesük van, akkor a leképzést az l egyenes mentén való *féleltolás*nak is nevezhetjük.

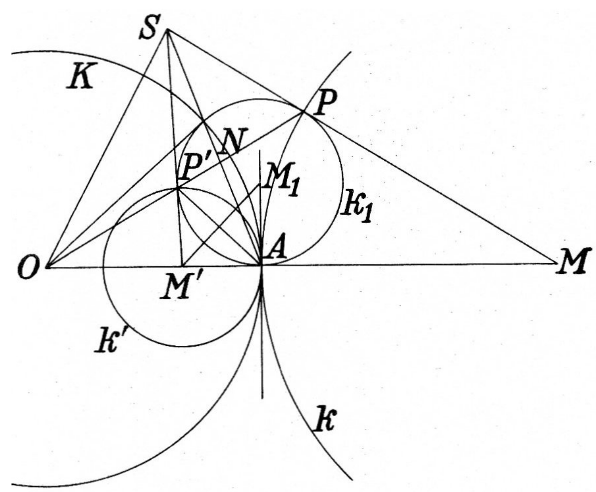
Figure 4: Generalized ‘Lotensatz’ in absolute geometry

We only illustrate in Figure 3 the basic theorem concerning half-rotations, the *Lotensatz.* Prof. Strommer extended this again to the half-rotation about a non-proper (ideal) starting point O (again new in the literature, Figure 4).

We see at Figure 5, left, the generalization of a half-rotation. First, a given rotation about a proper point O sends an arbitrary point A to A_2; then consider the midpoint A_1 of the segment AA_2. Thus, we get the mapping $A \mapsto A_1$, then similarly $B \mapsto B_1$, and $C \to C_1$. This mapping, called half-rotation, is no longer a congruent motion, but collinear points remain collinear (first proved by Hjelmslev), and there are also other nice properties. If we consider a rotation as a product of two reflections in lines incident to O, then by virtue of the basic axioms and the theorem of three reflections, this extends also to a non-proper starting point O as well. Moreover, the image of a proper point is always a proper one, but a non-proper point can also have a proper image.



8. ábra.



29. ábra.

Figure 5: On half-rotations

The other important topic of the first part is the extension of the circle inversion (circle reflection) onto the absolute plane (Figure 5, right), so that the images of lines and generalized circles (so-called *cycles*, as hypercycles, horocycles) are circles (or cycles, in general). Thus we can follow a strategy, similar to the Euclidean case. The intersection point of two lines or of a line and a circle can be constructed, after applying a circle inversion, via two circles. Their intersection point can be designed. Then this point can be reinverted to get the corresponding(occasionally non-proper) intersection point.

To this goal a well-prepared sequence of 'elementary constructions' (about 25 tasks) is necessary, sometimes with bravour solutions. We show here only the first two of them in Figure 6 (you may solve them voluntarily, please !!!).

Task 1: Double a given segment AB to $2\,AB = AC$ (Figure 6, left). Be careful, the traditional method is wrong now; it is applicable only in the Euclidean plane.

Task 2: Put a given segment CD onto a given half-line AB from the starting point A on ($AN = CD$) (Figure 6, right).

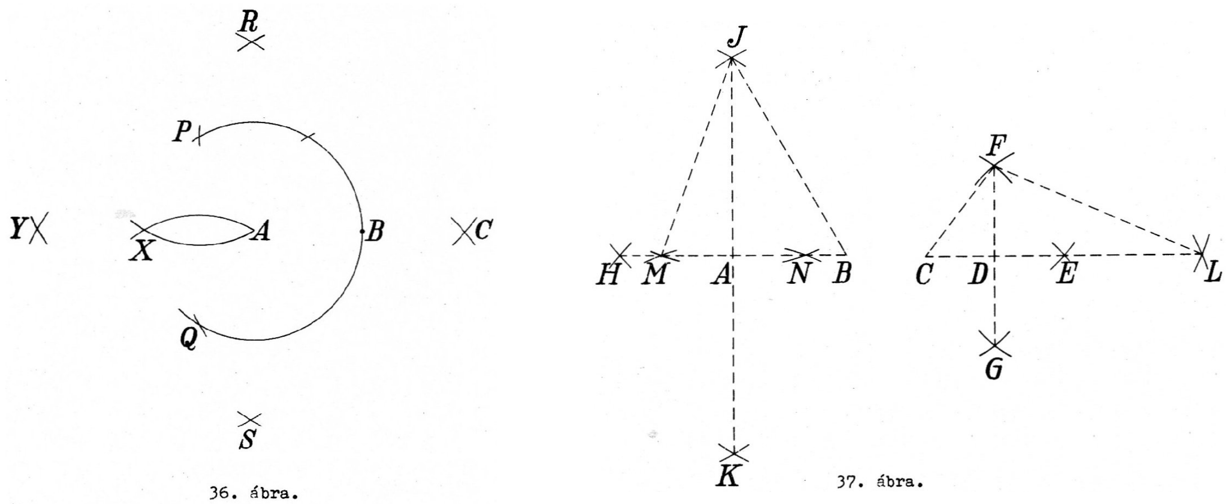


Figure 6: The first two elementary tasks

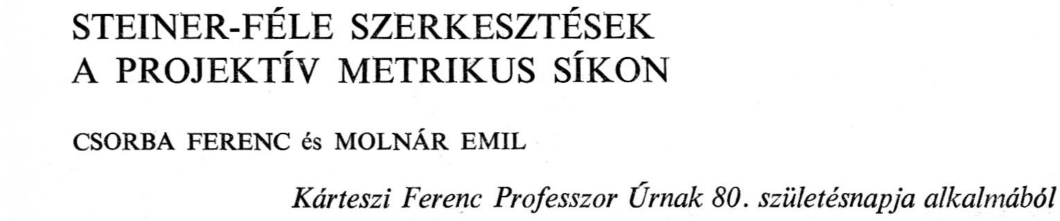
We emphasize here the role of the *Archimedian axiom*: For given segments $AB > CD$ there is a natural number n such that $n \cdot CD > AB$. It is a well-known result that, without assuming this axiom, the classical Euclidean constructions cannot be realized only by compass.

In the third part of his work Prof. Strommer dealt with ruler constructions, mainly with Poncelet-Steiner ones in the absolute plane. Namely, after giving a circle with its centre, we would like to realize all the traditional constructions only with a ruler. In the absolute plane, these are possible only with additional data. Those will be the topic of the next section where we also mention some resonances of the investigations.

3. Impact of the dissertation

Figure 7 shows the header of the author's joint work [2] with Ferenc Csorba, a leading teacher colleague in Györ/Hungary. It was a honour for us that Prof. Strommer was its referee.

Here we extended the usual real ($t \in \mathbb{R} \cup \infty$) parametrization of projective conics $(x_0; x_1, x_2) \sim (1; t^2, t)$ onto the constructing circle, calculation together with construction. Moreover, just in the sense of Hilbert's 'end-calculus' (German: Endenrechnung), we could transfer the theorem of Strommer to any construction cycle, i.e., to horocycles and hypercycles. Of course, we used here the complete projective embedding of the absolute plane. The problem can be raised, how can we make this as elementary as possible?



STEINER-FÉLE SZERKESZTÉSEK
A PROJEKTÍV METRIKUS SÍKON

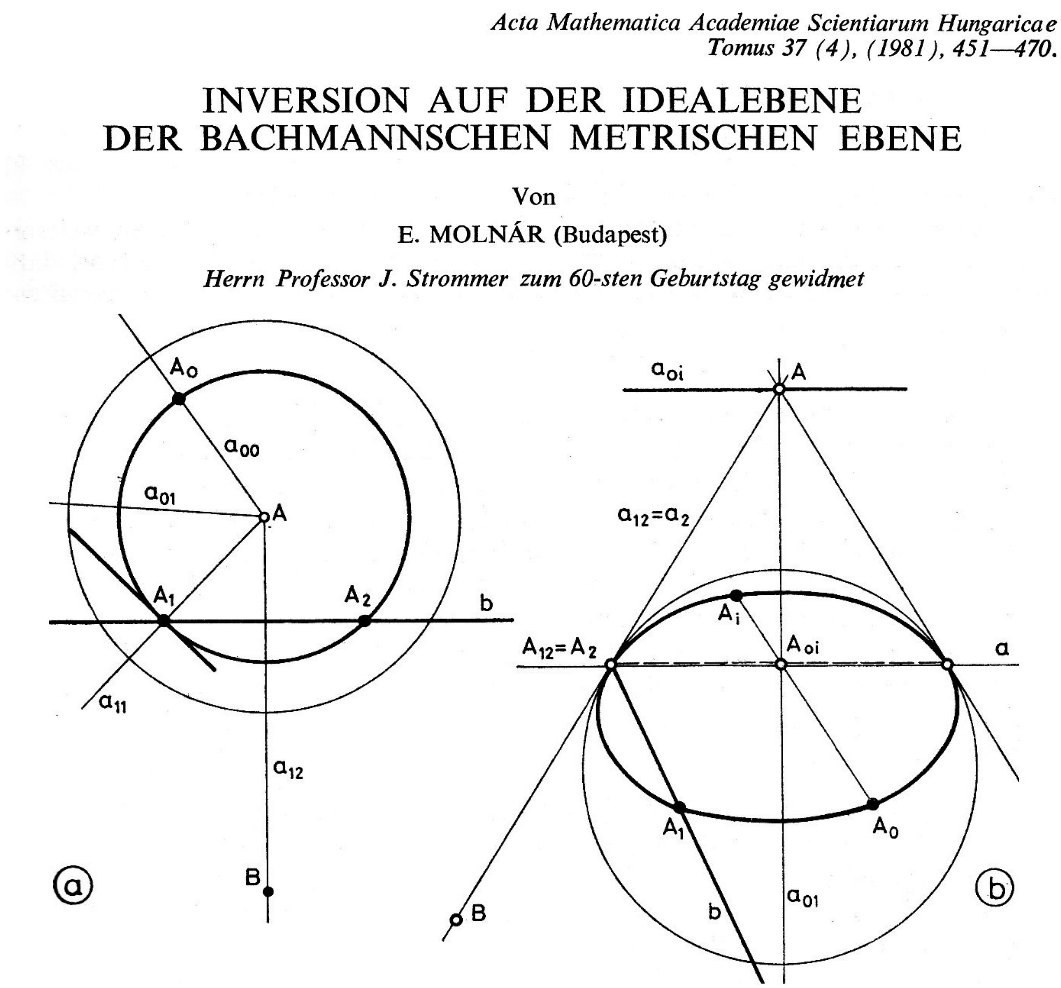
CSORBA FERENC és MOLNÁR EMIL

Kárteszi Ferenc Professzor Úrnak 80. születésnapja alkalmából

Figure 7: Generalized Poncelet-Steiner constructions with a ruler

The complete projective embedding was also used in the author's paper (Figure 8, top), as a result of the candidate dissertation (PhD) under the supervision of Prof. Strommer: How to define the inversion in the (ocasionally more extended) absolute plane, as generally as possible, as an involutory bijection, mapping cycles onto cycles? We followed the method of F. Bachmann and expressed everything in the language of *line reflections* (as a modern Cartesian coordinate calculus), but now also for the projective embedding.

Here we only mention the *concept of the cycle* (Figure 8, bottom). We take a (proper or ideal) centre A. Moreover, we take a point A_0 which we reflect in all the lines incident to A. All the reflection images of A_0 as a point set will be the cycle with centre A. Figure (a) shows when A is a proper point, Figure (b) shows the case when A is outside the absolute boundary



Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 37 (4), (1981), 451—470.

INVERSION AUF DER IDEALEBENE
DER BACHMANNSCHEN METRISCHEN EBENE

Von
E. MOLNÁR (Budapest)

Herrn Professor J. Strommer zum 60-sten Geburtstag gewidmet

Figure 8: On the generalized inversion, a) circle, b) hypercycle

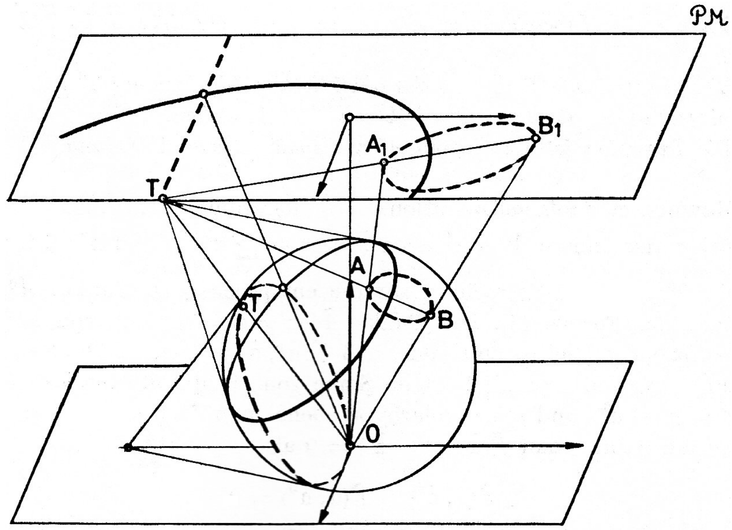


Figure 9: Euclidean inversion through stereographic projection

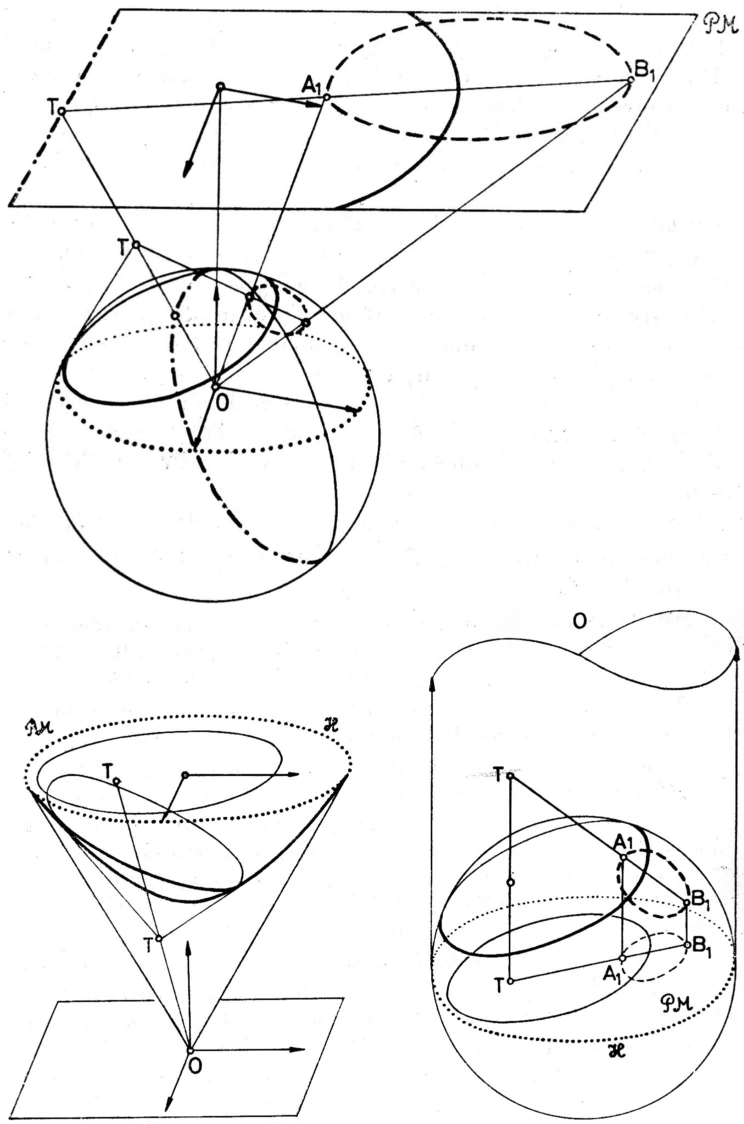


Figure 10: Elliptic (top) and hyperbolic (bottom) circle reflection through projection

$\mathcal{H}$ of the hyperbolic plane. Then the polar line of A will be the base of the distance line of the hypercycle. The third case is not depicted here: A is a boundary point and we obtain a horocycle which is a circle with infinite radius.

Finally it turns out that the extension of the base plane into the space gives us a better description, analogously to the classical Euclidean circle geometry, where we project the base plane stereographically onto a sphere (Figure 9). It is well-known that the Euclidean inversion is a stereographically transformed harmonic-projective sphere reflection, i.e., a spatial collineation with a pole as center and the polar plane axis plane.

Figure 10, top, shows the inversion of the elliptic plane. This is the analogue of the above mentioned Euclidean version; however, we project from an interior point of a sphere onto its polar plane. Figure 10, bottom, shows two variants for the inversion of the hyperbolic plane. Now we project from an exterior point onto its polar plane, which intersects the sphere just in the absolute $\mathcal{H}$.

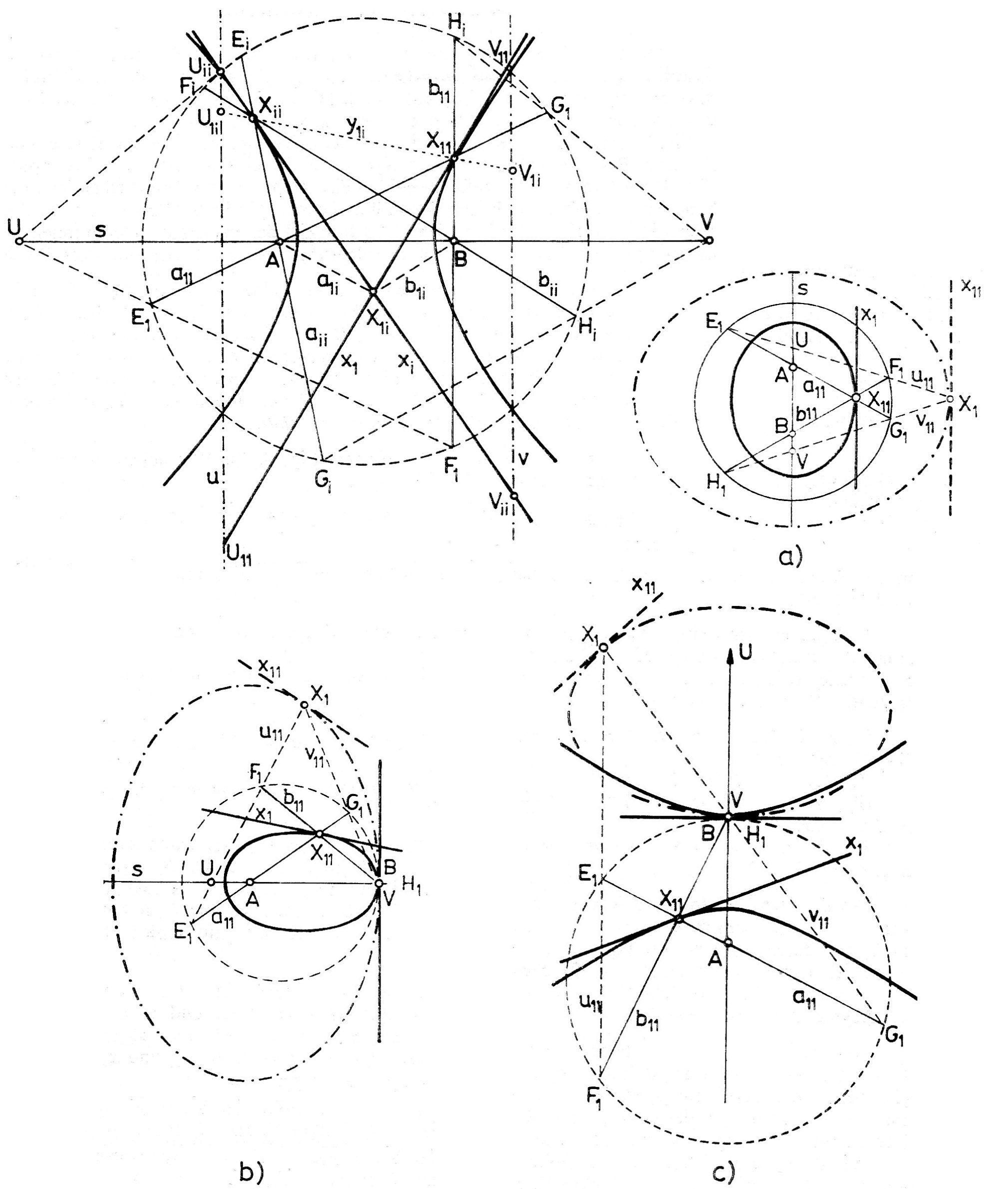


Figure 11: Hyperbolic conics in dual pairs

We did not mention some 'delicate things'. E.g., instead of sphere we can take an ellipsoid-like quadratic. The topic can be extended into higher dimensions ($d \geq 2$). Algebraically, the quadratic classes of the coordinate field (classically $\mathbb{R}$, the reals) of the projective embedding play decisive roles in the description.

4. On conic sections

Here we only mention that conics dropped out of the above discussions. E.g., we have 20 non-degenerated conics in the hyperbolic plane, classified onto dual pairs. Figure 11 shows some of them. Our colleague Géza Csima with his supervisor Jenö Szirmai described their isoptic curves recently.

5. Constructing the regular 17-gon and 257-gon, an epilogue

The young C.F. Gauss discovered — with algebraic methods, through quadratic field extensions — which regular prime sided polygons can be constructed with ruler and compass in the classical plane. Because of $2^4 + 1 = 17$ and $2^8 + 1 = 257$ the regular 17-gon and the 257-gon are among them. It was an attractive life-long task for Prof. Strommer to find a nice simple geometric procedure. A geometric analysis of the construction of the regular 17-gon, by the intention to be a part of a textbook, is the topic of the paper displayed in Figure 13. Of course, this indicates also delicate details, but one can follow his unified method also when one constructs the regular 257-gon (Figure 14).

This paper — under the management of our colleague István Prok — appeared sadly after Strommer's death. He attended together with us in Graz/Austria the birthday ceremonies of Prof. Hans Vogler. He drove his car with colleague Imre Vermes and the author. Turning back to our university, he was unusually tired. Imre accompanied him on the way home. On the other day, in the hospital, it turned out that his blood sugar level was 26 units. He did not return from hospital anymore (pancreatic cancer). Our brother Gyula ran a race with the time. But finally, he won.

Studia Scientiarum Mathematicarum Hungarica 30 (1995), 433–441

ZUR KONSTRUKTION DES REGULÄREN SIEBZEHNECKS

J. STROMMER

Meinem lieben Freund, Herrn Prof. H. Sachs zum 50. Geburtstag gewidmet

Abstract

At the construction of the regular 17-gon one has to solve a chain of dependent quadratic equations. All the authors of the various constructions have been concentrating on geometric representation of the roots of these equations. As F. Klein ([5], p. 19 and 26), F. Enriques ([2], p. 175), Th. Vahlen ([9], p. 155) and later also H. Lebesgue ([6], pp. 149–150) emphasized their wishes to construct the regular 17-gon on the base of purely geometric analysis. This is the intention of this paper giving such a discussion, which can be treated also in a textbook of plane geometry.

Figure 12: On the regular 17-gon

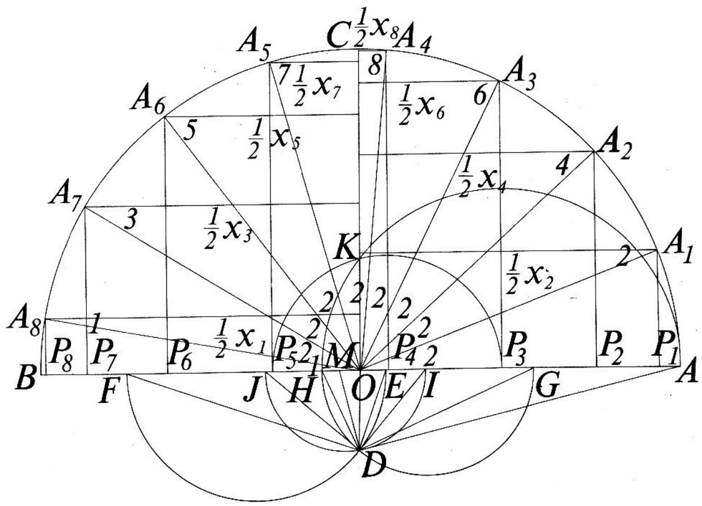


Fig. 1

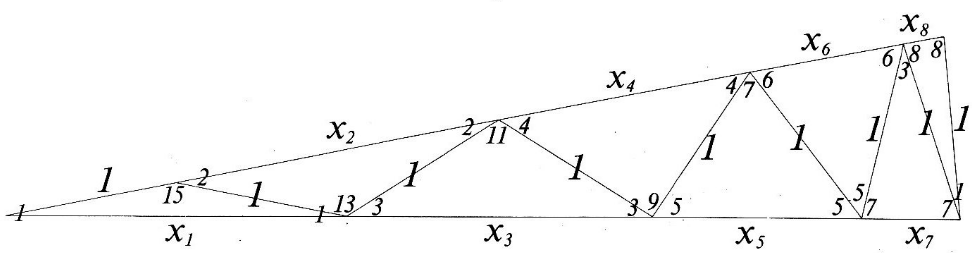


Fig. 2

Figure 13: Analysis for the construction of the regular 17-gon

Acta Math. Hungar.
70 (4) (1996), 259–292.

KONSTRUKTION DES REGULÄREN 257-ECKS MIT LINEAL UND STRECKENÜBERTRAGER

J. STROMMER (Budapest)*

Meinem lieben Freund, Herrn Prof. H. Vogler zum 60. Geburtstag gewidmet

1. Im Jahre 1832 hat F. J. Richelot, Professor der Mathematik an der Universität zu Königsberg, die Lösung der Gleichung

$$z^{257} - 1 = 0,$$

von der die Konstruktion des regelmäßigen 257-Ecks, oder die Teilung des Kreises in 257 gleiche Teile abhängt, in der umfangreichen Arbeit [8] nach zwei Methoden durchgeführt, an die sich dann noch die Arbeit [3] von A. Fischer anschloß.

Die Berechnung des regulären 257-Ecks behandeln gleichfalls die späteren Programmabhandlungen [7] von G. A. R. Maywald[1] und [10] von H. Schwendenwein[2], sowie die Arbeit [5] von K. Hagge, die verglichen mit der Richelotschen einige Vereinfachungen zeigen und deren wir uns im folgenden bedienen werden.

Figure 14: On the regular 257-gon

Acknowledgments

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