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Symbiotic Conics and Quartets of Four-Foci Orthogonal Circles

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Abstract. A quartet of orthogonal circles — one of them being imaginary — associated with a general point P taken on a given ellipse H is described. The mutual intersections of these circles, their intersections with Barlotti's circles $\left[x^2 + y^2 = (a \pm b)^2\right]$ and further, newly introduced points are peculiar under several aspects. A major result is the finding (Theorem 2.10) of a complete, cyclic quadrangle having two diagonal points in fixed positions on the ellipse minor axis; these diagonal points are concyclic with the ellipse foci, in spite of the dependence of the whole figure from the point P location. Two conics — the *symbiotic ellipse* H_Σ and *hyperbola* Y_Σ — are introduced, in association with P; such conics are characterized by the fact that they
(i) have P as center and the tangent and normal to H at P as axes of symmetry,
(ii) pass through the ellipse H center, and
(iii) admit the axes of symmetry of the ellipse H as tangent and normal.
Several relationships among these conics are described. The study of the symbiotic ellipse H_Σ reveals new properties of the ellipse H.

Key Words: Ellipse, Monge's circle, Barlotti's circles, concyclic points, collinear points, complete quadrangle

MSC 2010: 51M04, 51N20

1. Introduction

New developments of the author's research project [6, 7, 8] on the geometry of the ellipse are presented. In an orthogonal cartesian reference frame (Figure 1), let H be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b\,; \tag{1.1}$$

its general point — that is a point different from the apices — will be denoted throughout this paper by $P\,(a\cos\varepsilon;\ b\sin\varepsilon)$. For the reader's convenience, some geometrical objects frequently referred to are listed here:

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- the foci of the ellipse H:

$$F_1(-c, 0); \quad F_2(c, 0) \tag{1.2}$$

where, as usual, I have set $c = \sqrt{a^2 - b^2}$;

- the eccentric line e (the diameter with slope $m_e = \tan\varepsilon$):

$$y = x\tan\varepsilon\,; \tag{1.3}$$

- the symm-eccentric line e' (the diameter with slope $m_{e'} = -\tan\varepsilon$):

$$y = -x\tan\varepsilon\,; \tag{1.4}$$

- the tangent t to the ellipse at P:

$$y = -x\,\frac{b}{a}\cot\varepsilon + \frac{b}{\sin\varepsilon}\,; \tag{1.5}$$

- the intercepts of the tangent (1.5):

$$T_x\left(\frac{a}{\cos\varepsilon};\ 0\right), \quad T_y\left(0;\ \frac{b}{\sin\varepsilon}\right); \tag{1.6}$$

- the normal n to the ellipse at P:

$$y = x\,\frac{a}{b}\tan\varepsilon - \frac{c^2}{b}\sin\varepsilon\,; \tag{1.7}$$

- the intercepts of the normal (1.7):

$$N_x\left(\frac{c^2}{a}\cos\varepsilon\,;\ 0\right), \quad N_y\left(0;\ -\frac{c^2}{b}\sin\varepsilon\right); \tag{1.8}$$

- the points E and I where the normal (1.7) meets the eccentric (1.3) and the symm-eccentric (1.4) line of P, respectively:

$$E\left((a+b)\cos\varepsilon\,;\ (a+b)\sin\varepsilon\right), \quad I\left((a-b)\cos\varepsilon;\ -(a-b)\sin\varepsilon\right); \tag{1.9}$$

- the locus of point E (1.9); hereinafter, *Barlotti's external circle*:

$$B_e: \quad x^2 + y^2 = (a+b)^2\,; \tag{1.10}$$

- the locus of point I (1.9); hereinafter, *Barlotti's internal circle*:

$$B_i: \quad x^2 + y^2 = (a-b)^2\,. \tag{1.11}$$

The first mention of the circle (1.10) known to the author can be found in an exercise of Salmon ([4], Chapter XIII, Article 231, p. 221). Afterwards, both circles B_e (1.10) and B_i (1.11) have been studied by A. Barlotti [1], R. Fritsch [3], and Ternullo [6].

2. A triplet of orthogonal circles

Definition 1. The circle Φ_1 has its center at the intersection T_y (1.6) of the tangent (1.5) to H (1.1) at P with the minor axis and passes through the foci. It satisfies the equation

$$x^2 + \left(y - \frac{b}{\sin\varepsilon}\right)^2 = c^2 + \frac{b^2}{\sin^2\varepsilon}\,. \tag{2.1}$$

Definition 2. The circle Φ_2 has its center at the intersection N_y (1.8) of the normal (1.7) to H (1.1) at P with the minor axis and passes through the foci. It satisfies

$$x^2 + \left(y + \frac{c^2}{b}\sin\varepsilon\right)^2 = c^2 + \left(\frac{c^2}{b}\sin\varepsilon\right)^2. \tag{2.2}$$

Definition 3. The circle Φ_3 has its center at the intersection T_x (1.6) of the tangent (1.5) to H (1.1) at P with the major axis and passes through the points E and I (1.9) [where the normal (1.7) to H at P meets the eccentric (1.3) and the symm-eccentric (1.4) line of P, respectively]. Φ_3 satisfies

$$\left(x - \frac{a}{\cos\varepsilon}\right)^2 + y^2 = \frac{a^2\sin^2\varepsilon + b^2\cos^2\varepsilon}{\cos^2\varepsilon}\,. \tag{2.3}$$

Let Φ_i and Φ_j be two circles out of the triplet Φ_1 (2.1), Φ_2 (2.2) and Φ_3 (2.3). The corresponding radii r_i and r_j and the distance d_{ij} between the centers fulfill the relationship $r_i^2 + r_j^2 = d_{ij}^2$; accordingly, the following holds:

Theorem 2.1. *The circles* Φ_1 (2.1), Φ_2 (2.2), *and* Φ_3 (2.3) *are mutually orthogonal.*

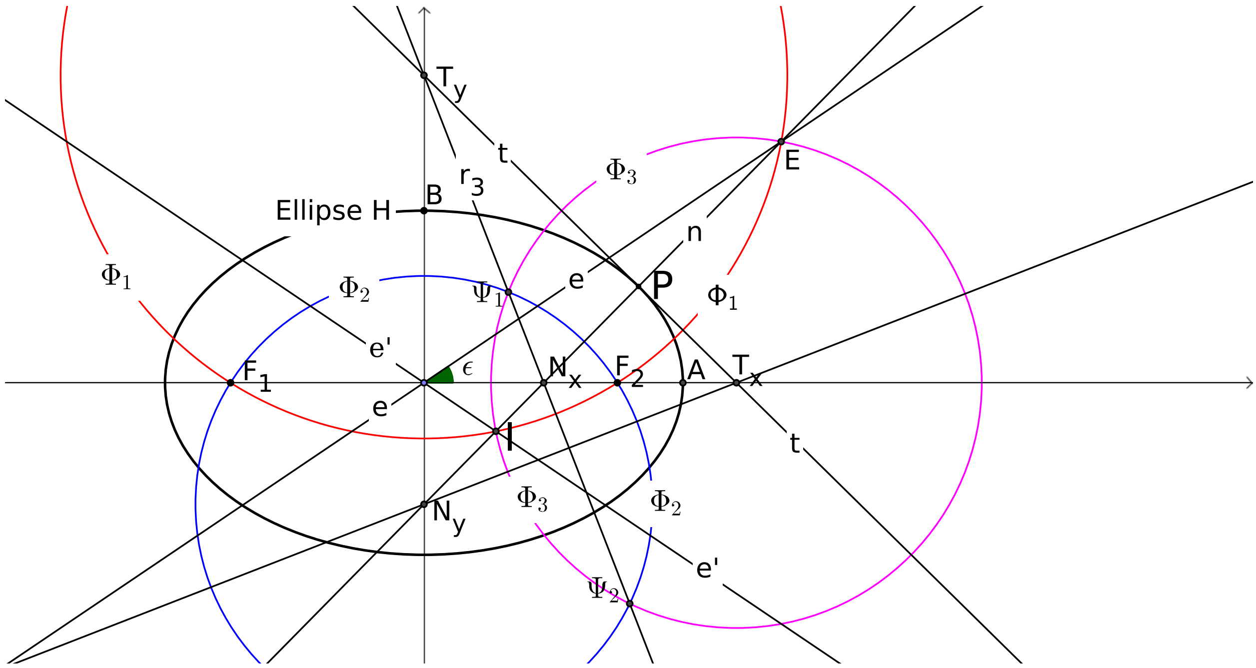

Figure 1: Illustrating Definitions 2.1, 2.2, and 2.3, Theorems 2.1 through 2.3, and Corollary 2.1: Taken a P on the ellipse H, its eccentric and symm-eccentric line [e (1.3) and e' (1.4), respectively], the tangent t and the normal n to H at P as well as the circles Φ_1 (2.1) [red], Φ_2 (2.2) [blue] and Φ_3 (2.3) [magenta] are shown.

The radical line of Φ_1 and Φ_2 is the x-axis, since both circles pass by definition through the foci. As regards the pairs (Φ_1, Φ_3) and (Φ_2, Φ_3), the following holds:

Lemma 2.1. *The radical line of the circles* Φ_1 (2.1) *and* Φ_3 (2.3) *is the normal* (1.7) *to* H *at* P. *The circles* Φ_1 *and* Φ_3 *meet at the points* E *and* I (1.9).

Lemma 2.1 and Definition 1 imply:

Theorem 2.2. *The points* E *and* I (1.9) *are concyclic with the ellipse foci* F_1 *and* F_2 (1.2) *on the circle* Φ_1 (2.1).

Further special points concyclic with E and I and the ellipse foci on the circle Φ_1 will be described in Theorem 3.6.

Lemma 2.2. *The radical line of the circles* Φ_2 *and* Φ_3 *is the line* r_3 *with the equation*

$$y = -\frac{ab}{c^2 \sin\varepsilon \cos\varepsilon} x + \frac{b}{\sin\varepsilon}\,. \tag{2.4}$$

The circles Φ_2 *and* Φ_3 *share the points*

$$\begin{aligned} &\Psi_1 \quad \left(\frac{(a - c\sin\varepsilon)\, c^2 \cos\varepsilon}{a^2\cos^2\varepsilon + b^2 \sin^2\varepsilon};\; \frac{(a - c\sin\varepsilon)\, bc}{a^2\cos^2\varepsilon + b^2\sin^2\varepsilon} \right) \quad \textit{and} \\ &\Psi_2 \quad \left(\frac{(a + c\sin\varepsilon)\, c^2 \cos\varepsilon}{a^2\cos^2\varepsilon + b^2 \sin^2\varepsilon};\; \frac{-(a + c\sin\varepsilon)\, bc}{a^2\cos^2\varepsilon + b^2\sin^2\varepsilon} \right). \end{aligned} \tag{2.5}$$

Since the radical lines of the couples (Φ_1, Φ_2) and (Φ_1, Φ_3) (the x-axis and the normal n to H at P, respectively) meet at N_x (1.8), by virtue of a theorem of Monge [4, Chapter VIII, Article 108], the radical line r_3 (2.4) of the couple (Φ_2, Φ_3) passes through the same point N_x (the *radical center* of the circles Φ_1, Φ_2, and Φ_3), too. Moreover, since the circle Φ_1 is orthogonal to both Φ_2 and Φ_3, the center of Φ_1, i.e., the point T_y (1.6), belongs to the radical line r_3 of the circles Φ_2 and Φ_3. Therefore, we may state the following:

Theorem 2.3. *The* y*-intercept* T_y (1.6) *of the tangent to* H *at* P *and the* x*-intercept* N_x (1.8) *of the normal to* H *at* P *belong to the radical line* r_3 (2.4) *of the circles* Φ_2 (2.2) *and* Φ_3 (2.3).

The segment $\Psi_1\Psi_2$ is a chord belonging to both circles Φ_2 and Φ_3; its normal bisector

$$y = \frac{c^2 \sin\varepsilon \cos\varepsilon}{ab} x - \frac{c^2}{b}\sin\varepsilon \tag{2.6}$$

passes, therefore, through the centers of Φ_2 and Φ_3, namely, through N_y and T_x, respectively. Accordingly, Theorem 2.3 implies the following:

Corollary 2.1. *The line* N_yT_x (2.6) *meets the line* r_3 (2.4) *orthogonally at the midpoint of the common chord* $\Psi_1\Psi_2$ *of the circles* Φ_2 *and* Φ_3.

By invoking the definition of the polarity with respect to a conic, the following relationships can easily be verified:

Lemma 2.3. *The ellipse major axis is*

(*i*) *the polar line of the* y*-intercept* N_y (1.8) *of the normal* (1.7) *with respect to the circle* Φ_1 (2.1), *and* (*ii*) *the polar line of the* y*-intercept* T_y (1.6) *of the tangent* (1.5) *with respect to the circle* Φ_2 (2.2).

Lemma 2.4. *The normal* (1.7) *to* H *at* P *is*

(*i*) *the polar line of the* x*-intercept* T_x (1.6) *of the tangent* (1.5) *with respect to the circle* Φ_1 (2.1), *and*

(*ii*) *the polar line of the* y*-intercept* T_y (1.6) *of the tangent* (1.5) *with respect to the circle* Φ_3 (2.3).

Lemma 2.5. *The line* r_3 (2.4) *is*

(*i*) *the polar line of the* x*-intercept* T_x (1.6) *of the tangent* (1.5) *with respect to the circle* Φ_2 (2.2), *and*

(*ii*) *the polar line of the* y*-intercept* N_y (1.8) *of the normal* (1.7) *with respect to the circle* Φ_3 (2.3).

Lemmas 1.2 and 2.2 and Theorem 2.3 imply

Theorem 2.4. *The following lines pass through the* y*-intercept* T_y (1.6) *of the tangent* (1.5) *to the ellipse* H *at* P*:*

(*i*) *the tangents to the circle* Φ_2 (2.2) *at the foci;*

(*ii*) *the tangents to the circle* Φ_3 (2.3) *at* E *and* I (1.9)*;*

(*iii*) *the radical line* r_3 (2.4) *of the circles* Φ_2 *and* Φ_3.

Lemmas 1.1 and 3.2 imply

Theorem 2.5. *The following lines meet at the* y*-intercept* N_y (1.8) *of the normal* (1.7) *to the ellipse* H *at* P*:*

(*i*) *the tangents to the circle* Φ_1 (2.1) *at the foci;*

(*ii*) *the tangents to the circle* Φ_3 (2.3) *at* Ψ_1 *and* Ψ_2 (2.5).

From Lemmas 2.1 and 3.1 follows

Theorem 2.6. *The following lines meet at the* x*-intercept* T_x (1.6) *of the tangent* (1.5) *to the ellipse* H *at* P*:*

(*i*) *the tangents to the circle* Φ_1 (2.1) *at* E *and* I (1.9)*;*

(*ii*) *the tangents to the circle* Φ_2 (2.2) *at* Ψ_1 *and* Ψ_2 (2.5).

The distances of the foci F_1 and F_2 (1.2) from the center T_x (1.6) of the circle Φ_3 (2.3) are $F_1T_x = a/\cos\varepsilon + c$ and $F_2T_x = a/\cos\varepsilon - c$, respectively; multiplying them yields $(a\sin^2\varepsilon + b\cos^2\varepsilon)/\cos^2\varepsilon$. Since this product equals the circle's Φ_3 squared radius, we conclude that the foci are conjugate with respect to (hereinafter 'w.r.t.', in brief) the circle Φ_3 (2.3). For symmetry reasons, the foci are also conjugate w.r.t. the reflection of the circle Φ_3 in the y-axis:

$$\left(x + \frac{a}{\cos\varepsilon}\right)^2 + y^2 = \frac{a^2\sin^2\varepsilon + b^2\cos^2\varepsilon}{\cos^2\varepsilon}. \tag{2.7}$$

Accordingly, we can state:

Theorem 2.7. *The foci are conjugate w.r.t. the circle* Φ_3 (2.3) *and its reflection in the* y*-axis* (2.7).

Monge's orthoptic circle (hereinafter denoted by M) is the following:

$$x^2 + y^2 = a^2 + b^2. \tag{2.8}$$

Taking pairs of the equations (2.1), (2.3) and (2.8), one easily finds that the radical lines of the couples (M, Φ_1) and (M, Φ_3) are $y = b\sin\varepsilon$ and $x = a\cos\varepsilon$, respectively. Since these lines meet at the ellipse point P, we may state:

Theorem 2.8. *The radical center of the circles* Φ_1 (2.1)*,* Φ_3 (2.3) *and* M (2.8) *coincides with the ellipse point* $P(a\cos\varepsilon\,;\, b\sin\varepsilon)$*.*

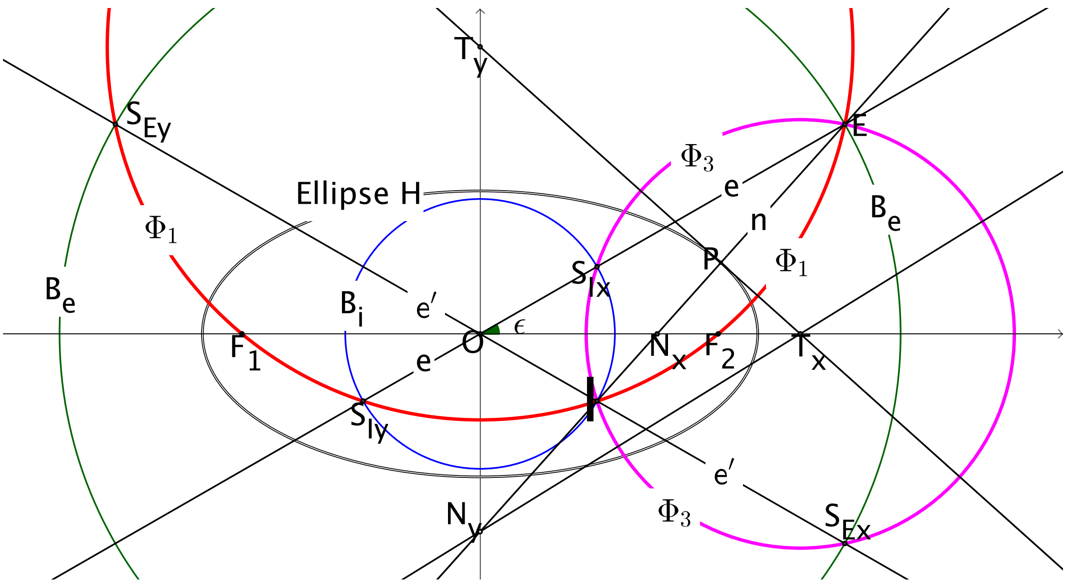

Figure 2: Illustrating Theorem 2.9: Barlotti's internal circle B_i [blue] meets the circle Φ_1 [red] at the points I and S_{Iy} and the circle Φ_3 [magenta] at I and S_{Ix}. The points S_{Iy} and S_{Ix} lie on the eccentric line OE. Analogously, Barlotti's external circle B_e [green] meets the circle Φ_1 at E and S_{Ey} and the circle Φ_3 at E and S_{Ex}. The points S_{Ey} and S_{Ex} lie on the symm-eccentric line OI.

Since the circle B_i (1.11) is (see Figure 2) the locus of point I [6, Theorem 2], and, on the other hand, both circles Φ_1 (2.1) and Φ_3 (2.3) pass through I (by virtue of Theorem 2.2 and by Definition 3, respectively), the circle B_i shares further points — let them be S_{Iy} and S_{Ix} — with Φ_1 and Φ_3, respectively. Analogously, as B_e (1.10) is the locus of E [6, Theorem 1], and both Φ_1 and Φ_3 pass through E, the circle B_e shares further points — let them be S_{Ey} and S_{Ex} — with Φ_1 and Φ_3, respectively. The following holds:

Theorem 2.9. [Figure 2] *The points* S_{Iy} *and* S_{Ix} *are symmetric to* I *w.r.t. the* y*- and the* x*-axis, respectively, and they are diametrically opposite on the circle* B_i (1.11)*. They belong to the eccentric line* (1.3) *of* P*.*
The points S_{Ey} *and* S_{Ex} *are symmetric to* E *w.r.t. the* y*- and the* x*-axis, respectively, and they are diametrically opposite on the circle* B_e (1.10)*. They belong to the symm-eccentric line* (1.4) *of* P*.*

Indeed, the symmetry properties (of I and S_{Iy} w.r.t. the y-axis and of I and S_{Ix} w.r.t. the x-axis) are obvious consequences of the symmetry of the circles B_i and Φ_1 w.r.t. the y-axis and of B_i and Φ_3 w.r.t. the x-axis. These symmetries imply, in turn, that S_{Iy} and S_{Ix} are symmetric w.r.t. O. Accordingly, S_{Iy} and S_{Ix} are diametrically opposite on the circle B_i and belong, moreover, to a unique diameter of the ellipse H. Since the line OI is the symm-eccentric line (1.4) and the line $S_{Iy}OS_{Ix}$ is symmetric to OI w.r.t. the x-axis, we conclude that the line $S_{Iy}OS_{Ix}$ coincides with the eccentric line (1.3). Similarly, the second proposition of Theorem 2.9 can be demonstrated.

The newly introduced four points are

$$S_{Iy}\,(-(a-b)\cos\varepsilon\,;\,-(a-b)\sin\varepsilon)\,,\quad S_{Ix}\,((a-b)\cos\varepsilon\,;\,(a-b)\sin\varepsilon)\,; \tag{2.9}$$

$$S_{Ey}\,(-(a+b)\cos\varepsilon\,;\,(a+b)\sin\varepsilon)\,,\quad S_{Ex}\,((a+b)\cos\varepsilon\,;\,-(a+b)\sin\varepsilon)\,. \tag{2.10}$$

The points S_{Ix} (2.9), S_{Ex} (2.10), and Ψ_1 and Ψ_2 (2.5) belong to the circle Φ_3. Accordingly, joining them in pairs, a *complete* cyclic quadrangle results (Figure 3). An unexpected property of this complete quadrangle is that two of its diagonal points are fixed points, in spite of the dependence of the figure on the point's P arbitrary location on the ellipse H:

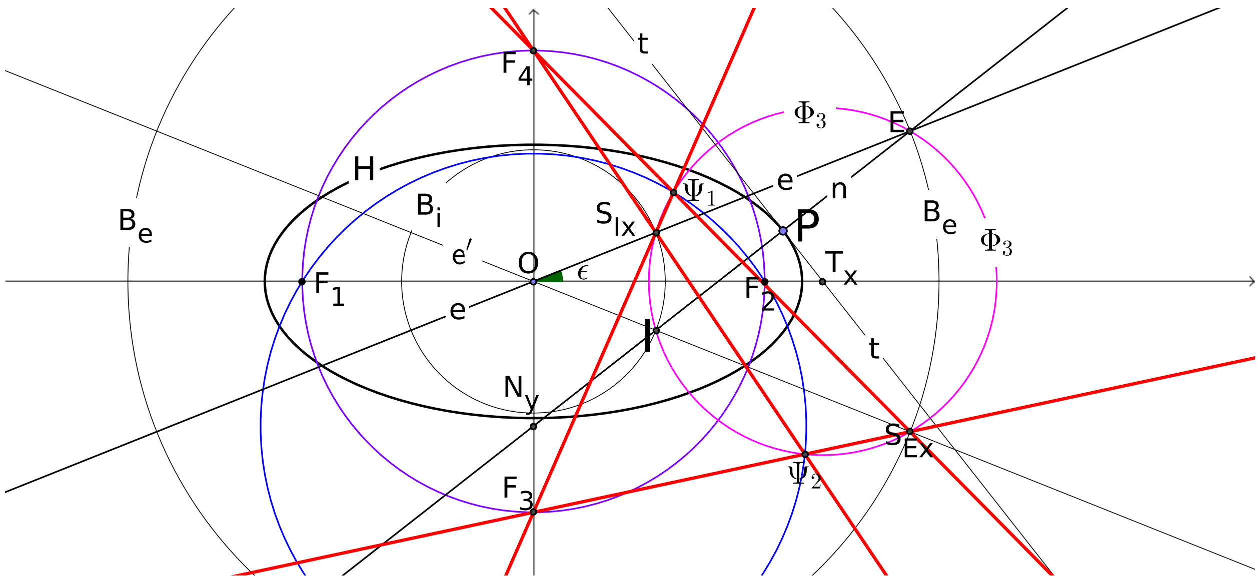

Figure 3: Illustrating Theorem 2.10: The points S_{Ix} (2.9), S_{Ex} (2.10), and Ψ_1 and Ψ_2 (2.5), belonging to the circle Φ_3 [magenta], define a complete cyclic quadrangle [red] (the lines $S_{Ix}S_{Ex}$ and $\Psi_1\Psi_2$ are not drawn), whose opposite sides ($S_{Ix}\Psi_1$, Ψ_2S_{Ex}) and ($S_{Ix}\Psi_2$, Ψ_1S_{Ex}) concur in the points F_3 and F_4, lying on the ellipse minor axis, at the invariant distance c from O, respectively.

Theorem 2.10. [Figure 3] *Let the complete quadrangle $S_{Ix}\Psi_1S_{Ex}\Psi_2$ be constructed. Its diagonal points have the following properties:*

- *the lines $S_{Ex}S_{Ix}$ and $\Psi_1\Psi_2$ meet at the x-intercept N_x of the normal to H at P;*
- *the lines $S_{Ex}\Psi_1$ and $S_{Ix}\Psi_2$ invariably meet at the point $F_4(0,\,c)$;*
- *the lines $S_{Ix}\Psi_1$ and $S_{Ex}\Psi_2$ invariably meet at the point $F_3(0,\,-c)$.*

Concerning the first item, observe that, as the points E and I (1.9) belong by definition to the normal (1.7), the line $S_{Ex}S_{Ix}$ mirrors the normal in the x-axis because the point S_{Ex} and S_{Ix} are the reflections of E and I w.r.t. the x-axis, respectively. Accordingly, the x-intercepts of both lines (normal and $S_{Ex}S_{Ix}$) coincide with N_x. Since the line $\Psi_1\Psi_2$ also passes through N_x (Theorem 2.3), the first item is demonstrated.

Concerning the second item, it suffices to construct the following determinants Δ_1 and Δ_2 by means of the coordinates of S_{Ex}, Ψ_1 and $F_4(0,\, c)$ for the former and S_{Ix}, Ψ_2 and $F_4(0,\, c)$ for the latter and verify that both determinants vanish identically:

$$\Delta_1 : \begin{vmatrix} (a+b)\cos\varepsilon & -(a+b)\sin\varepsilon & 1 \\ \dfrac{(a-c\sin\varepsilon)\,c^2\cos\varepsilon}{a^2\cos^2\varepsilon+b^2\sin^2\varepsilon} & \dfrac{(a-c\sin\varepsilon)\,bc}{a^2\cos^2\varepsilon+b^2\sin^2\varepsilon} & 1 \\ 0 & c & 1 \end{vmatrix}, \quad \Delta_2 : \begin{vmatrix} (a-b)\cos\varepsilon & (a-b)\sin\varepsilon & 1 \\ \dfrac{(a+c\sin\varepsilon)\,c^2\cos\varepsilon}{a^2\cos^2\varepsilon+b^2\sin^2\varepsilon} & -\dfrac{(a+c\sin\varepsilon)\,bc}{a^2\cos^2\varepsilon+b^2\sin^2\varepsilon} & 1 \\ 0 & c & 1 \end{vmatrix}.$$

Similarly, the next item is demonstrated. Of course, the quadrangle obtained by replacing $S_{Ix}\Psi_1 S_{Ex}\Psi_2$ by its reflection w.r.t. the x-axis has the same properties.

The points $F_3(0,-c)$ and $F_4(0,c)$ are concyclic with the foci on the following circle T:

$$x^2+y^2=a^2-b^2. \tag{2.11}$$

Theorem 2.11. *The circle* T (2.11) *is orthogonal to* Φ_3 (2.3). *The radical line of the circles* T *and* Φ_3 *contains the x-intercept* N_x (1.8) *of the normal.*

Indeed, as Φ_1 and Φ_2 are orthogonal to Φ_3 (Theorem 2.1), any circle belonging to the pencil determined by Φ_1 and Φ_2 — like T — is orthogonal to Φ_3, too. The comparison of (2.11) and (2.3) allows us to verify the second part of the statement.

3. Symbiotic conics

The fact that any point P on the ellipse H (1.1) is associated with a couple of orthogonal lines, namely, the tangent (1.5) and the normal (1.7) to H at P, has inspired to consider the conics for which the axes of symmetry of H on the one hand and the tangent and normal on the other hand exchange their roles. This is better told below.

Definition 4. [Figure 4] For any point P taken on the ellipse H, the *symbiotic conics* of H about P

(i) have P as center and the tangent and normal to H at P as axes of symmetry,

(ii) pass through the center O of the ellipse H, and

(iii) have the axes of symmetry of H as normal and tangent at O.

The major axis of H may be either normal or tangent to a symbiotic conic at O. Accordingly, two symbiotic conics exist for any P. By means of elementary and well-known methods they are determined as follows:

- the conic whose tangent at O is the minor axis of the ellipse H,

$$x^2\,\frac{a^2-b^2\cos^2\varepsilon}{a^2\cos^2\varepsilon}-2xy\,\frac{b}{a}\tan\varepsilon-2x\,\frac{c^2}{a\cos\varepsilon}+y^2=0, \tag{3.1}$$

- the conic whose tangent at O is the major axis of the ellipse H,

$$x^2\,\frac{b^2}{a^2}\tan^2\varepsilon-2xy\,\frac{b}{a}\tan\varepsilon+y^2\left(1-\frac{c^2}{a^2\cos^2\varepsilon}\right)+2y\,\frac{c^2}{a^2\cos^2\varepsilon}b\sin\varepsilon=0. \tag{3.2}$$

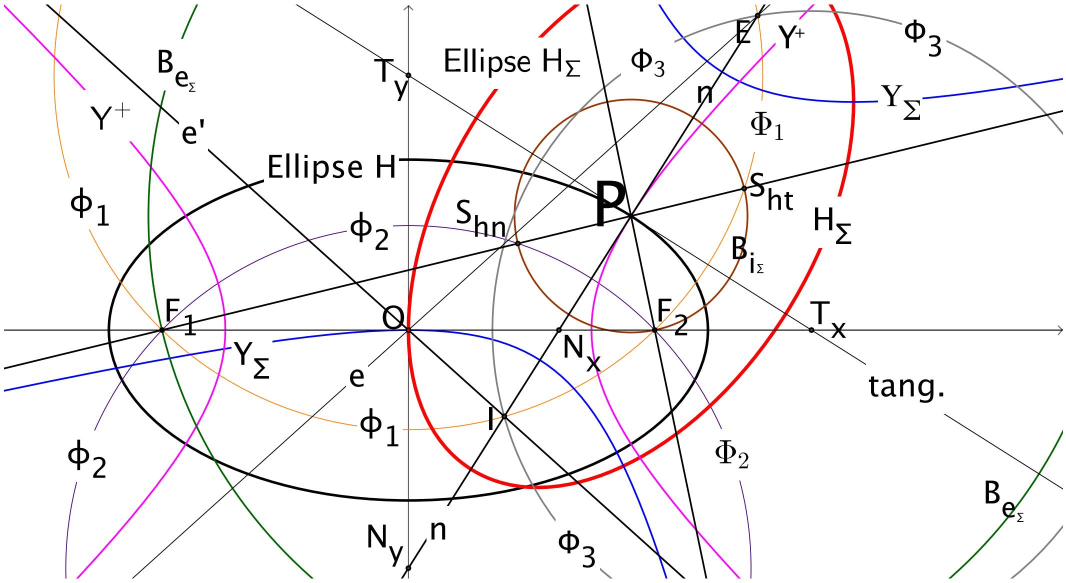

Figure 4: Illustrating the Definition of symbiotic conics and some related Theorems:
The ellipse H_Σ [red] and the hyperbola Y_Σ [blue] are the symbiotic conics of the ellipse H about its point P. Both H_Σ and Y_Σ share the tangent and normal to H at P as their axes of symmetry. They pass through the center O of H, admit here the axes of H as tangent and normal, and they are confocal (their foci E and I are the intersection of the normal n to H at P with the eccentric line e of P and the symm-eccentric e', respectively).
The eccentric and symm-eccentric lines of O, regarded as a point of H_Σ, are the focal radii PF_1 and PF_2 of the H point P. Barlotti's internal circle B_{i_Σ} [brown] associated with H_Σ passes through the *homolateral* focus F_2 of H, where it meets both circles Φ_1 [orange] and Φ_2 [indaco]. B_{i_Σ} further meets Φ_1 and Φ_2 at S_{ht} and S_{hn}, respectively. These points are symmetric to the homolateral focus F_2 w.r.t. the tangent and normal to H at P, respectively, and belong to the focal radius F_1P.
Barlotti's external circle B_{e_Σ} [green] passes through the *contralateral* focus F_1, where it meets both circles Φ_1 and Φ_2. B_{e_Σ} further meets Φ_1 and Φ_2 at S_{ct} and S_{cn}, respectively. These points are not displayed in the figure. S_{ct} and S_{cn} are symmetric to the contralateral focus F_1 w.r.t. the tangent and normal to H at P, respectively, and belong to the focal radius F_2P. The hyperbola Y_Σ has the focal radii PF_1 and PF_2 as asymptotes.
The symbiotic conics of the ellipse H_Σ about O are the ellipse H and the *adjoint hyperbola* Y^+ [magenta]. The asymptotes of Y^+ coincide with the eccentric and symm-eccentric line of P.

The conic (3.1) is an ellipse and (3.2) a hyperbola — hereinafter denoted by H_Σ and Y_Σ, respectively.

The semiaxes of the symbiotic ellipse H_Σ (3.1) are

$$a_{H_\Sigma} = a \quad \text{and} \quad b_{H_\Sigma} = c\cos\varepsilon\,. \tag{3.3}$$

The semiaxes of the symbiotic hyperbola Y_Σ (3.2) are

$$a_{Y_\Sigma} = b \quad \text{and} \quad b_{Y_\Sigma} = c\sin\varepsilon\,. \tag{3.4}$$

From (3.3) and (3.4) follows that the foci of both symbiotic conics H_Σ (3.1) and Y_Σ (3.2) lie at the same distance c_Σ from their common center P,

$$c_\Sigma = \sqrt{a_{H_\Sigma}^2 - b_{H_\Sigma}^2} = \sqrt{a_{Y_\Sigma}^2 + b_{Y_\Sigma}^2} = \sqrt{a^2\sin^2\varepsilon + b^2\cos^2\varepsilon}. \tag{3.5}$$

Therefore, the symbiotic conics H_Σ (3.1) and Y_Σ (3.2) are confocal and, accordingly, mutually orthogonal, too. On the other hand, since the points E and I (1.9) lie on the major axis n (1.7) of H_Σ, symmetric w.r.t. P, at the common distance c_Σ (3.5) from P, we may state:

Lemma 3.1. *The points E and I (1.9) coincide with the foci of the symbiotic conics H_Σ (3.1) and Y_Σ (3.2) of the ellipse H about P.*

The symbiotic ellipse H_Σ (3.1) admits, in turn, symbiotic conics about any of its general points. Definition 4 leads us to the identify the symbiotic ellipse of H_Σ about O with H (1.1). This conclusion implies that, if an object Ω (a point, line, circle etc.) is defined by a statement associated to the ellipse H and its point P and, on the other hand, an object Ω' plays the same role w.r.t. H_Σ and its point O — namely, if Ω' is *homologous to* Ω — then Ω is homologous to Ω'. The correspondence associating a geometrical object with its homologous one is involutory. On this basis, we may identify the points homologous to E and I with the foci of the ellipse H. Accordingly, the foci of H may be denoted also by E_Σ and I_Σ. Remembering that

(i) E and I have been introduced as the points where the normal to H at P meets the eccentric and the symm-eccentric lines of P, and that

(ii) the x-axis is the normal to H_Σ at O,

the eccentric and the symm-eccentric lines of O (regarded as a point of H_Σ) are to be identified with the lines linking the center P of H_Σ with the foci of H. More precisely, the focus of H which lies on the same side as P w.r.t. the minor axis (the homolateral focus F_h) coincides with I_Σ and the focal radius of H linking this focus with P coincides with the symm-eccentric line of O. The focus of H which lies on the other side (the contralateral focus F_c) and the associated focal radius of H coincide with E_Σ and the eccentric line of O, respectively.

As shown in [6, Theorems 1 and 2] by the author, the loci of E and I (1.9) are the external and internal circles B_e (1.10) and B_i (1.11). These circles are concentric with the ellipse H and have radii $a \pm b$. Analogously, Barlotti's external and internal circles for the ellipse H_Σ are the following circles (hereinafter, B_{e_Σ} and B_{i_Σ}, respectively):

$$(x - a\cos\varepsilon)^2 + (y - b\sin\varepsilon)^2 = (a + c\cos\varepsilon)^2\,, \tag{3.6}$$

$$(x - a\cos\varepsilon)^2 + (y - b\sin\varepsilon)^2 = (a - c\cos\varepsilon)^2\,. \tag{3.7}$$

The circles B_{e_Σ} (3.6) and B_{i_Σ} (3.7) are concentric with H_Σ; they have the radii $a_\Sigma \pm b_\Sigma$ and pass through E_Σ and I_Σ (namely, through the contralateral and the homolateral focus of H), respectively.

The asymptotes of the symbiotic hyperbola Y_Σ (3.2) are easily determined as follows:

$$y - b\sin\varepsilon = \frac{b\sin\varepsilon}{a\cos\varepsilon \pm c}\,(x - a\cos\varepsilon)\,. \tag{3.8}$$

They coincide with the focal radii PF_1 and PF_2 of the point P. The lines homologous to the focal radii PF_1 and PF_2 are the focal radii OE and OI of the point O (regarded as a point of H_Σ). The latter are, therefore, the asymptotes of the symbiotic hyperbola of H_Σ about O (hereinafter, the adjoint hyperbola Y^+). Since the symbiotic conics H_Σ and Y_Σ of a given ellipse H about P are confocal and orthogonal to each other, we conclude that the adjoint hyperbola Y^+ is confocal with and orthogonal to the ellipse H. It passes through P and admits the eccentric and symm-eccentric lines of P as its asymptotes,

$$\frac{x^2}{c^2\cos^2\varepsilon} - \frac{y^2}{c^2\sin^2\varepsilon} = 1\,. \tag{3.9}$$

The following statement summarizes the previous findings:

Theorem 3.1. *For any point P taken on the ellipse H (1.1), the following holds:*

(i) *There exist two symbiotic conics: the ellipse H_Σ (3.1) and the hyperbola Y_Σ (3.2)*[1]*. H_Σ and Y_Σ have as tangents at O the minor and major axis of the ellipse H, respectively, and share the foci, which are the points E and I (1.9));*
Barlotti's circles B_e (1.10)) and B_i (1.11)) associated with the ellipse H pass through the foci E and I of the ellipse H_Σ[2]*.*
The focal radii PF_1 and PF_2 of the point P are also

- *the eccentric and symm-eccentric line of O (regarded as a point of the ellipse H_Σ)*
- *and the asymptotes of the symbiotic hyperbola Y_Σ (3.2).*

(ii) *W.r.t. the ellipse H_Σ (3.1) and its point O, there exist two symbiotic conics: the ellipse H (1.1) and the adjoint hyperbola Y^+ (3.9)*[3]*. H and Y^+ admit as tangent at P the minor and major axis of the ellipse H_Σ, respectively, and share the foci F_1 and F_2.*
Barlotti's circles B_{e_Σ} (3.6) and B_{i_Σ} (3.7) associated with the ellipse H_Σ pass through the foci of the ellipse H.
The focal radii OE and OI of the point O are

- *the eccentric and symm-eccentric line of P (regarded as a point of the ellipse H)*
- *and the asymptotes of the adjoint hyperbola Y^+ (3.9).*

3.1. The effectiveness of the symbiotic conics approach

From any statement related to the ellipse H, a twin statement related to the ellipse H_Σ may be deduced, provided that any object entering the original statement is replaced by its homologous. In some cases, the new statement — far from being a trivial duplicate of the old one — may help us to see the geometrical facts under a new perspective, and even, to find new properties of H. This approach, which could be defined as *studying an ellipse by means of its symbiotic*, requires that further couples of homologous objects are identified.

(i) The point T_x, where the tangent t to H at P meets the major axis x of H, coincides with the point, where the normal to H_Σ at O (namely, the x-axis) meets the minor axis t of H_Σ. Therefore T_x plays, w.r.t. H_Σ and its point O, the same role as point N_y plays w.r.t. H and its point P. Analogously, the point N_y, where the normal n to H at P meets the minor axis y of H, coincides with the point, where the tangent to H_Σ at O (namely, the y-axis) meets the major axis n of H_Σ. Therefore N_y plays w.r.t. H_Σ and its point O the same role as point T_x w.r.t. H and its point P. Hence, points T_x and N_y are homologous to each other.

(ii) The point T_y, where the tangent t to H at P meets the minor axis y of H, coincides with the point, where the tangent y to H_Σ at O meets the minor axis t of H_Σ. As an effect of the exchange of roles between the lines t and y, the point T_y retains with H_Σ the same

[1] According to Definition 4, the center of both symbiotic conics H_Σ and Y_Σ is P. Their axes of symmetry are the normal and the tangent to H at P. H_Σ and Y_Σ pass through the center O of the ellipse H (1.1), admitting here the axes of H as tangent and normal.

[2] These properties of Barlotti's circles have already been described in [6]. They are repeated here to emphasize the symmetries between the ellipses H and H_Σ

[3] According to Definition 4, the center of both symbiotic conics H and Y^+ is O. Their axes of symmetry are the normal and tangent to H_Σ at O, namely, the x- and y- axis, respectively. H and Y^+ pass through the center P of the ellipse H_Σ, admitting here the axes of H_Σ as tangent and normal.

role as with H; it is an *auto-homologous* point. The same holds for the point N_x (because of the exchange of roles between the normal n to H at P and the major axis x of H) and, accordingly, for the whole line T_yN_x. Since this line coincides, by virtue of Theorem 2.3, with the radical line r_3 of Φ_1 and Φ_2, the points Ψ_1 and Ψ_2 – belonging to the line r_3 by definition – are auto-homologous, too.

(*iii*) As regards the circle Φ_1, its homologous should be (according to Definition 1) a circle with the center homologous to T_y, namely, T_y itself, and passing through points homologous to the H foci, namely, through the H_Σ foci E and I (Lemma 3.1). Since E and I have been proved (Theorem 2.2) to be concyclic with the H foci on the circle Φ_1, we conclude that the circle Φ_1 coincides with its homologous one. The circle homologous to Φ_2 coincides with Φ_3, since the latter admits as center T_x (which is homologous to the Φ_2 center N_y) and passes through the H_Σ foci E and I (Lemma 3.1). Indeed, Φ_2 and Φ_3 are homologous to each other.

Bearing these results in mind, the following may be easily understood:

(*i*) Since the normal n to H at P may be regarded as the major axis of the ellipse H_Σ and the focal radii of P may be regarded as the eccentric and symm-eccentric lines of O (considered as a point of the ellipse H_Σ), the Theorem stating that *"the normal to the ellipse H at P bisects the angle formed by the focal radii of P"* appears as a consequence of the symmetry of the eccentric and symm-eccentric lines of O about the major axis n of H_Σ.

(*ii*) The author has shown in [6, Theorems 1 and 2] that the circle K constructed on the segment T_yT_x of the tangent to H at P intercepted between the axes as diameter, passes through the points E and I (Figure 5). By virtue of these results and remembering that E and I have been shown to coincide with the foci of H_Σ, we may conclude that the circle K_Σ — which is constructed on the segment T_yN_y of the tangent ($x = 0$) to H_Σ at O intercepted between the axes of H_Σ (namely, between the tangent and normal to H at P) as diameter — passes through the foci of the ellipse H. In this way, we have refound a well known Theorem (for example, see [5, p. 86] or [2, Chapter VI, p. 215]).

The converse of the afore mentioned Theorem is the following:

Theorem 3.2. *Let r and s be the following orthogonal lines through P:*

$$y = b\sin\varepsilon + m\,(x - a\cos\varepsilon)\,; \quad y = b\sin\varepsilon - \frac{1}{m}\,(x - a\cos\varepsilon)\,. \tag{3.10}$$

If R_y and S_y denoted the y-intercepts of the lines r and s, the circle constructed on the segment R_yS_y as diameter passes through the ellipse foci if, and only if, the lines r and s are tangent and normal to the ellipse at P.

Indeed, the points R_y and S_y are $R_y\,(0;\, b\sin\varepsilon - am\cos\varepsilon)$ and $S_y\,(0;\, b\sin\varepsilon + a\cos\varepsilon/m)$, respectively. The circle with the segment R_yS_y as diameter is

$$x^2 + \left(y - \frac{2mb\sin\varepsilon + a\cos\varepsilon\,(1-m^2)}{2m}\right)^2 = \left(\frac{a\cos\varepsilon}{2m}\,\left(1+m^2\right)\right)^2. \tag{3.11}$$

Setting $x^2 = a^2 - b^2$ and $y = 0$ in (3.11) and solving it for m, we get

$$m_1 = \frac{a}{b}\tan\varepsilon\,; \quad m_2 = -\frac{b}{a}\cot\varepsilon\,. \tag{3.12}$$

The slopes (3.12) identify the normal and tangent to H at P.

(*iii*) Theorem 2.7 has shown that the foci of H are conjugate w.r.t. the circle Φ_3 (2.3). The replacement of the objects in such statement by their homologous ones results in the following

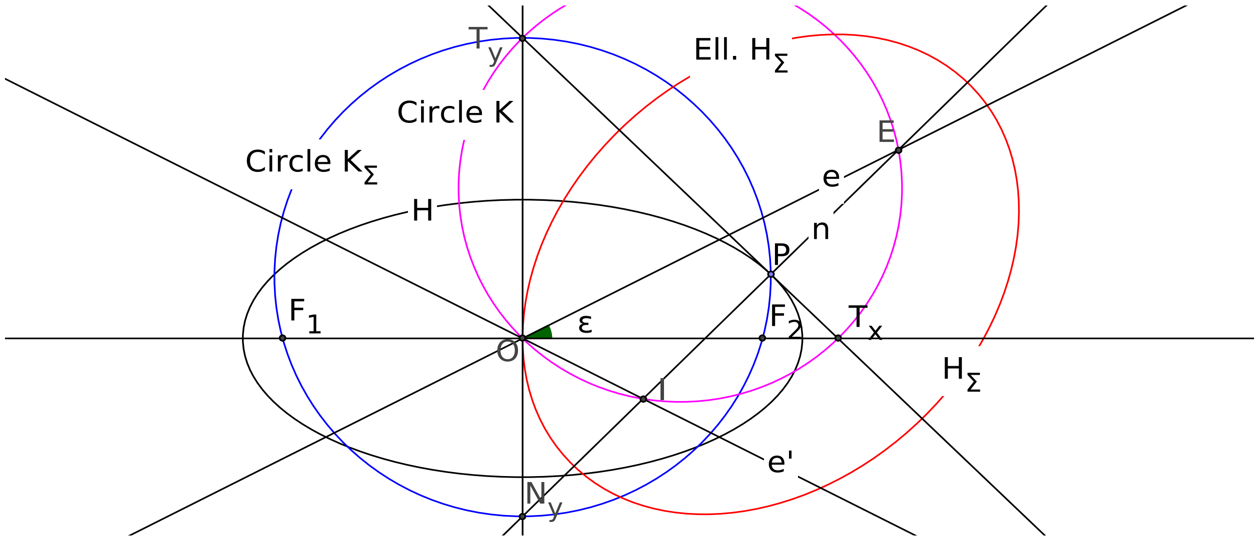

Figure 5: The circle K [magenta] is constructed taking as diameter the segment T_yT_x of the tangent to the ellipse H [black] at P intercepted between the axes. It passes through the foci E and I of the symbiotic ellipse H_Σ [red]. On the other hand, T_yN_y is a segment of the tangent ($x = 0$) to H_Σ at O, intercepted between the axes of H_Σ. Accordingly, the circle K_Σ [blue], constructed on the segment T_yN_y as diameter, is homologous to the circle K. Therefore, the circle K_Σ passes through the foci of the ellipse H.

Theorem 3.3. [Figure 4] *The points E and I* (1.9) *are conjugate w.r.t. the circle* Φ_2 (2.2).

(iv) Theorem 2.8 has shown that the radical center of Monge's circle M, Φ_1 (2.1) and Φ_3 (2.3) coincides with point P on the H ellipse. Remembering the expressions (3.3) for the ellipse's H_Σ semiaxes, Monge's circle M_Σ for H_Σ is

$$(x - a\cos\varepsilon)^2 + (y - b\sin\varepsilon)^2 = a^2 + c^2\cos^2\varepsilon\,. \tag{3.13}$$

Therefore, we may write the following statement, homologous to Theorem 2.8:

Theorem 3.4. *The radical center of the circles* Φ_1 (2.1), Φ_2 (2.2) *and* M_Σ (3.13) *coincides with the center O of the ellipse H.*

(v) Theorem 2.9 has described properties of the points S_{Iy} and S_{Ix} (2.9) and S_{Ey} and S_{Ex} (2.10), which represent intersections of Barlotti's circles B_e and B_i with the circles Φ_1 and Φ_3. The following Theorem 3.5 regards the homologous four points, which represent, in turn, intersections of the ellipse's H_Σ Barlotti's circles B_{e_Σ} (3.6) and B_{i_Σ} (3.7) with the circles Φ_1 and Φ_2. It should be stressed that the circles B_{e_Σ} and B_{i_Σ} may be redefined as circles about P, passing through the foci of H, avoiding any explicit reference to the symbiotic ellipse H_Σ.

Theorem 3.5. [Figure 4] *Taken a point P on the ellipse H, the following special points exist:*

$$S_{hn}\left(\frac{(c + a\cos\varepsilon)\cos\varepsilon - a\sin^2\varepsilon}{a + c\cos\varepsilon}c;\ \frac{2bc\sin\varepsilon\cos\varepsilon}{a + c\cos\varepsilon}\right);\quad S_{ht}\left(\frac{(a^2 + b^2)\cos\varepsilon + ac}{a + c\cos\varepsilon};\ \frac{2ab\sin\varepsilon}{a + c\cos\varepsilon}\right) \tag{3.14}$$

$$S_{cn}\left(\frac{(c - a\cos\varepsilon)\cos\varepsilon + a\sin^2\varepsilon}{a - c\cos\varepsilon}c;\ \frac{-2bc\sin\varepsilon\cos\varepsilon}{a - c\cos\varepsilon}\right);\ S_{ct}\left(\frac{(a^2 + b^2)\cos\varepsilon - ac}{a - c\cos\varepsilon};\ \frac{2ab\sin\varepsilon}{a - c\cos\varepsilon}\right) \tag{3.15}$$

(a) *S_{hn} is common to the circles B_{i_Σ}* (3.7) *and Φ_2* (2.2)*, belongs to the focal radius PF_c and is symmetric to the homolateral focus F_h w.r.t. the normal to H at P.*

(b) *S_{ht} is common to the circles B_{i_Σ}* (3.7) *and Φ_1* (2.1)*, belongs to the focal radius PF_c and is symmetric to the homolateral focus F_h w.r.t. the tangent to H at P.*

(c) *S_{cn} is common to the circles B_{e_Σ}* (3.6) *and Φ_2* (2.2)*, belongs to the focal radius PF_h and is symmetric to the contralateral focus F_c w.r.t. the normal to H at P.*

(d) *S_{ct} is common to the circles B_{e_Σ}* (3.6) *and Φ_1* (2.1)*, belongs to the focal radius PF_h and is symmetric to the contralateral focus F_c w.r.t. the tangent to H at P.*

The following should be explicited mentioned:

Corollary 3.1. *The points S_{hn} and S_{ht} are collinear on the focal radius PF_c. The points S_{cn} and S_{ct} are collinear on the focal radius PF_h.*

Theorem 2.2 may be integrated by Theorem 3.5, items (b) and (d), which results in the following new formulation:

Theorem 3.6. *The foci of H, the points E and I* (1.9)*, the point S_{ht}* (3.14) *(that is the point symmetric to the homolateral focus F_h w.r.t. the tangent to H at P), and the point S_{ct}* (3.15) *(that is the point symmetric to the contralateral focus F_c w.r.t. the tangent to H at P) are concyclic on the circle Φ_1* (2.1)*.*

Definition 2 and Theorem 3.5, and the items (a) and (c) may be synthesized as follows:

Theorem 3.7. *The foci of H and the points S_{hn}* (3.14) *and S_{cn}* (3.15) *are concyclic on the circle Φ_2* (2.2)*.*

(vi) In Section 2 it has been shown (Theorem 2.10, Figure 3) that any point P taken on the ellipse H may be associated with the quadrangle $S_{Ix}\Psi_1 S_{Ex}\Psi_2$ inscribed in the circle Φ_3. Analogously, the point O, taken on the ellipse H_Σ, may be associated (Figure 6) with the quadrangle $S_{hn}\Psi_1 S_{cn}\Psi_2$, which is inscribed in the circle Φ_2 and is homologous to $S_{Ix}\Psi_1 S_{Ex}\Psi_2$. Indeed, S_{hn} and S_{cn} correspond to S_{Ix} and S_{Ex}, respectively, while Ψ_1 and Ψ_2 correspond to themselves. Accordingly, Theorem 2.10 may be invoked for the complete quadrangle $S_{hn}\Psi_1 S_{cn}\Psi_2$, resulting in the following statement, where any explicit mention of the symbiotic ellipse H_Σ is avoided:

Theorem 3.8. [Figure 6] *Let us consider the complete quadrangle $S_{hn}\Psi_1 S_{cn}\Psi_2$. The opposite sides ($\Psi_1 S_{hn}$, $S_{cn}\Psi_2$), and ($\Psi_1 S_{cn}$, $S_{hn}\Psi_2$) meet in the points F_{3_Σ} and F_{4_Σ}, which lie on the tangent to H at P and are concyclic with E and I, on a circle denoted by T_Σ. The circle T_Σ is orthogonal to Φ_2.*

3.2. The fourth circle

Further symmetry relations among the circles Φ_1, Φ_2 and Φ_3 and the ellipses H and H_Σ appear, if we introduce the following objects:

- the imaginary foci of the ellipse H,

$$F_1^i\,(0,\, ic)\,, \quad F_2^i\,(0,\, -ic)\,; \tag{3.16}$$

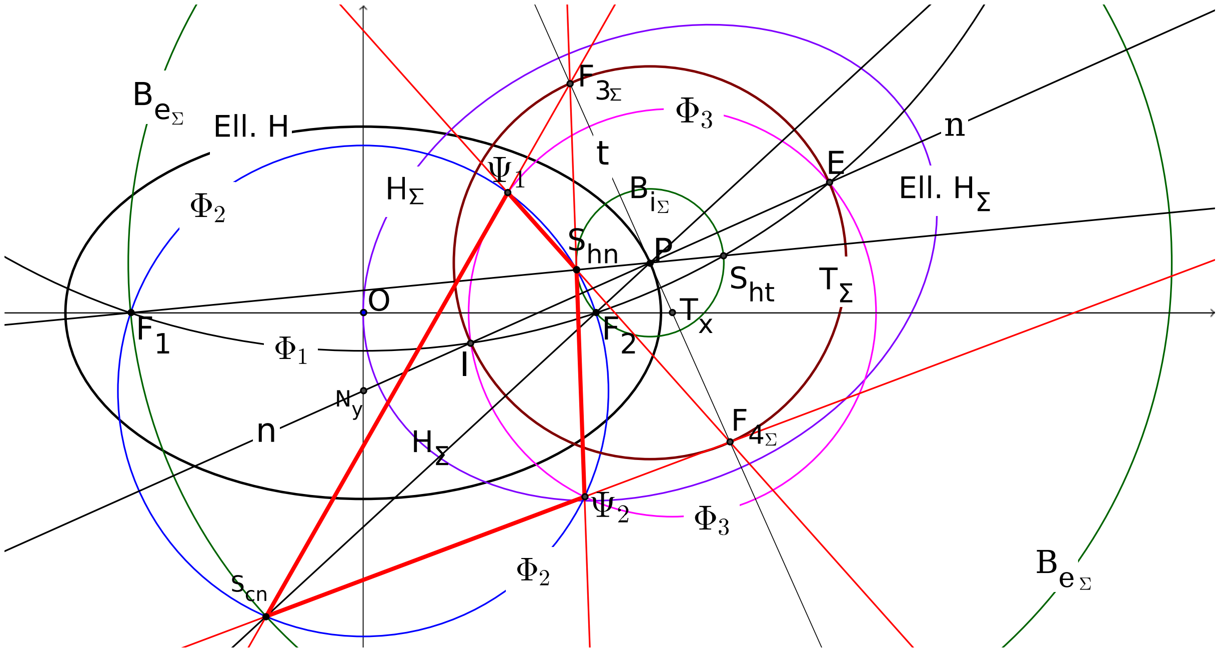

Figure 6: Illustrating Theorem 3.8: The points Ψ_1 and Ψ_2 (representing the intersections of the circles Φ_2 [blue] and Φ_3 [magenta]), and the points S_{hn} (3.14) (symmetric to the homolateral focus F_2 w.r.t. the normal n to H at P) and S_{cn} (3.15) (symmetric to the contralateral focus F_1 w.r.t. the normal n to H at P) define a complete quadrangle [red]; the opposite sides ($\Psi_1 S_{cn}$, $\Psi_2 S_{hn}$) and ($\Psi_1 S_{hn}$, $\Psi_2 S_{cn}$) concur in F_{3_Σ} and F_{4_Σ}, respectively. These points belong to the tangent to H at P, which is also the minor axis of the symbiotic ellipse H_Σ [blue]. F_{3_Σ} and F_{4_Σ} lie symmetrically w.r.t. P at the distance $c_\Sigma = PE = PI$. Accordingly, F_{3_Σ} and F_{4_Σ} are concyclic with the foci E and I of H_Σ on the circle T_Σ [brown], orthogonal to Φ_2.

- the imaginary foci of the ellipse H_Σ,

$$E^i_\Sigma\left(a(\cos\varepsilon - i\sin\varepsilon);\, b(\sin\varepsilon + i\cos\varepsilon)\right); \quad I^i_2\left(a(\cos\varepsilon + i\sin\varepsilon);\, b(\sin\varepsilon - i\cos\varepsilon)\right); \tag{3.17}$$

- the imaginary circle centered at the radical center N_x (1.8) of the circles Φ_1, Φ_2 and Φ_3, orthogonal to all of them (hereinafter, the circle Φ_4):

$$\left(x - \frac{c^2}{a}\cos\varepsilon\right)^2 + y^2 = -\frac{c^2}{a^2}\left(a^2\sin^2\varepsilon + b^2\cos^2\varepsilon\right). \tag{3.18}$$

Repeating, for the sake of completeness, some already stated facts[4], we may state the following Theorem, whose genuine novel parts can be demonstrated analytically:

Theorem 3.9. *The circles Φ_1, Φ_2, Φ_3, and Φ_4* (3.18) *have the following properties:*

(i) *the circle Φ_1 passes through the foci of the ellipses H and H_Σ;*

(ii) *the circle Φ_2 passes through the foci of H and the imaginary foci of H_Σ;*

(iii) *the circle Φ_3 passes through the foci of H_Σ and the imaginary foci of H;*

(iv) *the circle Φ_4* (3.18) *passes through the imaginary foci of the ellipses H and H_Σ.*

[4] Item 1 is just a new version of Theorem 2.2, reformulated so as to take into account the fact that the points E and I are the foci of the ellipse H_Σ. The passage of Φ_2 through the foci of H is part of the definition of the circle Φ_2. The passage of Φ_3 through the foci of H_Σ is part of the definition of the circle Φ_3, reformulated so as to take into account the fact that the points E and I are the foci of the ellipse H_Σ

Remark on Corollary 3.1. Because of the orthogonality of the ellipse H and the adjoint hyperbola Y^+ (3.9), the tangent and normal to H at P may be regarded as normal and tangent to Y^+ at P, respectively. Accordingly, the points symmetric to the focus F_h $[F_c]$ w.r.t. the tangent and the normal to Y^+ (3.9) at P are S_{hn} and S_{ht} $[S_{cn}$ and $S_{ct}]$, respectively. Such points are collinear on the focal radius PF_c $[PF_h]$. Therefore Corollary 3.1 holds also for P taken on a hyperbola. Moreover, if a point $P(x_0; y_0)$ is taken on the parabola Π : $y^2 = 2px$, the points symmetric to the focus $F\left(\frac{p}{2}; 0\right)$ w.r.t. the tangent and the normal to Π at P are $S_{Ft}\left(-\frac{p}{2}; y_0\right)$ and $S_{Fn}\left(2x_0 + \frac{p}{2}; y_0\right)$, respectively. The line connecting such two points is parallel to the parabola's axis of symmetry and passes, therefore, through the focus at infinity of the parabola.

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