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# Ruled and Quadric Surfaces Satisfying $\Delta^{III}\boldsymbol{x} = \varLambda\,\boldsymbol{x}$

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**Abstract.** We consider ruled and quadric surfaces in the 3-dimensional Euclidean space which are of coordinate finite type with respect to the third fundamental form $III$, i.e., their position vector $\boldsymbol{x}$ satisfies the relation $\Delta^{III}\boldsymbol{x} = \varLambda\,\boldsymbol{x}$ where $\varLambda$ is a square matrix of order 3. We show that helicoids and spheres are the only classes of surfaces mentioned above satisfying $\Delta^{III}\boldsymbol{x} = \varLambda\,\boldsymbol{x}$.

*Key Words:* surfaces in Euclidean space, surfaces of coordinate finite type, Beltrami operator

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## 1. Introduction

Let $S$ be a (connected) surface in a Euclidean 3-space $E^3$ referred to any system of coordinates $(u^1,\, u^2)$, which does not contain parabolic points. We denote by $b_{ij}$ the components of the second fundamental form $II = b_{ij}du^idu^j$ of $S$. Let $\varphi(u^1, u^2)$ and $\psi(u^1, u^2)$ be two sufficient differentiable functions on $S$. Then the first differential parameter of Beltrami with respect to the second fundamental form of $S$ is defined by

$$\nabla^{II}(\varphi, \psi) := b^{ij}\varphi_{/i}\psi_{/j},$$

where $\varphi_{/i} := \dfrac{\partial\varphi}{\partial u^i}$ and $(b^{ij})$ denotes the inverse tensor of $(b_{ij})$.

Let $e_{ij}$ be the components of the third fundamental form $III$ of $S$. Then the second differential parameter of Beltrami with respect to the third fundamental form of $S$ is defined by [9]

$$\Delta^{III}\varphi := -\frac{1}{\sqrt{e}}(\sqrt{e}\,e^{ij}\varphi_{/i})_{/j}{}^{1},$$

[1]with sign convention such that $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ for the metric $ds^2 = dx^2 + dy^2$.

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where $(e^{ij})$ denotes the inverse tensor of $(e_{ij})$ and $e := \det(e_{ij})$.

In [9], S. Stamatakis and H. Al-Zoubi showed for the position vector $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$ of $S$ the relation

$$\Delta^{III}\boldsymbol{x} = \nabla^{III}\left(\frac{2H}{K}, \boldsymbol{n}\right) - \frac{2H}{K}\,\boldsymbol{n}, \tag{1.1}$$

where $\boldsymbol{n}$ is the Gauss map, $K$ the Gauss curvature and $H$ the mean curvature of $S$. Moreover, in this context, the same authors proved that the surfaces $S:\ \boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$ satisfying the condition

$$\Delta^{III}\boldsymbol{x} = \lambda\boldsymbol{x}, \qquad \lambda \in \mathbb{R},$$

i.e., for which all coordinate functions are eigenfunctions of $\Delta^{III}$ with the same eigenvalue $\lambda$, are precisely either the minimal surfaces ($\lambda = 0$), or parts of spheres ($\lambda = 2$).

In [2] B.-Y. Chen introduced the notion of Euclidean immersions *of finite type*. In terms of B.-Y. Chen's theory, a surface $S$ is said to be of finite type, if its coordinate functions are a finite sum of eigenfunctions of the Beltrami operator $\Delta^{III}$. Therefore the two facts mentioned above can be stated as follows

- $S$ is minimal if and only if $S$ is of null type 1.
- $S$ lies in an ordinary sphere $\mathrm{S}^2$ if and only if $S$ is of type 1.

Following [2], we say that a surface $S$ is of finite type with respect to the fundamental form $III$, or briefly *of finite III-type* if the position vector $\boldsymbol{x}$ of $S$ can be written as a finite sum of nonconstant eigenvectors of the operator $\Delta^{III}$, i.e., if

$$\boldsymbol{x} = \boldsymbol{x}_0 + \sum_{i=1}^{m} \boldsymbol{x}_i, \qquad \Delta^{III}\boldsymbol{x}_i = \lambda_i\,\boldsymbol{x}_i, \qquad i = 1, \ldots, m\,, \tag{1.2}$$

where $\boldsymbol{x}_0$ is a fixed vector and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are eigenvalues of $\Delta^{III}$. When there are exactly $k$ nonconstant eigenvectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$ appearing in (1.2) which all belong to different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then $S$ is said to be *of III-type* $k$, otherwise $S$ is said to be *of infinite type*. When $\lambda_i = 0$ for some $i = 1, 2, \ldots, k$, then $S$ is said to be *of null III-type* $k$.

Up to now, very little is known about surfaces of finite *III*-type. Concerning this problem, the only known surfaces of finite *III*-type in $E^3$ are parts of spheres, the minimal surfaces and the parallel of the minimal surfaces which are of null *III*-type 2 (see [9]).

In this paper we shall be concerned with the ruled and quadric surfaces in $E^3$ which are connected, complete and which are of coordinate finite *III*-type, i.e., their position vector $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$ satisfies the relation

$$\Delta^{III}\boldsymbol{x} = \Lambda\,\boldsymbol{x}, \tag{1.3}$$

where $\Lambda$ is a square matrix of order 3.

In [6] F. Dillen, J. Pas and L. Verstraelen studied coordinate finite type with respect to the first fundamental form $I = g_{ij}du^idu^j$ and they proved

**Theorem 1.** *The only surfaces in* $\mathbb{R}^3$ *satisfying*

$$\Delta^{I}\boldsymbol{x} = A\,\boldsymbol{x} + B, \qquad A \in M(3,3), \qquad B \in M(3,1),$$

*are the minimal surfaces, the spheres and the circular cylinders.*

Here $M(m,n)$ denotes the set of all matrices of the type $(m,n)$. On the other hand, O. Garay showed in [7]

**Theorem 2.** *The only complete surfaces of revolution in* $\mathbb{R}^3$, *whose component functions are eigenfunctions of their Laplacian, are catenoids, spheres and right circular cylinders.*

Recently, H. Al-Zoubi and S. Stamatakis studied coordinate finite type with respect to the third fundamental form, more precisely, in [10] they proved

**Theorem 3.** *A surface of revolution* $S$ *in* $\mathbb{R}^3$ *satisfies* (1.3) *if and only if* $S$ *is a catenoid or a part of a sphere.*

## 2. Main results

Our main results are the following.

**Proposition 1.** *The only ruled surfaces in the 3-dimensional Euclidean space that satisfy* (1.3) *are the helicoids.*

**Proposition 2.** *The only quadric surfaces in the 3-dimensional Euclidean space that satisfy* (1.3) *are the spheres.*

Our discussion is local, which means that we show in fact that any open part of a ruled or a quadric satisfies (1.3) if it is an open part of a helicoid or an open part of a sphere, respectively.

Before starting the proof of our main results, we first show that the surfaces mentioned in the above propositions indeed satisfy the condition (1.3). On a helicoid the mean curvature vanishes, so, by virtue of (1.1), $\Delta^{III}\boldsymbol{x} = \mathbf{0}$. Therefore a helicoid satisfies (1.3), where $\Lambda$ is the null matrix in $M(3,3)$.

Let $\mathrm{S}^2(r)$ be a sphere of radius $r$ centered at the origin. If $\boldsymbol{x}$ denotes the position vector field of $\mathrm{S}^2(r)$, then the Gauss map $\boldsymbol{n}$ is given by $-\dfrac{\boldsymbol{x}}{r}$. For the Gauss curvature $K$ and the mean curvature $H$ of $\mathrm{S}^2(r)$ we have $K = \dfrac{1}{r^2}$ and $H = \dfrac{1}{r}$. So, by virtue of (1.1), we obtain

$$\Delta^{III}\boldsymbol{x} = 2\boldsymbol{x},$$

and we find that $\mathrm{S}^2(r)$ satisfies (1.3) with

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

## 3. Proof of Proposition 1

Let $S$ be a ruled surface in $E^3$. We suppose that $S$ is a non-cylindrical ruled surface. This surface can be expressed in terms of a directrix curve $\boldsymbol{\alpha}(s)$ and a unit vectorfield $\boldsymbol{\beta}(s)$ pointing along the rulings as

$$S\colon\ \boldsymbol{x}(s,t) = \boldsymbol{\alpha}(s) + t\,\boldsymbol{\beta}(s), \quad s \in J, \quad -\infty < t < \infty.$$

Moreover, we can take the parameter $s$ to be the arc length along the spherical curve $\boldsymbol{\beta}(s)$. Then we have

$$\langle \boldsymbol{\alpha}', \boldsymbol{\beta} \rangle = 0, \quad \langle \boldsymbol{\beta}, \boldsymbol{\beta} \rangle = 1, \quad \langle \boldsymbol{\beta}', \boldsymbol{\beta}' \rangle = 1, \tag{3.1}$$

where the prime denotes the derivative with respect to $s$. The first fundamental form of $S$ is

$$I = q\,ds^2 + dt^2,$$

while the second fundamental form is

$$II = \frac{p}{\sqrt{q}}\,ds^2 + \frac{2A}{\sqrt{q}}\,ds\,dt,$$

where

$$\begin{aligned} q &= \langle\boldsymbol{\alpha}',\boldsymbol{\alpha}'\rangle + 2\langle\boldsymbol{\alpha}',\boldsymbol{\beta}'\rangle t + t^2, \\ p &= (\boldsymbol{\alpha}',\boldsymbol{\beta},\boldsymbol{\alpha}'') + [(\boldsymbol{\alpha}',\boldsymbol{\beta},\boldsymbol{\beta}'') + (\boldsymbol{\beta}',\boldsymbol{\beta},\boldsymbol{\alpha}'')]\,t + (\boldsymbol{\beta}',\boldsymbol{\beta},\boldsymbol{\beta}'')\,t^2, \\ A &= (\boldsymbol{\alpha}',\boldsymbol{\beta},\boldsymbol{\beta}')\,. \end{aligned}$$

For convenience, we put

$$\begin{aligned} \kappa &:= \langle\boldsymbol{\alpha}',\boldsymbol{\alpha}'\rangle, & \lambda &:= \langle\boldsymbol{\alpha}',\boldsymbol{\beta}'\rangle, \\ \mu &:= (\boldsymbol{\beta}',\boldsymbol{\beta},\boldsymbol{\beta}'')\,, & \nu &:= (\boldsymbol{\alpha}',\boldsymbol{\beta},\boldsymbol{\beta}'') + (\boldsymbol{\beta}',\boldsymbol{\beta},\boldsymbol{\alpha}'')\,, \\ \rho &:= (\boldsymbol{\alpha}',\boldsymbol{\beta},\boldsymbol{\alpha}'')\,, \end{aligned}$$

and thus we have

$$q = t^2 + 2\lambda t + \kappa, \qquad p = \mu t^2 + \nu t + \rho.$$

For the Gauss curvature $K$ of $S$ we find

$$K = -\frac{A^2}{q^2}. \tag{3.2}$$

The Beltrami operator with respect to the third fundamental form can be expressed, after a lengthy computation, as follows.

$$\begin{aligned} \Delta^{III} &= -\frac{q}{A^2}\frac{\partial^2}{\partial s^2} + \frac{2qp}{A^3}\frac{\partial^2}{\partial s\partial t} - \left(\frac{q^2}{A^2} + \frac{qp^2}{A^4}\right)\frac{\partial^2}{\partial t^2} + \left(\frac{q_s}{2A^2} + \frac{qp_t}{A^3} - \frac{pq_t}{2A^3}\right)\frac{\partial}{\partial s} \\ &\quad + \left(\frac{qp_s}{A^3} - \frac{pq_s}{2A^3} - \frac{pqA'}{A^4} - \frac{qq_t}{2A^2} + \frac{p^2q_t}{2A^4} - \frac{2qpp_t}{A^4}\right)\frac{\partial}{\partial t} \\ &= Q_1\frac{\partial^2}{\partial s^2} + Q_2\frac{\partial^2}{\partial s\partial t} + Q_3\frac{\partial}{\partial s} + Q_4\frac{\partial}{\partial t} + Q_5\frac{\partial^2}{\partial t^2}, \end{aligned} \tag{3.3}$$

where

$$q_t = \frac{\partial q}{\partial t}, \qquad q_s = \frac{\partial q}{\partial s}, \qquad p_t = \frac{\partial p}{\partial t}, \qquad p_s = \frac{\partial p}{\partial s},$$

and $Q_1, Q_2, \ldots, Q_5$ are polynomials in $t$ with functions in $s$ as coefficients, and $\deg(Q_i) \leq 6$. More precisely we have

$$\begin{aligned} Q_1 &= -\frac{1}{A^2}\left[t^2 + 2\lambda t + \kappa\right], \\ Q_2 &= \frac{2}{A^3}\left[\mu t^4 + (2\lambda\mu + \nu)\,t^3 + (2\lambda\nu + \rho + \kappa\mu)\,t^2 + (2\lambda\rho + \kappa\nu)\,t + \kappa\rho\right], \\ Q_3 &= \frac{1}{A^3}\left[\mu t^3 + 3\lambda\mu t^2 + (\lambda\nu - \rho + 2\kappa\mu + \lambda' A)\,t + \frac{1}{2}\kappa' A - \lambda\rho + \kappa\nu\right], \end{aligned}$$

$$
\begin{aligned}
Q_4 \;=\; & \frac{1}{A^4}\Big[-3\mu^2 t^5 + (\mu' A - \mu A' - 4\mu\nu - 7\lambda\mu^2)\, t^4 \\
& + (\nu' A - \nu A' + 2\lambda\mu' A - 2\lambda\mu A' - \lambda'\mu A - A^2 - 10\lambda\mu\nu - 2\mu\rho - \nu^2 - 4\kappa\mu^2)t^3 \\
& + (\kappa\mu' A - \kappa\mu A' - \frac{1}{2}\kappa'\mu A + 2\lambda\nu' A - 2\lambda\nu A' - \lambda'\nu A \\
& - \rho A' + \rho' A - 3\lambda A^2 - 3\lambda\nu^2 - 6\lambda\mu\rho - 6\kappa\mu\nu)t^2 \\
& + (\kappa\nu' A - \kappa\nu A' - \frac{1}{2}\kappa'\nu A + 2\lambda\rho' A - 2\lambda\rho A' - \lambda'\rho A \\
& - \kappa A^2 - 2\lambda^2 A^2 - 2\kappa\nu^2 + \rho^2 - 2\lambda\nu\rho - 4\kappa\mu\rho)t \\
& + (\kappa\rho' A - \kappa\rho A' - \frac{1}{2}\kappa'\rho A + \lambda\rho^2 - \kappa\lambda A^2 - 2\kappa\nu\rho)\Big], \\
Q_5 \;=\; & -\frac{1}{A^4}\Big[\mu^2 t^6 + (2\mu\nu + 2\lambda\mu^2)\, t^5 + (2\mu\rho + \nu^2 + 4\lambda\mu\nu + \kappa\mu^2 + A^2)t^4 \\
& + (2\nu\rho + 4\lambda\mu\rho + 2\lambda\nu^2 + 2\kappa\mu\nu + 4\lambda A^2)t^3 + (\rho^2 + 4\lambda\nu\rho + 2\kappa\mu\rho + \kappa\nu^2 \\
& + 4\lambda^2 A^2 + 2\kappa A^2)t^2 + (2\lambda\rho^2 + 2\kappa\nu\rho + 4\lambda\kappa A^2)t + (\kappa\rho^2 + \kappa^2 A^2)\Big].
\end{aligned}
$$

Applying (3.3) for the position vector $\boldsymbol{x}$, one finds

$$\Delta^{III}\boldsymbol{x} = Q_1\boldsymbol{\alpha}'' + Q_2\boldsymbol{\beta}' + Q_3\boldsymbol{\alpha}' + Q_4\boldsymbol{\beta} + (Q_1\boldsymbol{\beta}'' + Q_3\boldsymbol{\beta}')\, t. \tag{3.4}$$

Let $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ be the coordinate functions of $\boldsymbol{x}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. By virtue of (3.4) we obtain

$$\Delta^{III} x_i = Q_1\alpha_i'' + Q_2\beta_i' + Q_3\alpha_i' + Q_4\beta_i + (Q_1\beta_i'' + Q_3\beta_i')\, t, \quad i = 1,2,3\,. \tag{3.5}$$

We denote the entries of the matrix $\Lambda$ by $\lambda_{ij}$ for $i, j = 1,2,3$. Using (3.5) and condition (1.3), we have for $i = 1,2,3$

$$Q_1\alpha_i'' + Q_2\beta_i' + Q_3\alpha_i' + Q_4\beta_i + (Q_1\beta_i'' + Q_3\beta_i')t = \lambda_{i1}\alpha_1 + \lambda_{i2}\alpha_2 + \lambda_{i3}\alpha_3 + (\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3)t.$$

Consequently

$$
\begin{aligned}
& -3\mu^2\beta_i\, t^5 + \big[(\mu' A - \mu A' - 4\mu\nu - 7\lambda\mu^2)\beta_i + 3\mu A\beta_i'\big]t^4 + \big[\mu A\alpha_i' - A^2\beta_i'' + (2\nu A + 7\lambda\mu A)\beta_i' \\
& + (\nu' A - \nu A' + 2\lambda\mu' A - 2\lambda\mu A' - \lambda'\mu A - A^2 - 10\lambda\mu\nu - 2\mu\rho - \nu^2 - 4\kappa\mu^2)\beta_i\big]t^3 \\
& + \big[(\kappa\mu' A - \kappa\mu A' - \tfrac{1}{2}\kappa'\mu A + 2\lambda\nu' A - 2\lambda\nu A' - \lambda'\nu A - \rho A' + \rho' A - 3\lambda A^2 - 3\lambda\nu^2 \\
& - 6\lambda\mu\rho - 6\kappa\mu\nu)\beta_i + 3\lambda\mu A\alpha_i' - 2\lambda A^2\beta_i'' - A^2\alpha_i'' + (\lambda' A + 5\lambda\nu + 4\kappa\mu + \rho)A\beta_i'\big]t^2 \\
& + \big[(\tfrac{1}{2}\kappa' A + 3\kappa\nu + 3\lambda\rho)A\beta_i' + (\kappa\nu' A - \kappa\nu A' - \tfrac{1}{2}\kappa'\nu A + 2\lambda\rho' A - 2\lambda\rho A' - \lambda'\rho A \\
& - \kappa A^2 - 2\lambda^2 A^2 - 2\kappa\nu^2 + \rho^2 - 2\lambda\nu\rho - 4\kappa\mu\rho)\beta_i - 2\lambda A^2\alpha_i'' - \kappa A^2\beta_i'' \\
& + (\lambda\nu - \rho + 2\kappa\mu + \lambda' A)A\alpha_i'\big]t - A^4\big(\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3\big)t \\
& + (\kappa\rho' A - \kappa\rho A' - \tfrac{1}{2}\kappa'\rho A + \lambda\rho^2 - \kappa\lambda A^2 - 2\kappa\nu\rho)\beta_i - \kappa A^2\alpha_i'' + 2\kappa\rho A\beta_i' \\
& + (\tfrac{1}{2}\kappa' A - \lambda\rho + \kappa\nu)A\alpha_i' - A^4(\lambda_{i1}\alpha_1 + \lambda_{i2}\alpha_2 + \lambda_{i3}\alpha_3) = 0.
\end{aligned} \tag{3.6}
$$

For $i = 1,2,3$, the left hand side of (3.6) is a polynomial in $t$ with functions in $s$ as coefficients . This implies that the coefficients of the powers of $t$ in (3.6) must be zeros, so we obtain, for $i = 1,2,3$, the following equations.

$$3\mu^2\beta_i = 0, \tag{3.7}$$

$$
\begin{aligned}
(\mu' A - \mu A' - 4\mu\nu - 7\lambda\mu^2)\beta_i + 3\mu A\beta_i' &= 0, \\
\mu A\alpha_i' - A^2\beta_i'' + (2\nu A + 7\lambda\mu A)\beta_i' + (\nu' A - \nu A' + 2\lambda\mu' A & \\
- 2\lambda\mu A' - \lambda'\mu A - A^2 - 10\lambda\mu\nu - 2\mu\rho - \nu^2 - 4\kappa\mu^2)\beta_i &= 0,
\end{aligned} \tag{3.8}
$$

$$\begin{aligned}(\kappa\mu' A - \kappa\mu A' - \tfrac{1}{2}\kappa'\mu A + 2\lambda\nu' A - 2\lambda\nu A' - \lambda'\nu A - \rho A' + \rho' A - 3\lambda A^2 - 3\lambda\nu^2 \\ -6\lambda\mu\rho - 6\kappa\mu\nu)\beta_i + 3\lambda\mu A\alpha_i' - 2\lambda A^2\beta_i'' - A^2\alpha_i'' + (\lambda' A + 5\lambda\nu + 4\kappa\mu + \rho)A\beta_i' = 0,\end{aligned} \tag{3.9}$$

$$\begin{aligned}(\kappa\nu' A - \kappa\nu A' - \tfrac{1}{2}\kappa'\nu A + 2\lambda\rho' A - 2\lambda\rho A' - \lambda'\rho A - \kappa A^2 - 2\lambda^2 A^2 - 2\kappa\nu^2 \\ + \rho^2 - 2\lambda\nu\rho - 4\kappa\mu\rho)\beta_i - 2\lambda A^2\alpha_i'' - \kappa A^2\beta_i'' + (\tfrac{1}{2}\kappa' A + 3\kappa\nu + 3\lambda\rho)A\beta_i' \\ + (\lambda\nu - \rho + 2\kappa\mu + \lambda' A)A\alpha_i' = A^4(\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3),\end{aligned} \tag{3.10}$$

$$\begin{aligned}(\kappa\rho' A - \kappa\rho A' - \tfrac{1}{2}\kappa'\rho A + \lambda\rho^2 - \kappa\lambda A^2 - 2\kappa\nu\rho)\beta_i + 2\kappa\rho A\beta_i' \\ + (\tfrac{1}{2}\kappa' A - \lambda\rho + \kappa\nu)A\alpha_i' - \kappa A^2\alpha_i'' = A^4(\lambda_{i1}\alpha_1 + \lambda_{i2}\alpha_2 + \lambda_{i3}\alpha_3).\end{aligned} \tag{3.11}$$

From (3.7) one finds

$$\mu = (\boldsymbol{\beta}', \boldsymbol{\beta}, \boldsymbol{\beta}'') = 0, \tag{3.12}$$

which implies that the vectors $\boldsymbol{\beta}'$, $\boldsymbol{\beta}$ and $\boldsymbol{\beta}''$ are linearly dependent, and hence there exist two functions $\sigma_1 = \sigma_1(s)$ and $\sigma_2 = \sigma_2(s)$ such that

$$\boldsymbol{\beta}'' = \sigma_1\boldsymbol{\beta} + \sigma_2\boldsymbol{\beta}'. \tag{3.13}$$

Upon differentiating $\langle\boldsymbol{\beta}', \boldsymbol{\beta}'\rangle = 1$, we obtain $\langle\boldsymbol{\beta}', \boldsymbol{\beta}''\rangle = 0$. So from (3.13) we have

$$\boldsymbol{\beta}'' = \sigma_1\boldsymbol{\beta}. \tag{3.14}$$

By taking the derivative of $\langle\boldsymbol{\beta}, \boldsymbol{\beta}\rangle = 1$ twice, we find that

$$\langle\boldsymbol{\beta}', \boldsymbol{\beta}'\rangle + \langle\boldsymbol{\beta}, \boldsymbol{\beta}''\rangle = 0.$$

But $\langle\boldsymbol{\beta}', \boldsymbol{\beta}'\rangle = 1$, and taking into account (3.14) we find that $\sigma_1(s) = -1$. Thus (3.14) becomes $\boldsymbol{\beta}'' = -\boldsymbol{\beta}$ which implies that

$$\beta_i'' = -\beta_i, \qquad i = 1, 2, 3\,. \tag{3.15}$$

Using (3.12) and (3.15), Equation (3.8) reduces to

$$2\nu A\beta_i' + (\nu' A - \nu A' - \nu^2)\beta_i = 0, \qquad i = 1, 2, 3\,,$$

or, in vector notation,

$$2\nu A\boldsymbol{\beta}' + (\nu' A - \nu A' - \nu^2)\boldsymbol{\beta} = \mathbf{0}. \tag{3.16}$$

By taking the derivative of $\langle\boldsymbol{\beta}, \boldsymbol{\beta}\rangle = 1$, we find that the vectors $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ are linearly independent, and so from (3.16) we obtain that $\nu A = 0$. We note that $A \neq 0$, since from (3.2) the Gauss curvature vanishes, so we are left with $\nu = 0$. Then Equation (3.9) becomes

$$-A^2\alpha_i'' + (\lambda' A + \rho)A\beta_i' + (\rho' A - \rho A' - \lambda A^2)\beta_i = 0, \qquad i = 1, 2, 3\,,$$

or, in vector notation,

$$-A^2\boldsymbol{\alpha}'' + (\lambda' A + \rho)A\boldsymbol{\beta}' + (\rho' A - \rho A' - \lambda A^2)\boldsymbol{\beta} = \mathbf{0}. \tag{3.17}$$

Taking the inner product of both sides of the above equation with $\boldsymbol{\beta}'$, we find in view of (3.1) that

$$-A^2\langle\boldsymbol{\alpha}'', \boldsymbol{\beta}'\rangle + \rho A + \lambda' A^2 = 0. \tag{3.18}$$

On differentiating $\lambda = \langle\boldsymbol{\alpha}', \boldsymbol{\beta}'\rangle$ with respect to $s$, by virtue of (3.15) and (3.1), we get

$$\lambda' = \langle\boldsymbol{\alpha}'', \boldsymbol{\beta}'\rangle + \langle\boldsymbol{\alpha}', \boldsymbol{\beta}''\rangle = \langle\boldsymbol{\alpha}'', \boldsymbol{\beta}'\rangle - \langle\boldsymbol{\alpha}', \boldsymbol{\beta}\rangle = \langle\boldsymbol{\alpha}'', \boldsymbol{\beta}'\rangle. \tag{3.19}$$

Hence, (3.18) reduces to $\rho A = 0$, which implies that $\rho = 0$. Thus the vectors $\boldsymbol{\alpha}'$, $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}''$ are linearly dependent, and so there exist two functions $\sigma_3 = \sigma_3(s)$ and $\sigma_4 = \sigma_4(s)$ such that

$$\boldsymbol{\alpha}'' = \sigma_3\boldsymbol{\beta} + \sigma_4\boldsymbol{\alpha}'. \tag{3.20}$$

Taking the inner product of both sides of the last equation with $\boldsymbol{\beta}$, we find in view of (3.1) that $\sigma_3 = \langle\boldsymbol{\alpha}'', \boldsymbol{\beta}\rangle$.

Now, by taking the derivative of $\langle\boldsymbol{\alpha}', \boldsymbol{\beta}\rangle = 0$, we find $\langle\boldsymbol{\alpha}'', \boldsymbol{\beta}\rangle + \langle\boldsymbol{\alpha}', \boldsymbol{\beta}'\rangle = 0$, that is

$$\langle\boldsymbol{\alpha}'', \boldsymbol{\beta}\rangle + \lambda = 0, \tag{3.21}$$

and hence $\sigma_3 = -\lambda$. Taking again the inner product of both sides of Equation (3.20) with $\boldsymbol{\beta}'$, we find in view of (3.1) that

$$\langle\boldsymbol{\alpha}'', \boldsymbol{\beta}'\rangle = \sigma_4\lambda. \tag{3.22}$$

Using (3.19), we find $\lambda' = \sigma_4\lambda$. Thus $\sigma_4 = \dfrac{\lambda'}{\lambda}$. Therefore

$$\boldsymbol{\alpha}'' = -\lambda\boldsymbol{\beta} + \frac{\lambda'}{\lambda}\boldsymbol{\alpha}'. \tag{3.23}$$

We distinguish two cases.

**Case 1**, $\lambda = 0$. Because of $\rho = 0$ Equation (3.17) would yield $A = 0$, which is clearly impossible for the surfaces under consideration.

**Case 2**, $\lambda \neq 0$. From (3.17), (3.23) and $\rho = 0$ we find that

$$-\frac{\lambda'}{\lambda}A^2\boldsymbol{\alpha}' + \lambda' A^2\boldsymbol{\beta}' = \mathbf{0}$$

which implies that $\lambda'(\boldsymbol{\alpha}' - \lambda\boldsymbol{\beta}') = \mathbf{0}$.

If $\lambda' \neq 0$, then $\boldsymbol{\alpha}' = \lambda\boldsymbol{\beta}'$. Hence $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ are linearly dependent, and so $A = 0$ which contradicts our previous assumption. Thus $\lambda' = 0$. From (3.23) we have

$$\boldsymbol{\alpha}'' = -\lambda\boldsymbol{\beta}. \tag{3.24}$$

On the other hand, by taking the derivative of $\kappa$ and using the last equation we obtain that $\kappa$ is constant. Hence Equations (3.10) and (3.11) reduce to

$$\begin{aligned} \lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3 &= 0, \\ \lambda_{i1}\alpha_1 + \lambda_{i2}\alpha_2 + \lambda_{i3}\alpha_3 &= 0, \end{aligned} \qquad i = 1, 2, 3\,,$$

and so $\lambda_{ij} = 0$ for $i, j = 1, 2, 3$.

Since the parameter $s$ is the arc length of the spherical curve $\boldsymbol{\beta}(s)$, and because of (3.12) we suppose, without loss of generality, that the parametrization of $\boldsymbol{\beta}(s)$ is

$$\boldsymbol{\beta}(s) = (\cos s,\ \sin s,\ 0).$$

Integrating (3.24) twice, we get

$$\boldsymbol{\alpha}(s) = (c_1 s + c_2 + \lambda\cos s,\ c_3 s + c_4 + \lambda\sin s,\ c_5 s + c_6),$$

where $c_i$ are constants for $i = 1, 2, \ldots, 6$ . Since $\kappa = \langle\boldsymbol{\alpha}', \boldsymbol{\alpha}'\rangle$ is constant, it's easy to show that $c_1 = c_3 = 0$. Hence $\boldsymbol{\alpha}(s)$ reduces to

$$\boldsymbol{\alpha}(s) = (c_2 + \lambda\cos s,\ c_4 + \lambda\sin s,\ c_5 s + c_6).$$

Thus we have

$$S\colon\ \boldsymbol{x}(s,t) = (c_2 + (\lambda + t)\cos s,\ c_4 + (\lambda + t)\sin s,\ c_5 s + c_6)$$

which is a helicoid.

# 4. Proof of Proposition 2

Let now $S$ be a quadric surface in the Euclidean 3-space $E^3$. Then $S$ is either ruled, or of one of the following two kinds

$$z^2 - ax^2 - by^2 = c, \qquad abc \neq 0, \tag{4.1}$$

or

$$z = \frac{a}{2}\,x^2 + \frac{b}{2}\,y^2, \qquad a > 0,\ b > 0. \tag{4.2}$$

If $S$ is ruled and satisfies (1.3), then by Proposition 1 $S$ is a helicoid. We first show that a quadric of the kind (4.1) satisfies (1.3) if and only if $a = -1$ and $b = -1$, which means that $S$ is a sphere. Next we show that a quadric of the kind (4.2) is never satisfying (1.3).

## 4.1. Quadrics of the first kind

This kind of quadric surfaces can be parametrized as follows

$$\boldsymbol{x}(u, v) = \left(u,\ v,\ \sqrt{c + au^2 + bv^2}\right).$$

Let's denote the function $c + au^2 + bv^2$ by $\omega$ and the function $c + a(a+1)u^2 + b(b+1)v^2$ by $T$. Then the components $g_{ij}$, $b_{ij}$ and $e_{ij}$ of the first, second and third fundamental tensors in (local) coordinates are the following

$$\begin{aligned}
g_{11} &= 1 + \frac{(au)^2}{\omega}, & g_{12} &= \frac{abuv}{\omega}, & g_{22} &= 1 + \frac{(bv)^2}{\omega}, \\
b_{11} &= \frac{a(c + bv^2)}{\omega\sqrt{T}}, & b_{12} &= -\frac{abuv}{\omega\sqrt{T}}, & b_{22} &= \frac{b(c + au^2)}{\omega\sqrt{T}},
\end{aligned}$$

and

$$\begin{aligned}
e_{11} &= \frac{a^2}{\omega T^2}\left[(buv)^2 + (bv^2 + c)^2 + b^2v^2\omega\right], \\
e_{12} &= -\frac{ab}{\omega T^2}\left[c(a+b)uv + abuv(u^2 + v^2 + \omega)\right], \\
e_{22} &= \frac{b^2}{\omega T^2}\left[(auv)^2 + (au^2 + c)^2 + a^2u^2\omega\right].
\end{aligned}$$

Notice that $\omega$ and $T$ are polynomials in $u$ and $v$. If for simplicity we put

$$\begin{aligned}
C(u, v) &:= (buv)^2 + (bv^2 + c)^2 + b^2v^2\omega, \\
B(u, v) &:= uv\left[c(a+b) + ab(u^2 + v^2 + \omega)\right], \\
A(u, v) &:= (auv)^2 + (au^2 + c)^2 + a^2u^2\omega,
\end{aligned}$$

then the third fundamental tensors $e_{ij}$ turns into

$$e_{11} = \frac{a^2}{\omega T^2}\,C(u, v), \quad e_{12} = -\frac{ab}{\omega T^2}\,B(u, v), \quad e_{22} = \frac{b^2}{\omega T^2}\,A(u, v).$$

Hence the Beltrami operator $\Delta^{III}$ of $S$ can be expressed as follows.

$$\begin{aligned}\Delta^{III} = &-\frac{T}{(abc)^2}\left[b^2A\frac{\partial^2}{\partial u^2}+2abB\frac{\partial^2}{\partial u\partial v}+a^2C\frac{\partial^2}{\partial v^2}\right]\\ &-\frac{T}{(abc)^2}\left[b\left(b\frac{\partial A}{\partial u}+a\frac{\partial B}{\partial v}\right)\frac{\partial}{\partial u}+a\left(a\frac{\partial C}{\partial v}+b\frac{\partial B}{\partial u}\right)\frac{\partial}{\partial v}\right]\\ &+\frac{T}{(abc)^2}\left[\frac{ab^2}{\omega}(uA+vB)\frac{\partial}{\partial u}+\frac{a^2b}{\omega}(uB+vC)\frac{\partial}{\partial v}\right]\\ &+\frac{1}{(abc)^2}\left[ab^2\left((a+1)uA+(b+1)vB\right)\frac{\partial}{\partial u}+a^2b\left((b+1)vC+(a+1)uB\right)\frac{\partial}{\partial v}\right].\end{aligned} \tag{4.3}$$

We remark that

$$\begin{aligned} b\frac{\partial A}{\partial u}+a\frac{\partial B}{\partial v} &= au\left[5ab(a+1)u^2+5ab(b+1)v^2+c(3ab+5b+a)\right],\\ a\frac{\partial C}{\partial v}+b\frac{\partial B}{\partial u} &= av\left[5ab(a+1)u^2+5ab(b+1)v^2+c(3ab+5a+b)\right],\\ uA+vB &= \left[c+a(a+1)u^2+a(b+1)v^2\right]u\omega,\\ uB+vC &= \left[c+b(a+1)u^2+b(b+1)v^2\right]v\omega,\\ (a+1)uA+(b+1)vB &= u\left[c(a+1)+a(a+1)u^2+a(b+1)v^2\right]T,\\ (b+1)vC+(a+1)uB &= v\left[c(b+1)+b(a+1)u^2+b(b+1)v^2\right]T.\end{aligned}$$

We denote by $\lambda_{ij}$ for $i,j=1,2,3$ the entries of the matrix $\Lambda$. On account of (1.3) we get

$$\Delta^{III}x_1 = \Delta^{III}u = \lambda_{11}u+\lambda_{12}v+\lambda_{13}\sqrt{\omega}, \tag{4.4}$$
$$\Delta^{III}x_2 = \Delta^{III}v = \lambda_{21}u+\lambda_{22}v+\lambda_{23}\sqrt{\omega}, \tag{4.5}$$
$$\Delta^{III}x_3 = \Delta^{III}\sqrt{\omega} = \lambda_{31}u+\lambda_{32}v+\lambda_{33}\sqrt{\omega}.$$

Applying (4.3) on the coordinate functions $x_i$, $i=1,2$, of the position vector $\boldsymbol{x}$ and by virtue of (4.4) and (4.5), we find respectively

$$\Delta^{III}u = -\frac{uT}{c^2}\left[3(a+1)u^2+3(b+1)v^2+\frac{c(3b+a+2ab)}{ab}\right] = \lambda_{11}u+\lambda_{12}v+\lambda_{13}\sqrt{\omega}, \tag{4.6}$$
$$\Delta^{III}v = -\frac{vT}{c^2}\left[3(a+1)u^2+3(b+1)v^2+\frac{c(b+3a+2ab)}{ab}\right] = \lambda_{21}u+\lambda_{22}v+\lambda_{23}\sqrt{\omega}. \tag{4.7}$$

Putting $v=0$ in (4.6), we obtain that

$$-\frac{3a(a+1)^2}{c^2}u^5-\frac{(a+1)(6b+a+2ab)}{bc}u^3-\frac{(3b+a+2ab)}{ab}u = \lambda_{11}u+\lambda_{13}\sqrt{c+au^2}.$$

Since $a\neq 0$ and $c\neq 0$ this implies that $a=-1$. Similarly, if we put $u=0$ in (4.7) we obtain that

$$-\frac{3b(b+1)^2}{c^2}v^5-\frac{(b+1)(b+6a+2ab)}{ac}v^3-\frac{(b+3a+2ab)}{ab}v = \lambda_{22}v+\lambda_{23}\sqrt{c+bv^2}.$$

This implies that $b=-1$. Hence $S$ must be a sphere.

### 4.2. Quadrics of the second kind

For this kind of surfaces we can consider a parametrization

$$\boldsymbol{x}(u,v) = \left(u,\ v,\ \frac{a}{2}\,u^2 + \frac{b}{2}\,v^2\right).$$

Then the components $g_{ij}$, $b_{ij}$ and $e_{ij}$ of the first, second and third fundamental tensors are the following.

$$\begin{aligned}
g_{11} &= 1+(au)^2, & g_{12} &= abuv, & g_{22} &= 1+(bv)^2,\\
b_{11} &= \frac{a}{\sqrt{g}}, & b_{12} &= 0, & b_{22} &= \frac{b}{\sqrt{g}},\\
e_{11} &= \frac{a^2}{g^2}(1+b^2v^2), & e_{12} &= -\frac{a^2b^2}{g^2}\,uv, & e_{22} &= \frac{b^2}{g^2}(1+a^2u^2),
\end{aligned}$$

where $g = \det(g_{ij}) = 1+(au)^2+(bv)^2$.

A straightforward computation shows that the Beltrami operator $\Delta^{III}$ of $S$ takes the following form.

$$\Delta^{III} = -\frac{g(1+a^2u^2)}{a^2}\frac{\partial^2}{\partial u^2} - \frac{g(1+b^2v^2)}{b^2}\frac{\partial^2}{\partial v^2} - 2uvg\frac{\partial^2}{\partial u\partial v} - 2ug\frac{\partial}{\partial u} - 2vg\frac{\partial}{\partial v}\,. \tag{4.8}$$

On account of (1.3) we get

$$\Delta^{III}x_1 = \Delta^{III}u = \lambda_{11}u + \lambda_{12}v + \lambda_{13}\left(\frac{a}{2}\,u^2 + \frac{b}{2}\,v^2\right), \tag{4.9}$$

$$\Delta^{III}x_2 = \Delta^{III}v = \lambda_{21}u + \lambda_{22}v + \lambda_{23}\left(\frac{a}{2}\,u^2 + \frac{b}{2}\,v^2\right), \tag{4.10}$$

$$\Delta^{III}x_3 = \Delta^{III}\sqrt{\omega} = \lambda_{31}u + \lambda_{32}v + \lambda_{33}\left(\frac{a}{2}\,u^2 + \frac{b}{2}\,v^2\right).$$

Applying (4.8) on the coordinate functions $x_i$, $i=1,2$, of the position vector $\boldsymbol{x}$ and by virtue of (4.9) and (4.10) we find respectively

$$\Delta^{III}u = -2ug = \lambda_{11}u + \lambda_{12}v + \lambda_{13}\left(\frac{a}{2}\,u^2 + \frac{b}{2}\,v^2\right), \tag{4.11}$$

$$\Delta^{III}v = -2vg = \lambda_{21}u + \lambda_{22}v + \lambda_{23}\left(\frac{a}{2}\,u^2 + \frac{b}{2}\,v^2\right). \tag{4.12}$$

Putting $v=0$ in (4.11), we obtain that

$$-2a^2u^3 - 2u = \lambda_{11}u + \lambda_{13}\,\frac{a}{2}\,u^2.$$

This implies that $a$ must be zero. Putting $u=0$ in (4.12), we obtain that

$$-2b^2v^3 - 2v = \lambda_{22}v + \lambda_{23}\,\frac{b}{2}\,v^2.$$

This implies that $b$ must be zero, which is clearly impossible, since $a>0$ and $b>0$.

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