Journal for Geometry and Graphics
Volume 20 (2016), No. 2, 173–184.

Normal Forms of Triangles and Quadrangles up to Similarity

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Abstract. In this paper the problem of introducing new normal forms of mathematical objects is considered in the case of Euclidean geometry. Normal forms of plane geometry objects such as triangles up to similarity are considered. Several normal forms for triangles and a normal form for quadrangles of special case are described. Normal forms of simple plane objects such as triangles can be used in mathematics education, computations and research.

Key Words: normal forms, similarity, triangle, quadrangle

MSC 2010: 51M04, 97G50, 97D99

1. Introduction

Many problems and applications of classical Euclidean geometry consider objects up to similarity, explicitly or implicitly. Understanding and using similarity is an important geometry competence feature for schoolchildren and all practitioners using geometry.

Recall that two geometric figures A and B are *similar* if B can be obtained from A after a finite composition of translations, rotations, reflections, and dilations (homotheties). Similarity is an equivalence relation and thus, for example, the set of all triangles in a plane is partitioned into similarity equivalence classes which can be identified with *similarity types* of triangles.

In many areas of mathematics objects are studied up to equivalence relations. The problem of finding distinguished (canonical or normal) representatives of equivalence classes of objects is posed. Alternatively, it is the problem of mapping the quotient set injectively back into the original set.

Let X be a set with an equivalence relation $\sim$ or, equivalently, $R \subseteq X \times X$, and denote the equivalence class of $x \in X$ by $[x]$. Let $\pi: X \to X/R$ be such that $\pi(x) = [x]$ is the canonical projection map. We call a map $\sigma: X/R \to X$ *normal object map* if $\pi \circ \sigma = \mathrm{id}_{X/R}$. For example, there are various normal forms of matrices, such as the Jordan normal form. See the examples of normal forms in algebra Shafarevich [4] and Paolini [3] for related

ISSN 1433-8157/\$ 2.50

recent work. Normal objects are designed for educational reasons, for pure research (e.g., for classification) and for applications. Normal objects are constructed as objects of simple, minimalistic design, to show essential properties and parameters of original objects. Often it is easier to solve a problem for normal objects first and extend the solution to arbitrary objects afterwards. Normal objects, which are initially designed for education, pure research or problem solving purposes, are also used to optimize computations.

In elementary Euclidean geometry the normal map approach does not seem to be popular working with simple objects such as triangles. We can pose the problem of introducing and using normal forms of triangles up to similarity. This means, to describe a set S of mutually non-similar triangles such that any triangle would be similar to a triangle in S.

Thus, our goal in this paper is to describe uniquely defined representatives of similarity classes of triangles instead of studying properties of members of these classes in an invariant way, for example, using homogeneous, trilinear or other coordinates.

We assume that Cartesian coordinates are introduced in the plane; S is designed using the Cartesian coordinates. For triangles we offer three normal forms based on side lengths. Using these normal forms, the set of triangle similarity forms is bijectively mapped to a fixed plane domain bounded by lines and circles. For these forms two vertices are fixed, and the third vertex belongs to this domain. We call them *the one vertex normal forms.* One vertex normal forms are also generalized to quadrangles. Another normal form for triangles is based on angles and circumscribed circles. For this form one constant vertex is fixed on the unit circle, and two other variable vertices also belong to the unit circle; we call this form *the circle normal form.*

These normal forms may be useful in solving geometry problems involving similarity and teaching geometry. The paper may be useful for mathematics educators interested in developing and improving mathematics teaching.

2. Main results

2.1. Normal forms of triangles

2.1.1. Notations

Consider $\mathbb{R}^2$ with a Cartesian system of coordinates (x, y) with the origin O. We think of classical triangles as being encoded by their vertices. Strictly speaking, by the *triangle* $\triangle XYZ$ we mean the multiset $\{\{X, Y, Z\}\}$ of three points in $\mathbb{R}^2$, each point having a multiplicity of at most 2. A triangle is called *degenerate* if the points lie on a line. Given $\triangle ABC$, we denote $\angle BAC = \alpha$, $\angle ABC = \beta$, $\angle ACB = \gamma$, $|BC| = a$, $|AC| = b$, and $|AB| = b$. We exclude multisets of type $\{\{X, X, X\}\}$.

We will use the following affine transformations of $\mathbb{R}^2$: 1) translations, 2) rotations, 3) reflections in an axis, 4) dilations (given by the rule $(x, y) \to (cx, cy)$ for some $c \in \mathbb{R}\backslash\{0\}$) (see Audin [1] and Venema [5] for comprehensive expositions). It is known that these transformations generate the *dilation group* of $\mathbb{R}^2$, denoted by some authors as $IG(2)$ (see Hazewinkel [2] or Paolini [3]). Two triangles T_1 and T_2 are *similar* if there exists $g \in IG(2)$ such that $g(T_1) = T_2$ (as multisets). If triangles T_1 and T_2 are similar, we write $T_1 \sim T_2$ or $\triangle X_1Y_1Z_1 \sim \triangle X_2Y_2Z_2$.

We use normal letters to denote fixed objects and calligraphic letters to denote objects as function values.

2.1.2. The C-vertex normal form

A normal form can be obtained by transforming the longest side of the triangle into a unit interval of the x-axis. We call it the *C-normal form.* In this subsection we set $A = (0,0)$ and $B = (1,0)$.

Definition 2.1. Let $S_C \subseteq \mathbb{R}^2$ be the domain in the first quadrant bounded by the lines $y = 0$, $x = \frac{1}{2}$ and the circle $x^2 + y^2 = 1$ (see Figure 1).
In other terms, S_C is the set of solutions of the system of inequalities

$$\begin{cases} y \geq 0, \\ x \geq \frac{1}{2}, \\ x^2 + y^2 \leq 1 . \end{cases}$$

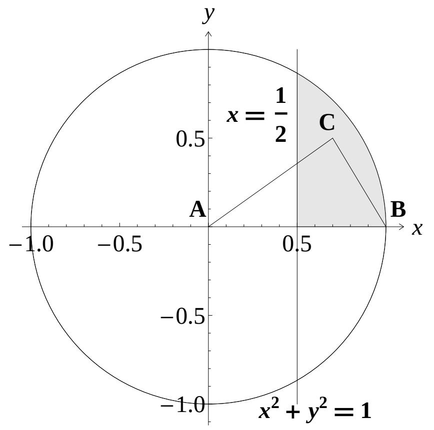

Figure 1: The domain S_C

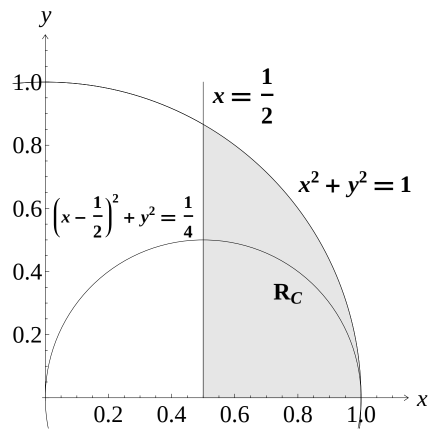

Figure 2: The subdomains of S_C corresponding to obtuse and acute triangles

Theorem 2.2. *Every triangle $\triangle UVW$ in $\mathbb{R}^2$ (including degenerate triangles) is similar to a triangle $\triangle AB\mathcal{C}$, where $A = (0,0)$, $B = (1,0)$ and $\mathcal{C} \in S_C$.*

Proof. Let the $\triangle UVW$ have side lengths a, b and c, satisfying $a \leq b \leq c$. Perform the following sequence of transformations:

1. translate and rotate the triangle so that the longest side is on the x-axis; one vertex has coordinates $(0,0)$ and another vertex has coordinates $(c,0)$ where $c > 0$;
2. if the third vertex has a negative y-coordinate reflect the triangle in the x-axis;
3. do the dilation with coefficient $\dfrac{1}{c}$; note that after this the vertices on the x-axis have coordinates $(0,0)$ and $(1,0)$; the third vertex has coordinates (x'_C, y'_C), where $x'^2_C + y'^2_C \leq 1$ and $(x'_C - 1)^2 + y'^2_C \leq 1$;
4. if $x'_C < \frac{1}{2}$ then reflect the triangle in the line $x = \frac{1}{2}$; denote the third vertex by $\mathcal{C} = (x_C, y_C)$; by construction we have that $\mathcal{C} \in S_C$.

The image of the initial triangle $\triangle UVW$ is the triangle $\triangle AB\mathcal{C}$, where $\mathcal{C} \in S_C$. All transformations preserve the similarity type; therefore $\triangle UVW \sim \triangle AB\mathcal{C}$. □

Theorem 2.3. *If $C_1 \in S_C$, $C_2 \in S_C$ and $C_1 \neq C_2$ then $\triangle ABC_1 \not\sim \triangle ABC_2$.*

Proof. If $\angle C_1AB = \angle C_2AB$ and $C_1 \neq C_2$ then $\angle C_1BA \neq \angle C_2BA$. By the equality of angles for similar triangles it follows that $\triangle ABC_1 \not\sim \triangle ABC_2$.
Let $\angle C_1AB \neq \angle C_2AB$. The angle $\angle C_iAB$ is the smallest angle in $\triangle ABC_i$. By the equality of angles for similar triangles it again follows that $\triangle ABC_1 \not\sim \triangle ABC_2$. $\square$

Definition 2.4. If $\triangle AB\mathcal{C} \sim \triangle UVW$ with $\mathcal{C} \in S_C$, then $\triangle AB\mathcal{C}$ is called *C-vertex normal form* of $\triangle UVW$, and $\mathcal{C}$ is its *C-normal point.*

Remark 2.5. Denote by R_C the intersection of the circle $(x-\frac{1}{2})^2+y^2 = (\frac{1}{2})^2$ and S_C. Points of R_C correspond to right angle triangles. Points below and above R_C correspond to, respectively, obtuse and acute triangles (see Figure 2).
Points on the line $x = \frac{1}{2}$ with $0 < y < \frac{1}{2}$ correspond to isosceles obtuse (nondegenerate) triangles. Points on the line $x = \frac{1}{2}$ with $\frac{1}{2} < y \leq \frac{\sqrt{3}}{2}$ and the boundary of S_C with $x \geq \frac{1}{2}$ and $y > 0$ correspond to isosceles acute triangles. Points in the interior of S_C correspond to scalene triangles. The point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ corresponds to the equilateral triangle. Points on the intersection of the line $y = 0$ and S_C correspond to degenerate triangles. $\mathcal{C} = B$ holds for triangles with side lengths 0, c and c.

2.1.3. The B-vertex normal form

Another normal form can be obtained by transforming the median length side (in the sense of ordering) of the triangle into a unit interval of the x-axis. By analogy it is called the *B-normal form.* In this subsection we set $A = (0, 0)$ and $C = (1, 0)$.

Definition 2.6. Let $S_B \subseteq \mathbb{R}^2$ be the domain in the first quadrant bounded by the line $y = 0$ and the circles $x^2 + y^2 = 1$ and $(x-1)^2 + y^2 = 1$ (see Figure 3).
In other terms, S_B is the set of solutions of the system of inequalities

$$\begin{cases} y \geq 0, \\ x^2 + y^2 \geq 1, \\ (x-1)^2 + y^2 \leq 1\,. \end{cases}$$

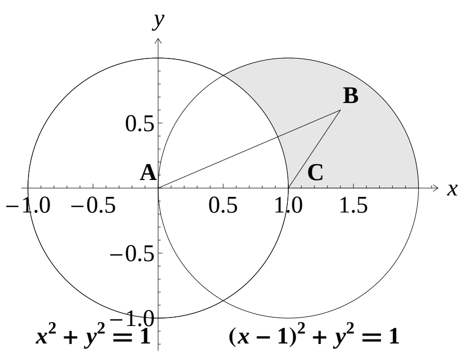

Figure 3: The domain S_B

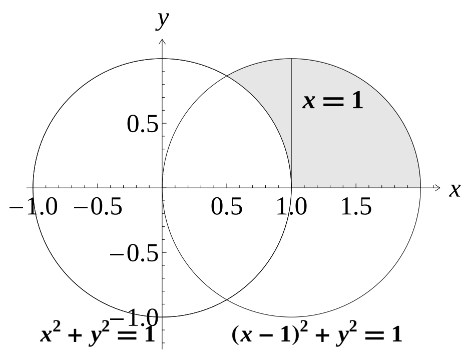

Figure 4: The subdomains of S_B corresponding to obtuse and acute triangles

Theorem 2.7. *Every triangle $\triangle UVW$ in $\mathbb{R}^2$ (including degenerate triangles) is similar to a triangle $\triangle A\mathcal{B}C$, where $A = (0, 0)$, $C = (1, 0)$ and $\mathcal{B} \in S_B$.*

Proof. Let $\triangle UVW$ have the side lengths a, b and c satisfying $a \leq b \leq c$. Perform the following sequence of transformations:

1. translate and rotate the triangle so that the side with length b is on the x-axis, one vertex has the coordinates $(0,0)$ and another vertex has the coordinates $(b,0)$ with $b>0$; the side with length c is incident to the vertex $(0,0)$;
2. if the third vertex has a negative y-coordinate reflect the triangle in the x-axis;
3. do the dilation with the coefficient $\frac{1}{b}$; note that the vertices on the x-axis have the coordinates $(0,0)$ and $(1,0)$; at this point the third vertex $\mathcal{B}$ has coordinates (x'_B, y'_B), where $y'_B \geq 0$, $x'^2_B + y'^2_B \geq 1$ or $(x'_B - 1)^2 + y'^2_B \leq 1$.

The image of the initial triangle $\triangle UVW$ is the triangle $\triangle A\mathcal{B}C$, where $\mathcal{B} \in S_B$. All transformations preserve the similarity type; therefore $\triangle UVW \sim \triangle A\mathcal{B}C$. □

Theorem 2.8. *If $B_1 = (x_i, y_i) \in S_B$, $B_2 = (x_2, y_2) \in S_B$ and $B_1 \neq B_2$ then $\triangle AB_1C \not\sim \triangle AB_2C$.*

Proof. The angle $\angle B_iAC$ is the smallest angle in the triangle $\triangle AB_iC$.
If $\angle B_1AC \neq \angle B_2AC$ then, since these are the smallest angles in the triangles, it follows that $\triangle AB_1C \not\sim \triangle AB_2C$.
If $\angle B_1AC = \angle B_2AC$ and $B_1 \neq B_2$ then $\angle ACB_1 \neq \angle ACB_2$. The angle $\angle ACB_i$ is the biggest in $\triangle AB_iC$; therefore $\angle ACB_1 \neq \angle ACB_2$ implies $\triangle AB_1C \not\sim AB_2C$. □

Definition 2.9. If $\triangle A\mathcal{B}C \sim \triangle UVW$ with $\mathcal{B} \in S_B$ then $\triangle A\mathcal{B}C$ is called *B-vertex normal form* of $\triangle UVW$, and $\mathcal{B}$ is its *B-normal point.*

Remark 2.10. Denote by R_B the intersection of the line $x=1$ and S_B. Points of R_B correspond to right angle triangles. Points to the right and left of R_B correspond to, respectively, obtuse and acute triangles (see Figure 4).
Points on the boundary of S_B with $x<1$ correspond to isosceles acute triangles. Points on the boundary of S_B with $1<x<2$ and $y>0$ correspond to isosceles obtuse (nondegenerate) triangles. Points in the interior of S_B correspond to scalene triangles. The point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ corresponds to the equilateral triangle. Points on the intersection of the line $y=0$ and S_B correspond to degenerate triangles. $\mathcal{B} = C$ holds for triangles with side lengths 0, c and c.

2.1.4. The *A*-vertex normal form

Finally a normal form can be obtained by transforming the shortest side of the triangle into a unit interval of the x-axis. By analogy, it is called *the A-normal form.* In this case again the two vertices on the x-axis are $(0,0)$ and $(1,0)$; the domain S_A of possible positions of the third vertex is unbounded. In this subsection we set $B=(0,0)$ and $C=(1,0)$.

Definition 2.11. Let $S_A \subseteq \mathbb{R}^2$ be the unbounded domain in the first quadrant bounded by the lines $y=0$, $x=\frac{1}{2}$ and the circle $(x-1)^2+y^2=1$ (see Figure 5).
In other terms, S_A is the set of solutions of the system of inequalities

$$\begin{cases} y \geq 0, \\ x \geq \frac{1}{2}, \\ (x-1)^2 + y^2 \geq 1 . \end{cases}$$

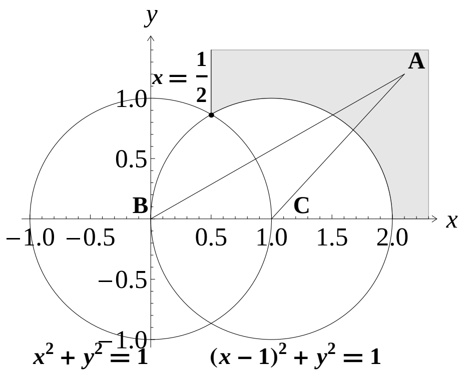

Figure 5: The domain S_A

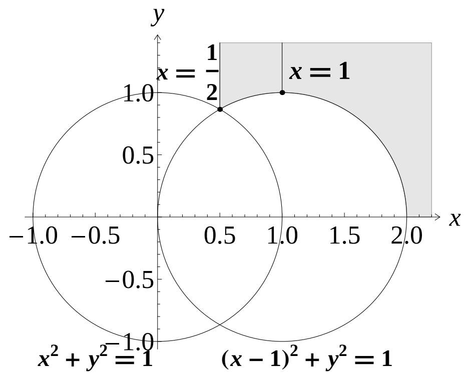

Figure 6: The subdomains of S_A corresponding to obtuse and acute triangles

Theorem 2.12. *Every triangle $\triangle UVW$ in $\mathbb{R}^2$ (including degenerate triangles, but excluding the similarity type having side lengths 0, c and c) is similar to a triangle $\triangle \mathcal{A}BC$, where $B = (0,0)$, $C = (1,0)$ and $\mathcal{A} \in S_A$.*

Proof. Let $\triangle UVW$ have side lengths a, b and c satisfying $a \leq b \leq c$. Perform the following sequence of transformations:

1. translate and rotate the triangle so that the side of length a lies on the x-axis, one vertex has the coordinates $(0,0)$ and another vertex has the coordinates $(a,0)$ with $a > 0$; the side of length c is incident to the vertex $(0,0)$;
2. if the third vertex has a negative y-coordinate reflect the triangle in the x-axis;
3. apply the dilation with coefficient $\dfrac{1}{a}$; note that the vertices on the x-axis have the coordinates $(0,0)$ and $(1,0)$.

The image of the initial triangle $\triangle UVW$ is the triangle $\triangle \mathcal{A}BC$, where $\mathcal{A} \in S_A$. All transformations preserve the similarity type; therefore $\triangle UVW \sim \triangle \mathcal{A}BC$. □

Theorem 2.13. *Let $B = (0,0)$ and $C = (1,0)$. If $A_1 = (x_i, y_i) \in S_A$, $A_2 = (x_2, y_2) \in S_A$ and $A_1 \neq A_2$ then $\triangle A_1BC \not\sim \triangle A_2BC$.*

Proof. The angle $\angle BCA_i$ is the largest in the triangle $\triangle A_iBC$.
If $\angle BCA_1 \neq \angle BCA_2$ then, since these are the largest angles in the triangles, it follows that $\triangle A_1BC \not\sim \triangle A_2BC$.
If $\angle BCA_1 = \angle BCA_2$ and $A_1 \neq A_2$ then $\angle BA_1C \neq \angle BA_2C$. The angle $\angle BA_iC$ is the smallest in $\triangle A_iBC$; therefore $A_1BC \neq A_2BC$ implies $\triangle AB_1C \not\sim AB_2C$. □

Definition 2.14. If $\triangle \mathcal{A}BC \sim \triangle UVW$ with $\mathcal{A} \in S_A$ then $\triangle \mathcal{A}BC$ is called *A-vertex normal form* of $\triangle UVW$, and $\mathcal{A}$ is its *A-normal point.*

Remark 2.15. Denote by R_A the intersection of the line $x = 1$ with S_A. Points of R_A correspond to right angle triangles. Points to the right and left of R_A correspond to, respectively, obtuse and acute triangles (see Figure 6).
Points on the boundary of S_A with $x < 1$ correspond to isosceles acute triangles. Points on the boundary of S_A with $x > 1$ and $y > 0$ correspond to isosceles obtuse (nondegenerate) triangles. Points in the interior of S_A correspond to scalene triangles. The point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$

corresponds to the equilateral triangle. Points on the intersection of the line $y = 0$ and S_A correspond to degenerate triangles excluding the similarity type with side lengths 0, c and c, which corresponds to the point at infinity. In contrast to the C-vertex and B-vertex normal forms, the triangles of the A-vertex normal form are not bounded.

2.1.5. The circle normal form

Consider $\mathbb{R}^2$ with a Cartesian system of coordinates (x, y) with the origin O. We also consider polar coordinates $[r, \varphi]$ introduced in the standard way: the polar angle φ is measured from the positive x-axis going counterclockwise.

Note that the angles α, β and γ of a nondegenerate triangle with $\alpha \leq \beta \leq \gamma$ satisfy the system of inequalities

$$\begin{cases} 0 < \alpha \leq \dfrac{\pi}{3}, \\ \alpha \leq \beta \leq \dfrac{\pi - \alpha}{2}. \end{cases}$$

Similarity types of nondegenerate triangles are parametrized by one point in the domain in the (α, β)-plane determined by the system

$$\begin{cases} \alpha > 0, \\ \beta \geq \alpha, \\ \beta \leq \dfrac{\pi}{2} - \dfrac{\alpha}{2} \end{cases}$$

(see Figure 7).

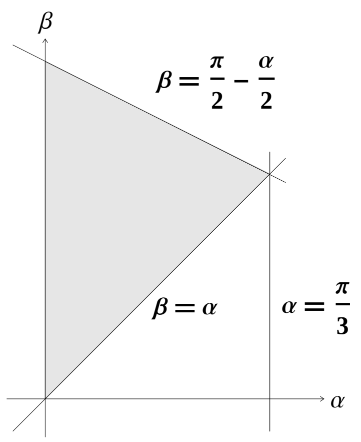

Figure 7: The parametrization of the similarity types by (α, β)

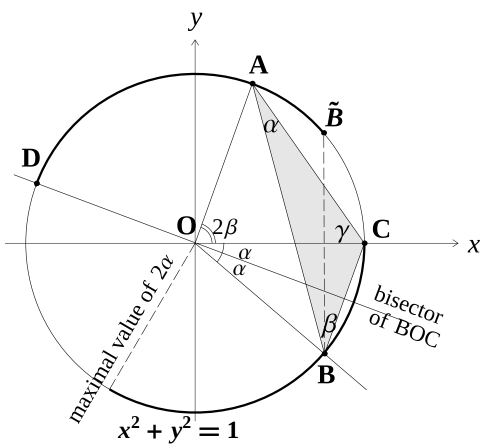

Figure 8: Construction of a normal circle triangle

For the normal form described in this subsection, the vertex with the biggest angle will be fixed at $(1, 0)$, and, in order to be consistent with the previous notations, we define $C = (1, 0)$. For this normal form only nondegenerate triangles are considered.

In this case all normal form triangles are inscribed in the unit circle $\mathbb{U} = \{x^2 + y^2 = 1\}$ having C as one of the vertices.

Definition 2.16. A triangle $\triangle ABC$ inscribed in $\mathbb{U}$ is called a *normal circle triangle* if

1. $0 \leq \alpha \leq \frac{\pi}{3}$,
2. $\alpha \leq \beta \leq \frac{\pi}{2} - \frac{\alpha}{2}$,

3. $C = (1, 0)$,
4. the point A is above the x-axis,
5. the point B is below the x-axis.

Remark 2.17. A normal circle triangle with given angles $\alpha \leq \beta \leq \gamma$ can be constructed in the following way (see Figure 8):

1. choose a point B below the y-axis with the argument equal to 2α, where $0 \leq 2\alpha \leq \frac{2\pi}{3}$;
2. draw the bisector of $\angle BOC$; let D denote the intersection of this bisector with the arc BC with the center angle $2\pi - 2\alpha$;
3. find the point $\widetilde{B}$ which is symmetric to B with respect to the x-axis;
4. choose a point A at the shorter arc $\widetilde{B}D$.

Theorem 2.18. *For every nondegenerate triangle $\triangle UVW$ there exists a normal circle triangle $\triangle \mathcal{ABC}$ such that $\triangle UVW \sim \triangle \mathcal{ABC}$.*

Proof. Suppose that $\triangle UVW$ has angles with $\alpha \leq \beta \leq \gamma$. Let $\mathcal{B} \in \mathbb{U}$ be the point with polar coordinates $[1, -2\alpha]$. Let $\mathcal{A} \in \mathbb{U}$ be the point with polar coordinates $[1, 2\beta]$. Then, since $\triangle \mathcal{ABC}$ is inscribed in $\mathbb{U}$, we have $\angle \mathcal{BAC} = \alpha$, $\angle \mathcal{ABC} = \beta$ and thus $\triangle \mathcal{ABC} \sim \triangle UVW$. $\square$

Theorem 2.19. *Let $\triangle A_1BC_1$ and $\triangle A_2BC_2$ be two distinct normal circle triangles with $A_1 \neq A_2$ or $B_1 \neq B_2$. Then $\triangle A_1BC_1 \not\sim \triangle A_2BC_2$.*

Proof. If $A_1 \neq A_2$, then $\angle B_1A_1C \neq \angle B_2A_2C$. Since angle $\angle B_iA_iC$ is the smallest in $\triangle A_iB_iC$, we have $\triangle A_1B_1C \not\sim \triangle A_2B_2C$.
If $B_1 \neq B_2$ and $A_1 = A_2$ then $\angle A_1CB_1 \neq \angle A_2CB_2$. The angle $\angle A_iCB_i$ is the largest in $\triangle A_iB_iC$. Therefore, $\angle A_1CB_1 \neq \angle A_2CB_2$ implies $\triangle A_1B_1C \not\sim \triangle A_2B_2C$. $\square$

Definition 2.20. The normal triangle $\triangle \mathcal{ABC}$ such that $\triangle UVW \sim \triangle \mathcal{ABC}$ is called *circle normal form* of $\triangle UVW$.

Remark 2.21. The only isosceles normal circle triangles are normal circle triangles of type $\triangle B\widetilde{B}C$ and $\triangle BDC$. Right normal circle triangles are normal circle triangles with AB passing through O. Acute/obtuse normal circle triangles are normal circle triangles with O inside/outside $\triangle ABC$. In contrast to the one vertex normal forms, the side lengths at triangles of the circle normal form can be arbitrarily small.

2.1.6. Conversions

Definition 2.22. According to the previous Definitions 2.4, 2.9 and 2.14, let X be the X-normal point in the cases $X = A$, B or C. Then the pair of coordinates of X in terms of the side lengths a, b and c is denoted by $N_X(a, b, c)$. Note that N_X is a symmetric function. We can also think of arguments of N_X as multisets and think that $N_X(a, b, c) = N_X(L)$, where L is the multiset $\{\{a, b, c\}\}$.

Theorem 2.23. *Let $\triangle ABC$ have side lengths $a \le b \le c$. Then*

1. $N_C(a,b,c) = \left(\dfrac{-a^2+b^2+c^2}{2c^2}, \dfrac{\sqrt{-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2)}}{2c^2}\right)$;
2. $N_B(a,b,c) = \left(\dfrac{-a^2+b^2+c^2}{2b^2}, \dfrac{\sqrt{-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2)}}{2b^2}\right)$;
3. $N_A(a,b,c) = \left(\dfrac{a^2-b^2+c^2}{2a^2}, \dfrac{\sqrt{-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2)}}{2a^2}\right)$.

Proof. 1. Translate, rotate and reflect $\triangle ABC$ so that $A = (0,0)$, $B = (c,0)$, and point $C = (x,y)$ lies in the first quadrant. For (x,y) we have the system

$$\begin{cases} x^2+y^2 = b^2, \\ (c-x)^2+y^2 = a^2 \end{cases}$$

and find

$$\begin{cases} x = \dfrac{-a^2+b^2+c^2}{2c}, \\[2mm] y = \dfrac{\sqrt{-a^4-b^4-c^4+2(a^2b^2+a^2c^2+b^2c^2)}}{2c}. \end{cases}$$

After the dilation with the coefficient $\frac{1}{c}$ we get the given formula.
2. and 3. can be proved are in a similar way. □

Theorem 2.24. *Let the triangle T have the angles $\alpha \le \beta \le \gamma$. Then*

1. *its C-normal point is* $N_C\left(\dfrac{\sin\alpha}{\sin\gamma}, \dfrac{\sin\beta}{\sin\gamma}, 1\right)$;
2. *if T has the C-normal point (x,y) then it has the angles* $\alpha = \arctan\dfrac{y}{x}$, $\beta = \arctan\dfrac{y}{1-x}$ *and* $\gamma = \pi - \arctan\dfrac{y}{x} - \arctan\dfrac{y}{1-x}$;
3. *its B-normal point is* $N_B\left(\dfrac{\sin\alpha}{\sin\beta}, \dfrac{\sin\gamma}{\sin\beta}, 1\right)$;
4. *if T has the B-normal point (x,y) then it has the angles* $\alpha = \arctan\dfrac{y}{x}$, $\beta = -\arctan\dfrac{y}{x} + \arctan\dfrac{y}{x-1}$ *and* $\gamma = \pi - \arctan\dfrac{y}{x-1}$;
5. *its A-normal point is* $N_A\left(\dfrac{\sin\beta}{\sin\alpha}, \dfrac{\sin\gamma}{\sin\alpha}, 1\right)$;
6. *if T has the A-normal point (x,y) then it has the angles* $\alpha = -\arctan\dfrac{y}{x} + \arctan\dfrac{y}{x-1}$, $\beta = \arctan\dfrac{y}{x}$ *and* $\gamma = \pi - \arctan\dfrac{y}{x-1}$.

Proof. 1. Let $\triangle ABC$ be the C-normal triangle with the angles $\alpha \le \beta \le \gamma$, i.e., $|AB| = 1$. By the law of sines we have $b = |AC| = \dfrac{\sin\beta}{\sin\gamma}$ and $a = \dfrac{\sin\alpha}{\sin\gamma}$. By definition, C has the coordinates $N_C\left(\dfrac{\sin\alpha}{\sin\gamma}, \dfrac{\sin\beta}{\sin\gamma}, 1\right)$.
2. Let CD be a height of $\triangle ABC$. Formulas for angles are obtained considering $\triangle ACD$ and $\triangle BCD$.
3., 4., 5., and 6. are proved similarly. □

2.2. Normal forms of quadrangles

In this subsection we consider multisets of four points in a plane. A multiset of four points can be interpreted as a quadrangle. We exclude the case of one point with multiplicity 4. The multiset $Q = \{\{X, Y, Z, T\}\}$ is also denoted as $\square XYZT$. We define $Q_1 \sim Q_2$ provided that there is an element of the dilation group g such that $g(Q_1) = Q_2$.

A set of four points defines a set of 6 mutual distances. Choosing any two points, we can translate, rotate, reflect, and dilate the given four -point configuration so that the chosen two points get coordinates $A = (0, 0)$ and $B = (1, 0)$. Different normal forms can be obtained choosing pairs with different relative metric properties. In this paper we only consider the simplest case — two points having the maximal distance are mapped to the x-axis.

2.2.1. Longest distance normal form

Suppose we are given a quadrangle $\square XYZT$ such that $|XY| > |XZ|$, $|XY| > |XT|$, $|XY| > |YZ|$, $|XY| > |YT|$, and $|XY| > |ZT|$. We map X and Y by an affine transformation to the x-axis (to $A = (0, 0)$ and $B = (1, 0)$) and determine what are the positions of the two remaining vertices C and D so that $\square XYZT \sim \square ABCD$.

Definition 2.25. If $p \in \mathbb{R}^2$ is mapped under reflections in the x-axis and the line $x = \frac{1}{2}$ into the domain $y \geq 0$, $x \geq \frac{1}{2}$ then its image is denoted by p_s.

Definition 2.26. The point (x_1, y_1) is *lexicographically smaller* than the point (x_2, y_2) and denoted as $(x_1, y_1) \prec (x_2, y_2)$ if $(x_1 < x_2)$ or $(x_1 = x_2$ and $y_1 < y_2)$.
The lexicographical order of points can be extended to a lexicographical ordering of sequences of points: the sequence of points $[p_1, p_2]$ is *lexicographically smaller* than the sequence $[q_1, q_2]$, denoted by $[p_1, p_2] \prec [q_1, q_2]$, if $(p_1 \prec q_1)$ or $(p_1 = q_1$ and $p_2 \prec q_2)$.

Definition 2.27. Let $p, p' \in \mathbb{R}^2$. We say that p is *quasilexicographically smaller or equal* to p', denoted by $p \lhd p'$, if $p_s \prec p'_s$ or $p_s = p'_s$. Given two pairs $[p, q]$ and $[p', q']$, we write $[p, q] \lhd [p', q']$ if $(p_s \prec p'_s)$ or $(p_s = p'_s$ and $q \lhd q')$.

Definition 2.28. Let $S_D(x_0, y_0) \subseteq \mathbb{R}^2$ with $(x_0, y_0) \in S_C$ (for the definition of S_C see Section 2.1.2) be the set of solutions of the following system of inequalities (see Figure 9):

$$\begin{cases} x^2 + y^2 \leq 1, \\ (x-1)^2 + y^2 \leq 1, \\ (x-x_0)^2 + (y-y_0)^2 \leq 1, \\ |x - \frac{1}{2}| \leq |x_0 - \frac{1}{2}|, \\ \text{if } \ |x - \frac{1}{2}| = |x_0 - \frac{1}{2}| \ \text{ then } \ |y| \leq |y_0|. \end{cases} \tag{1}$$

Remark 2.29. The conditions for $p \in S_D(x_0, y_0)$ consist of two parts:

1. the distance from p to A, B and (x_0, y_0) is less than or equal to 1;
2. $p_s \lhd (x_0, y_0)$.

Theorem 2.30. *Every $\square UVWZ$ in $\mathbb{R}^2$ as described in the beginning of Subsection 2.2.1 (including multisets with multiplicities at most 3) is similar to $\square AB\mathcal{C}\mathcal{D}$, where $A = (0, 0)$, $B = (1, 0)$, $\mathcal{C} \in S_C$, and $\mathcal{D} \in S_D(\mathcal{C})$.*

Proof. Let $UVWZ$ be a multiset of points in $\mathbb{R}^2$ with at least two distinct elements. Perform the following sequence of transformations:

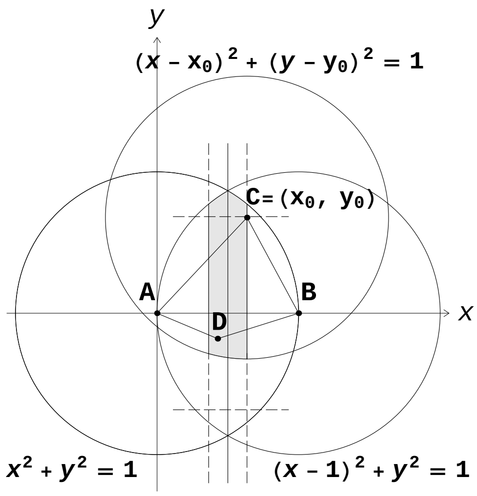

Figure 9: Example of the domain S_D

1. Translate and rotate the plane such that two points with the longest distance between are mapped to the x-axis, one vertex obtaining the coordinates $(0,0)$ and the other obtaining the coordinates $(d,0)$ where $d > 0$.
2. Apply the dilation with coefficient $\frac{1}{d}$. Note that the vertices on the x-axis get the coordinates $(0,0)$ and $(1,0)$; suppose that the two remaining vertices have coordinates (x_C, y_C) and (x_D, y_D).
3. If $|x_C - \frac{1}{2}| \neq |x_D - \frac{1}{2}|$ then put the point with the maximal $|x - \frac{1}{2}|$ value into S_C by reflections in the x-axis and the line $x = \frac{1}{2}$.
4. If $|x_C - \frac{1}{2}| = |x_D - \frac{1}{2}|$ then put the point with the maximal $|y|$-value into S_C by reflections in the x-axis and the line $x = \frac{1}{2}$.
5. If $|x_C - \frac{1}{2}| = |x_D - \frac{1}{2}|$ and $|y_C| = |y_D|$ then map any of the points into S_C.

Let $C = (x_C, y_C)$ denote the point which has been mapped to S_C by this sequence of transformations and let the fourth point be $D = (x_D, y_D)$. For any $C = (x_0, y_0) \in S_C$ we have that $S_D(x_0, y_0) \neq \emptyset$.
We check that $D \in S_D(x_C, y_C)$. From the conditions $|AD| \leq 1$, $|BD| \leq 1$ and $|CD| \leq 1$ it follows that D satisfies the first three inequalities of the system 1. If $|y_D| > |y_C|$ then $|x_D - \frac{1}{2}| < |x_C - \frac{1}{2}|$ due to the quasilexicographic order condition. $\square$

Definition 2.31. The *longest distance normal form* of $\square UVWZ$ is $\square ABCD$ with $A = (0,0)$, $B = (1,0)$, $C = (x_C, y_C) \in S_C$, and $D \in S_D(x_C, y_C)$, constructed according to the algorithm given in the proof of Theorem 2.30.

Theorem 2.32. *Let $\square ABC_1D_1$ and $\square ABC_2D_2$ be two quadrangles constructed according to the longest distance normal form algorithm.*
If $C_1 \neq C_2$ or $D_1 \neq D_2$ then $\square ABC_1D_1 \not\sim \square ABC_2D_2$.

Proof. $D_i \lhd C_i$; therefore, if $C_1 \neq C_2$ then $\square ABC_1D_1 \not\sim \square ABC_2D_2$.
In the remaining case $C_1 = C_2$ we use an indirect proof. Suppose there is a similarity with $\square ABC_1D_1 \mapsto \square ABC_2D_2$. If a similarity mapping fixes three noncollinear points A, B and C_i then it must fix any other point of the plane. If A, B and C_i are on the x-axis then D_i must

also be on the x-axis and must be fixed. Therefore $D_1 \neq D_2$ implies $\square ABC_1D_1 \not\sim \square ABC_2D_2$ in this case. $\square$

3. Possible uses of normal forms in education

One vertex normal forms of triangles can be used to represent all similarity types of triangles in a single picture with all triangles having a fixed side, especially C-vertex and B-vertex normal forms. It may be useful to have an example for students showing that the similarity type of any triangle can be parametrized by coordinates of a single point. Similarity types of triangles having specific properties (e.g., isosceles triangles) may correspond to subsets of normal points; this may stimulate interest and advances in the coordinatization of mathematical concepts. One vertex normal forms can also be used in considering quadrangles.

The circle normal form of triangles may be useful for teaching properties of circumscribed circles, e.g., inscribing triangles with given angles in a circle in a canonical way.

Normal forms of triangles can also be used to teach the idea of normal (canonical) objects using a case of simple and popular geometric constructions.

4. Conclusion and further development

It is relatively easy to define normal forms of triangles up to similarity. Only simple approaches, which may be used in teaching and applications, are considered in this paper. One approach is to map one side to the x-axis and to use dilations and reflections in order to position the third vertex in a unique way. In this approach normal triangles are parametrized by one vertex. This approach can be generalized for quadrangles. Another approach considered in this paper is to design normal triangles as triangles inscribed in a unit circle. Further developments in this direction may be related to using other figures related to a given triangle, for example, the inscribed circle, medians, altitudes, or bisectors.

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Received November 6, 2015; final form June 8, 2016