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On the Equality of Cevians: Beyond the Steiner-Lehmus Theorem

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Abstract. The aim of the present work is to investigate the relations in a triangle in order to have two cevians equal, given the fact that they intersect in a point of a third cevian. Obviously the Steiner Lehmus theorem deals with the specific case of cevians being angle-bisectors. All possible combinations of external or internal cevians, plus the possibilities of equicevian points are examined.

Key Words: cevians, A-equicevian points, equicevian points, Steiner-Lehmus Theorem

MSC 2010: 51M04

1. Introduction

Prologue. Fortyone years ago, while I was a student at the National Technical University of Athens, I was given a problem (Appendix 1) to solve by one of the most famous final year students (famous mainly for his mathematical experience and skills). He said to me that "only ten people in Greece can solve this problem, using Euclidean Geometry", since analytic geometry was an anathema for us purists. That problem took me an embarrassingly long time to solve, or so I thought, because when I showed my 10–15 pages long "solution" to my mathematical genius friend, he said dryly "too long" and he didn't bother to look further, although he never solved it as far as I know. The funny thing was, as I found out after I swallowed and digested my pride, that he was right. I finally came to deal with this problem fifteen years ago and that forms the basis of the present work.

Let M be a point in the plane of $\triangle ABC$ and let AH, BD and CE be three cevians through M, inside the triangle as in Figure 1 or outside as in Figure 2. Let $BH = a$, $CH = b$, $AH = V$ and $MH = x$, also let the angle $\widehat{AHB} = \widehat{\phi}$. These four entities are the defining parameters of the triangle and of the cevians in order to calculate the conditions and the relations which control the equality of cevians BD and CD. Also let $a < b$.

As shown in Figure 2, the cevians intersect on the extension of AH in the region P1, which is bounded by BC and the extensions of AB and AC, while the external cevian CE exists in this part of the plane and the external cevian BD exists in the region P2 bounded by AB

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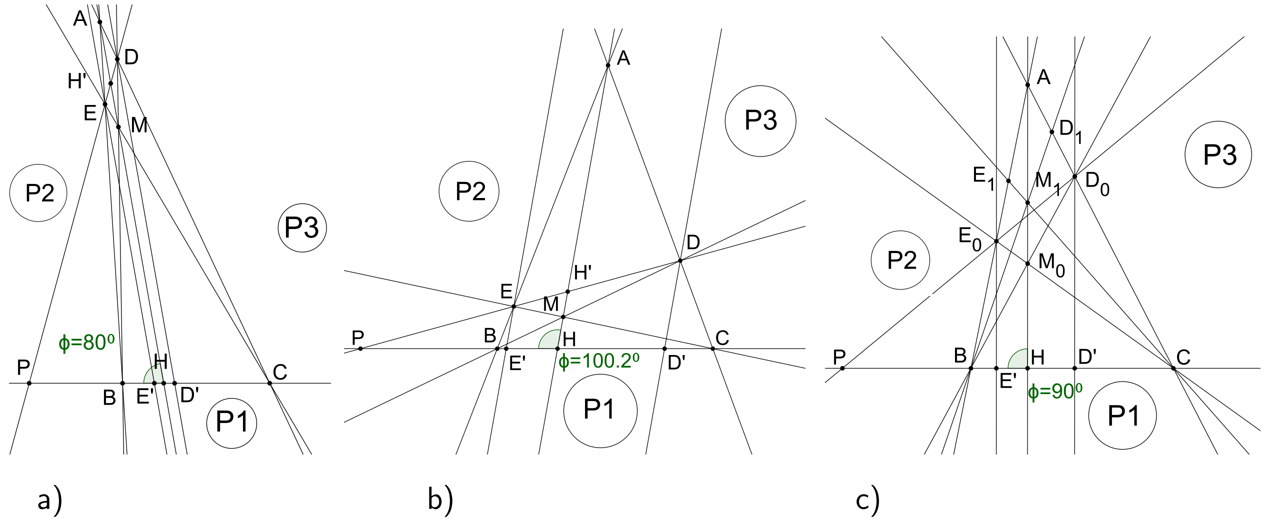

Figure 1: Cevians BD, CE and point M in the triangle ABC:
a) $\widehat{\phi} < 90°$, b) $\widehat{\phi} > 90°$, c) $\widehat{\phi} = 90°$.

and the extensions of BC and AC. Obviously, the cevian AH is internal. If the cevians are external they can exist in three regions of the plane of $\triangle ABC$, in P1 and P2 as defined above and in P3, which is bounded by AC and the extensions of AB and BC.

As can easily be seen, when the two cevians are external, the third one has to be internal and this is why in this work the cevian going through A is *assumed to be always internal.* (For example, in Figure 2, the cevian BD is in P1 and P2, the cevian CE is in P3 and P1; so if BD and CE intersect this has to take place in P1, so AH is internal and its extension is in P1). The external cevian BD can be either in P2 or in P1, the external cevian CE either in P3 or P1 and the angle $\widehat{\phi}$ can have two ranges of values, $0 < \widehat{\phi} < 90°$ or $180° > \widehat{\phi} > 90°$, or the specific value $\widehat{\phi} = 90°$. All the above can be summed up in Table 1 which also indicates which combination allows an equality of cevians.

Table 1: Equality of cevians: + if possible, − if impossible.

Cases	$0 < \widehat{\phi} < 90°$	$\widehat{\phi} = 90°$	$90° < \widehat{\phi} < 180°$
1. BD, CE, in triangle	Figure 1a +	Figure 1c +	Figure 1b +
2. BD in P2, CE in P1	Figure 2a +	Figure 2c +	Figure 2b +
3. BD in P1, CE in P1	Figure 3a −	Figure 3c −	Figure 3b +
4. BD in P2, CE in P3	Figures 4a, 7a +	Figure 4c −	Figures 4b, 8a +
5. BD in P1, CE in P3	−	−	−

The case $\widehat{\phi} = 90°$ is fundamental for the study of equal cevians; so, it is examined in Section 2 for internal cevians and in Section 3 for external cevians. In [2] an interesting work is presented, related to equicevian points on the altitude of the vertex A (A-equicevian points, as stated in [2], [3] and [4]). The same results are obtained in the present paper with the Theorems 1 and 2, however by using a different method. In [2] it is also proved that if A-equicevian points exist then angle $\widehat{A} \leq 45°$ at $\triangle ABC$.

In Section 4 the Case 1 with $\widehat{\phi} \neq 90°$ and with internal cevians is examined, and in Section 5 a specific version of Case 4 (where $\widehat{\phi} > 90°$) with external cevians (because of common geometrical conditions and equations). In Section 5 the rest of the cases are examined where $\widehat{\phi} \neq 90°$ and BD, CD are external cevians. Section 6 deals with the case of angle-bisectors as cevians (Steiner-Lehmus theorem). For all figures in this work the parameters are listed in Table 2.

The famous Steiner-Lehmus (S-L) theorem is a topic which has gathered much interest and on which much work has been accumulated, and this body of work is still growing. In [7] a collection of 9 proofs of this theorem is presented. In [12] the problem of equal external angle-bisectors (external S-L theorem) is presented and solved. In [9], by virtue of an algorithmic method (Gröbner Cover), all possible cases of equal internal and external angle-bisectors are discussed. In [5] one can find a very handsome "indirect" proof of the S-L theorem (the Schizoid Scissors, based on Coxeter and Greitzer's *"Geometry Revisited"* [6]), which, after some very eloquent arguments of the authors, appears to be quite "direct", and additionally all cases of external S-L theorems are examined. In [8] all cases of the internal and external S-L theorem are analysed using trigonometric functions.

In [3] one can find a thorough study on the existence and properties of the equicevian points of a triangle and their related equations, closely related to Marden's theorem and Steiner's circumellipse, together with an insightful review of the existing bibliography. This work is complemented by [4] (where the length of each real and imaginary cevian is calculated and the focal points of the Steiner circumellipse are related to the equicevian points).

In Section 7 an attempt is made to investigate the properties of equicevian points (based on [3]), using the theorems and equations of this work. This approach could be further exploited in future. Finally, in Section 8, a calculation of the angle of $\triangle A'BC$ (related to $\triangle ABC$ in Figure 9c) is presented, inspired by work on [2], extending the inequality $\widehat{A} \leq 45°$ in $\triangle ABC$ to the cases where the third cevian AH is not an altitude of the triangle. The approach in the present work, based on $\widehat{\phi}$ and on $BH = a$, $CH = b$, $AH = V$ and $MH = x$, may not be the most suitable one for studying the S-L theorem. However, when studying A-equicevian cases (such as the ones in Table 1), it appears to provide a flexible environment which allows an uniform approach to all the cases listed in this table.

This work is just one example of the strong attraction the quest of finding two or three equal cevians in a triangle exerts over many mathematically-minded people. The quest becomes even more intense as time goes by for anyone who endeavours to pursue it, because the relevant mathematical environment is surprisingly complex and interesting. It seems that the equality of cevians is still a very fertile ground for further research and it still can generate mathematical enjoyment for many participants.

2. Equality of internal cevians when $\widehat{\phi} = 90°$

Case 1, *BD and CE in the triangle ABC:*
In Figure 1c we have the case of $\triangle ABC$ with $\widehat{\phi} = 90°$ and internal cevians. From Appendix 2, which holds for the general case $0 < \widehat{\phi} < 180°$, we get $EE' = xV(a+b)/(Vb+xa)$. Similarly we get $DD' = xV(a+b)/(Va+xb)$. From Figure 1c we also get $EE'/x = EC/MC$ and $MC = (x^2+b^2)^{1/2}$, hence $EC = (EE'/x)(x^2+b^2)^{1/2}$. Thus we obtain

$$EC = \left((a+b)V(x^2+b^2)^{1/2}\right)/(Vb+xa) \quad \text{and} \tag{1}$$

$$BD = \left((a+b)V(x^2+a^2)^{1/2}\right)/(Va+xb). \tag{2}$$

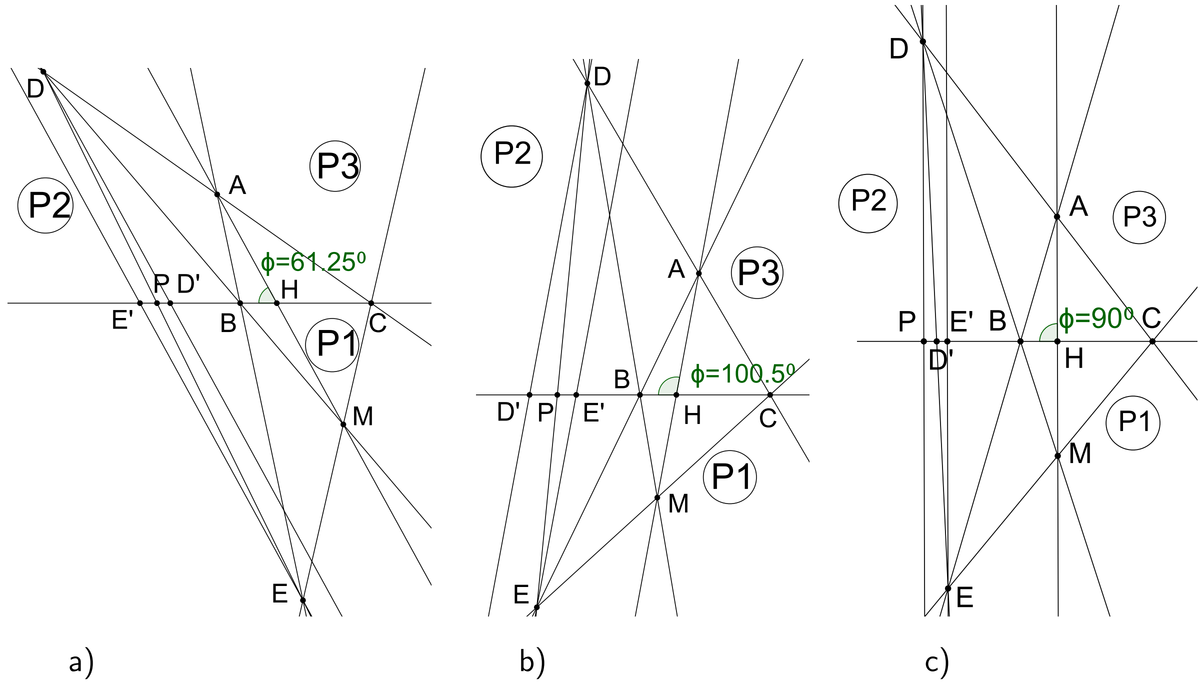

Figure 2: Cevian BD in P2, cevian CE and point M in P1,
a) $\widehat{\phi} < 90°$, b) $\widehat{\phi} > 90°$, c) $\widehat{\phi} = 90°$.

From the Eqs. (1) and (2) we obtain in the case $BD = CE$

$$(x^2 + a^2)^{1/2}/(x^2 + b^2)^{1/2} = (Va + xb)/(Vb + xa)$$

and therefore

$$b^2x^4 + V^2a^2x^2 + 2abx^3V + b^4x^2 + 2ab^3Vx = a^2x^4 + V^2b^2x^2 + 2abx^3V + a^4x^2 + 2a^3bVx.$$

For $b \neq a$ and $x \neq 0$ follows

$$f(x) = x^3 - x\left(V^2 - (a^2 + b^2)\right) + 2abV = 0. \tag{3}$$

This equation controls the conditions for the equality of cevians in a non-isosceles triangle $\triangle ABC$ and where these cevians intersect exactly on AH ($MH = x$). The condition $f'(x_d) = 0$ gives

$$x_d = \pm\left[\left(V^2 - (a^2 + b^2)\right)/3\right]^{1/2}. \tag{4}$$

If $V < (a^2 + b^2)^{1/2}$, it is obvious that Eq. (3) does not have any positive solution; so there are no equal cevians apart from the case where $a = b$. We can further narrow the range of values of V, a and b which give us equal internal cevians by plugging the positive value of x_d from Eq. (4) into Eq. (3) and solving it in order to find the relations between V, a, b which give us a double solution (double point x_d, $f'(x_d) = f(x_d) = 0$), as shown in Appendix 4. Thus we get

$$V = V_d = (a^{2/3} + b^{2/3})^{3/2}, \tag{5}$$

$$x_d = (ab)^{1/3}(a^{2/3} + b^{2/3})^{1/2}. \tag{6}$$

The two equations above show beauty and harmony to a certain extent. In Appendix 4 the value of $BD_d = CE_d$ is also calculated.

Obviously, for the case $V > (a^{2/3} + b^{2/3})^{3/2}$, we get from (3) three distinct algebraic roots x_1, x_2 and x_3. The product of these three roots gives $x_1x_2x_3 = -2abV < 0$. Taking into account that $f'(x) = 0$ for $x = x_d = \pm\left[(V^2 - (a^2 + b^2))/3\right]^{1/2}$, we get

$$x_3 < -\left[(V^2 - (a^2 + b^2))/3\right]^{1/2} < x_2 < \left[(V^2 - (a^2 + b^2))/3\right]^{1/2} < x_1,$$

and, since $x_1x_2x_3 < 0$, we have $x_1 > 0$ and $x_2 > 0$. Hence, for this case we always have two solutions from Eq. (3) which produce equal cevians BD and CE. Also by setting $x = V$ and by using (3) we get

$$f(x) = V^3 - V\left(V^2 - (a^2 + b^2)\right) + 2abV > 0.$$

So we always have $0 < x_2 < x_1 < V$.

All the above leads to the following theorem:

Theorem 1. *Let $\triangle ABC$ be a triangle with the height AH and two internal cevians BD and CE, intersecting at the point M on AH. Let $BH = a$, $HC = b$, $AH = V$, $MH = x$, and the angle $\widehat{AHB} = \widehat{\phi} = 90°$. There are three conditions related to the equality of cevians BD and CE.*

1. If $V < (a^{2/3} + b^{2/3})^{3/2}$ the two cevians BD and CE can be equal only when $a = b$, i.e., when the triangle is isoceles ($AB = AC$).

2. If $V = V_d = (a^{2/3} + b^{2/3})^{3/2}$ then for $MH = x_d = (ab)^{1/3}\left(a^{2/3} + b^{2/3}\right)^{1/2}$ the two cevians BD and CE are equal for any value of a and b. For $x \neq x_d$ the cevians can be equal only when $a = b$.

3. If $V > V_d = \left(a^{2/3} + b^{2/3}\right)^{3/2}$ there are always two solutions x_1 and x_2 ($0 < x_2 < x_1 < V$) of (3) which give us $MH_1 = x_1$ and $MH_2 = x_2$ for which $BD_1 = CE_1$ and $BD_2 = CE_2$, respectively, for any value of a and b. For $x \neq x_1$ and $x \neq x_2$, the two cevians BD and CE are equal only when $a = b$.

Results equivalent to Theorem 1 are obtained in [2] using a different method, in particular without Eqs. (5) and (6) for V_d and x_d, as given above.

3. Equality of external cevians when $\widehat{\phi} = 90°$

Case 2, *BD in* P2, *CE in* P1.

Figure 2c shows $\triangle ABC$ with $\widehat{AHB} = \widehat{\phi} = 90°$ and external cevians BD in P2 and CE in P1. (Necessary conditions for Case 2 are: $\widehat{ABM} > (180° - \widehat{BAC})$ for BD in P2 and $\widehat{ACM} < (180° - \widehat{BAC})$ for CE in P1). From Appendix 3, which is valid for the general case $0 < \widehat{AHB} = \widehat{\phi} < 180°$, we get $EE' = xV(a + b)/(Vb - xa)$. Similarly we get $DD' = xV(a + b)/(Va - xb)$. We have also $EE'/x = EC/MC$ and $MC = (x^2 + b^2)^{1/2}$, hence $EC = (EE'/x)\,(x^2 + b^2)^{1/2}$. So we get:

$$EC = \left((a + b)V(x^2 + b^2)^{1/2}\right)/(Vb - xa), \tag{7}$$

$$BD = \left((a + b)V(x^2 + a^2)^{1/2}\right)/(Va - xb). \tag{8}$$

In the case $BD = CE$ we conclude from (7) and (8)

$$(x^2 + a^2)^{1/2}/(x^2 + b^2)^{1/2} = (Va - xb)/(Vb - xa),$$

hence

$$b^2x^4 + V^2a^2x^2 - 2abx^3V + b^4x^2 - 2ab^3Vx = a^2x^4 + V^2b^2x^2 - 2abx^3V + a^4x^2 - 2a^3bVx.$$

The above equation gives for $b \neq a$ and $x \neq 0$

$$f(x) = x^3 - x(V^2 - (a^2 + b^2)) - 2abV = 0. \tag{9}$$

Generally, Eq. (9) has three algebraic roots x_1, x_2 and x_3. The product of the three roots is $x_1x_2x_3 = 2abV > 0$. Taking in account that $f'(x) = 0$ for $x = \pm\left[(V^2 - (a^2 + b^2))/3\right]^{1/2}$ (if $(V^2 - (a^2 + b^2)) > 0$), we get

$$x_3 < -\left[(V^2 - (a^2 + b^2))/3\right]^{1/2} < x_2 < \left[(V^2 - (a^2 + b^2))/3\right]^{1/2} < x_1.$$

Since $x_1x_2x_3 > 0$, we have $x_1 > 0$, $x_2 < 0$ and $x_3 < 0$. In the case $(V^2 - (a^2 + b^2)) < 0$ there is only one positive real solution. Therefore we always have one positive solution of (9) which produces equal cevians BD and CE.

In Figure 2c, if we consider MH becoming equal to AH, we have EE' coinciding with DD', $DC = CE$ and, since $DD' \perp BC$, we get $BD < DC = CE$. Also generally, when MH increases, BD decreases and CE increases. Therefore, in a case where the external cevians BD and CE are equal for a specific MH, we always have $MH < AH$ or $x < V$. We summarize in the following theorem:

Theorem 2. *Let $\triangle ABC$ be a triangle with the height AH and two external cevians, BD in* P2 *and CE in* P1 *intersecting at point M on AH in* P1 *(as in Figure 2c). Let $BH = a$, $HC = b$, $AH = V$, $MH = x$, and angle $\widehat{AHB} = \widehat{\phi} = 90°$. The following condition deals with the equality of the cevians BD and CE (apart from the isosceles case $a = b$):*

For any value of a, b and V there is always one solution $x < V$ of Eq. (9) for which the two cevians BD and CE are equal.

In [2] results equivalent to Theorem 2 are presented, however derived with a different method.

Case 3, *BD and CE in* P1.

In Figure 3c we have the case of $\widehat{AHB} = \widehat{\phi} = 90°$ and external cevians BD and CE in P1. From Appendix 5 follows $BD < CE$; therefore no equal cevians are possible in this case.

Case 4, *BD in* P2*, CE in* P3.

In Figure 4c we have the case $\widehat{AHB} = \widehat{\phi} = 90°$ with external cevians BD and CE in P2 and P3, respectively. From Appendix 6 follows $BD < CE$; therefore again no equality of cevians can exist in this case.

Case 5, *BD in* P1*, CE in* P3.

In Appendix 7 it is shown that in the case $BH < HC$ and $0 < \widehat{AHB} = \widehat{\phi} < 180°$ it is impossible to have BD in P1 and CE in P3.

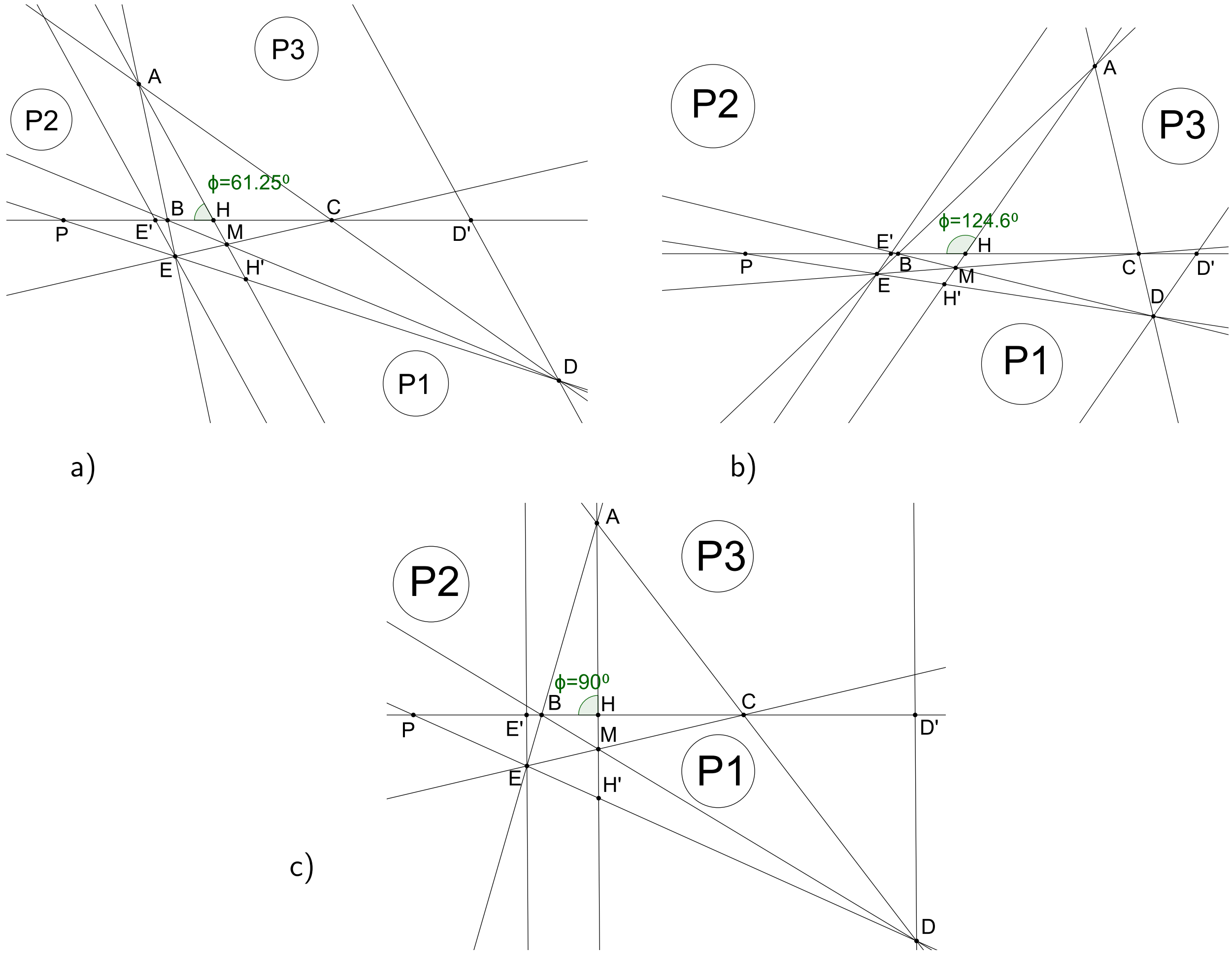

Figure 3: Cevians BD, CE and point M in P1, a) $\widehat{\phi} < 90°$, b) $\widehat{\phi} > 90°$, c) $\widehat{\phi} = 90°$.

4. Equality of internal cevians when $\widehat{\phi} \neq 90°$

Case 1, *BD and CE in the triangle ABC.*
In Figures 1a and 1b we have the cases of $\triangle ABC$ with internal cevians and $\widehat{\phi} < 90°$ and $\widehat{\phi} > 90°$, respectively. From Appendix 2, which is valid for the general case of $0 < \widehat{\phi} < 180°$, we get $EE' = xV(a+b)/(Vb+xa)$. Similarly we get $DD' = xV(a+b)/(Va+xb)$. Also from the Figures 1a and 1b we obtain $EE'/x = EC/MC$ and $MC = \left(x^2+b^2+2xb\cos\widehat{\phi}\right)^{1/2}$, hence $EC = (EE'/x)(x^2+b^2+2xb\cos\widehat{\phi})^{1/2}$. Thus we get

$$EC = \left((a+b)V(x^2+b^2+2xb\cos\widehat{\phi})^{1/2}\right)/(Vb+xa), \tag{10}$$

$$BD = \left((a+b)V(x^2+a^2-2xa\cos\widehat{\phi})^{1/2}\right)/(Va+xb). \tag{11}$$

From Eqs. (10) and (11) follows for $BD = CE$

$$\left(x^2+a^2-2xa\cos\widehat{\phi}\right)^{1/2}/(x^2+b^2+2xb\cos\widehat{\phi})^{1/2} = (Va+xb)/(Vb+xa)$$

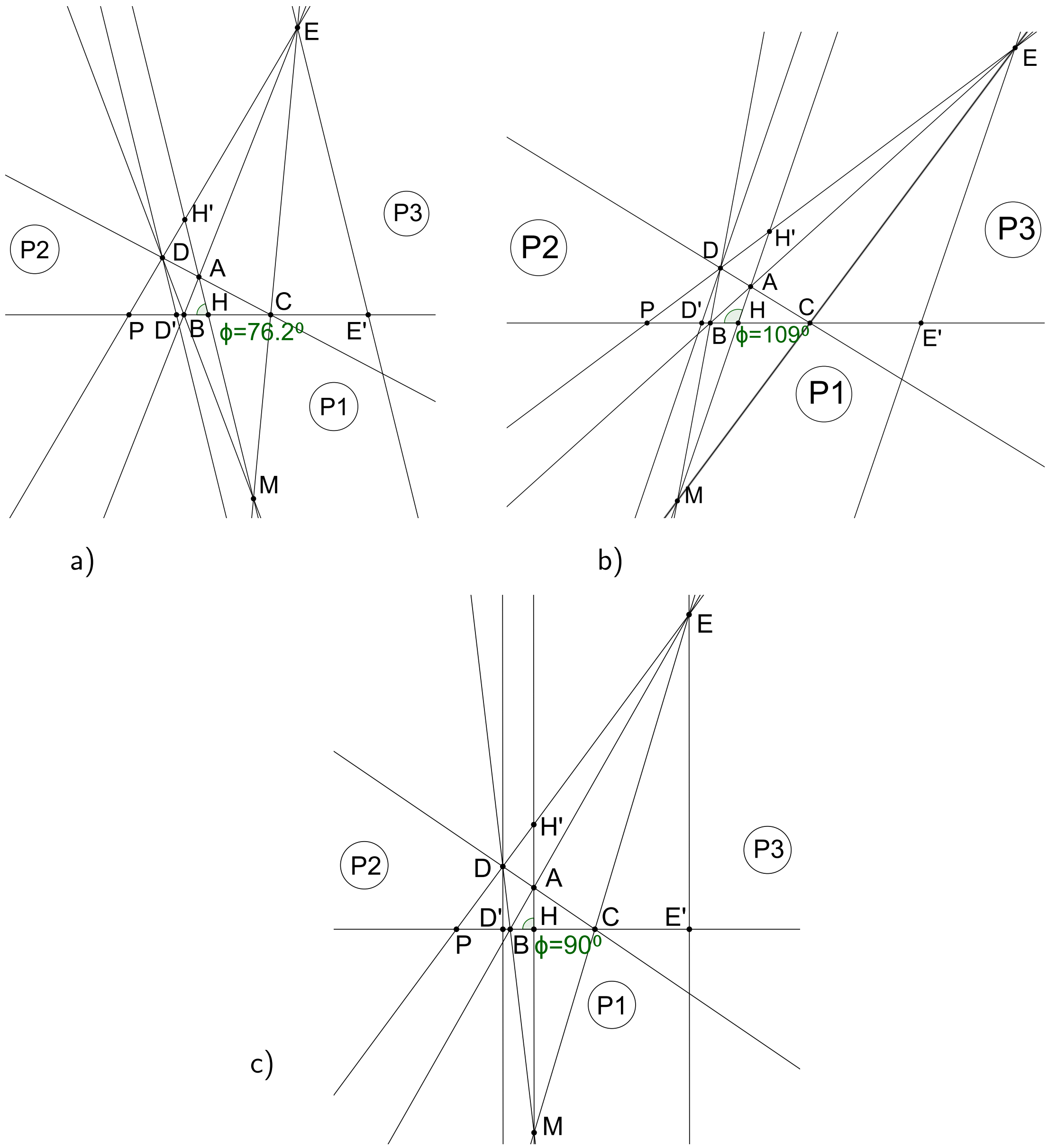

Figure 4: Cevian BD in P2, cevian CE in P3 and point M in P1,
a) $\widehat{\phi} < 90°$, b) $\widehat{\phi} > 90°$, c) $\widehat{\phi} = 90°$.

and further, explicitely,

$$\begin{aligned}
&b^2x^4 + V^2a^2x^2 + 2abx^3V + b^4x^2 + 2ab^3Vx - 2axb^2V^2\cos\widehat{\phi} - 4a^2bx^2V\cos\widehat{\phi} - 2a^3x^3\cos\widehat{\phi} \\
&= a^2x^4 + V^2b^2x^2 + 2abx^3V + a^4x^2 + 2a^3bVx + 2bxa^2V^2\cos\widehat{\phi} + 4ab^2x^2V\cos\widehat{\phi} + 2b^3x^3\cos\widehat{\phi}.
\end{aligned}$$

The equation above yields for $x \neq 0$

$$\begin{aligned}
f(x) \;=\;& x^3(a-b) - 2x^2(a^2+b^2-ab)\cos\widehat{\phi} \\
&+ x\left[(a-b)(a^2+b^2-V^2) - 4abV\cos\widehat{\phi}\right] + 2abV(a-b-V\cos\widehat{\phi}) = 0.
\end{aligned} \tag{12}$$

For $\widehat{\phi} = 90°$ and $a \neq b$ this equation gives again Eq. (3). Also this equation controls the conditions which allow an equality of cevians for a non-isosceles $\triangle ABC$ and where these

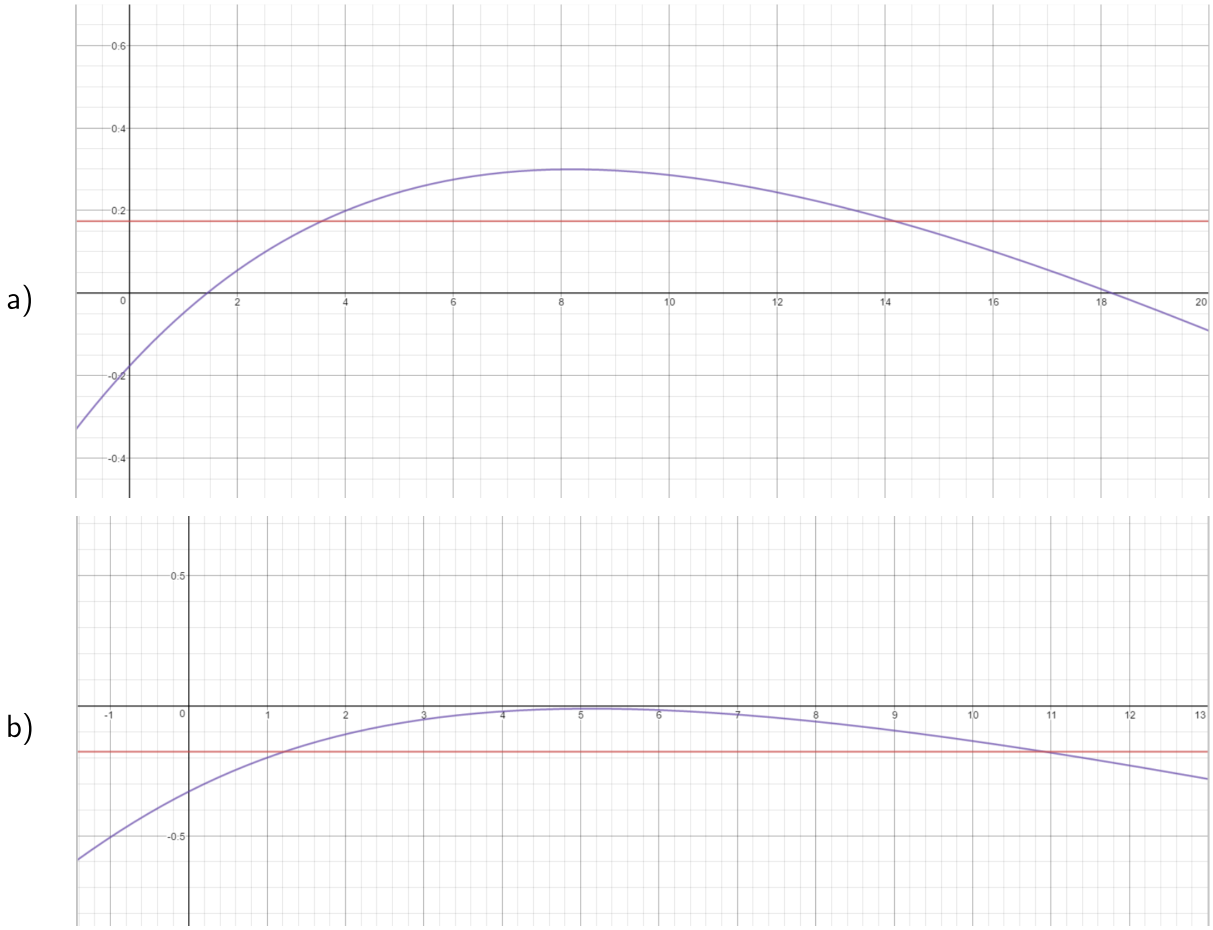

Figure 5: Graph of Eq. (13), x-axis with values of x, a) with parameters relevant to Figure 1a, b) only negative values of y because of the parameters relevant to Figure 1b.

cevians intersect exactly on AH ($MH = x$). There are three governing parameters a, b, and V. We solve (12) for $\cos\widehat{\phi}$ and obtain

$$\cos\widehat{\phi} = F_1(x)/G_1(x), \tag{13}$$

where

$$F_1(x) = \left[x^3 - x(V^2 - (a^2+b^2)) + 2abV\right](a-b) \quad \text{and} \tag{14}$$

$$G_1(x) = 2\left[(a^2+b^2-ab)x^2 + 2abVx + abV^2\right]. \tag{15}$$

A typical example of the graph of Eq. (13) is shown in Figure 5a for the case $\widehat{\phi} < 90°$ and in Figure 5b for $\widehat{\phi} > 90°$.

Equation (14) is equal to Eq. (3) multiplied by the constant $a-b$ which is negative because of $a < b$, as assumed from the beginning. The discriminant for Eq. (15) is $\Delta = -4V^2ab(b-a)^2 < 0$ and therefore $G_1(x) \neq 0$ for any value of x and also $G_1'(x) = 0$ for $x < 0$; hence $G_1(x) > 0$, and $G_1(x)$ increases for increasing $0 < x$.

From the above it can be deduced that Eq. (13), which is another form of Eq. (12), is a continuous function of x for given parameters a, b, V, and it allows to calculate (by using tools such as GraphSketch or Desmos) for which value of $\widehat{\phi}$ we have equal cevians for any given x, or vice versa, as can be seen in Figures 5a and 5b. The roots r_1, r_2 of Eq. (12) and also of Eq. (13) for $\widehat{\phi} < 90°$ are such that $0 < x_2 < r_2 \leq r_1 < x_1 < V$, where x_1, x_2 are the roots of Eq. (3) which is an obvious fact, as seen in Figure 5a. Alternatively, for $\widehat{\phi} > 90°$ we have $0 < r_2 \leq x_2 < x_1 \leq r_1$ (see Figure 5a). Obviously, if Eq. (3) has no solution (as analysed in Theorem 1) and $\widehat{\phi} < 90°$ then neither Eq. (12) nor (13) has one, whereas, if $\widehat{\phi} > 90°$ this does not affect the Eqs. (12) and (13) (Figure 5b). In the case $\widehat{\phi} > 90°$ it is possible that $r_1 > V$, in which case one of the solutions of (13) involves external cevians, BD in P2 and CE in P3 (Case 4, which is mentioned also in Section 5), intersecting at M on the extension of AH, opposite to P1 (see Figure 8a).

For a given set of values of the parameters a, b, V we can calculate the value of $\cos\widehat{\phi}_d$ which gives us a double solution ($r_d = r_2 = r_1$), by taking the parallel to x-axis tangent to the curve which represents (13), as shown in Figure 5a, with two necessary conditions:
a) $-1 < \cos\widehat{\phi}_d < 1$, and
b) either in the case of $\widehat{\phi} < 90°$ the relevant Eq. (3) has solutions (see Figure 5a), or, in the opposite case $\widehat{\phi} > 90°$, Eq. (3) does not have any solution (see Figure 5b).

This calculation can be done by using tools like GraphSketch or Desmos. All the above leads the following theorem:

Theorem 3. *Let M be a point in the plane of $\triangle ABC$ and let AH, BD and CE be three internal cevians through M, inside the triangle. Let $BH = a$, $HC = b$, $AH = V$, and $MH = x$. Then the following three conditions control the equality of cevians BD and CE (if $\triangle ABC$ is not isosceles).*

1. For $\widehat{AHB} = \widehat{\phi} < 90°$, if $V \leq (a^{2/3} + b^{2/3})^{3/2}$ then the two cevians BD and CE can never be equal.

2. For $\widehat{AHB} = \widehat{\phi} < 90°$, if $V > (a^{2/3} + b^{2/3})^{3/2}$ then for any $\widehat{\phi} \leq \widehat{\phi}_d$ or $\widehat{\phi} < 90°$ ($\widehat{\phi}_d$ has been defined in the previous paragraph) there are always two solutions r_1 and r_2 of (12) (if x_1 and x_2 are two solutions of (3), we have $0 < x_2 < r_2 \leq r_1 < x_1 < V$), which give us $MH_1 = r_1$ and $MH_2 = r_2$ for which $BD_1 = CE_1$ and $BD_2 = CE_2$, respectively, for any value of a and b.

3. For $\widehat{AHB} = \widehat{\phi} > 90°$, for any $\widehat{\phi} > \widehat{\phi}_d$ or $\widehat{\phi} > 90°$ there are always two solutions r_1 and r_2 of (12), which give $MH_1 = r_1$ and $MH_2 = r_2$ for which $BD_1 = CE_1$ and $BD_2 = CE_2$, respectively, for any value of a and b. It is possible that one of the solutions r_1 (belonging to Case 4 and mentioned in Theorem 6) gives external cevians BD_1 in P2 *and CE_1 in* P3*, intersecting at M on the extension of AH, but not in* P1.

5. Equality of external cevians when $\widehat{\phi} \neq 90°$

Case 2, *BD in* P2, *CE in* P1.
In the Figures 2a and 2b we have the case of $\triangle ABC$ with external cevians BD in P2 and CE in P1, $\widehat{\phi} < 90°$ and $\widehat{\phi} > 90°$, respectively (the same necessary conditions as in Case 2 and $\widehat{\phi} = 90°$). From Appendix 3, which is valid for the general case $0 < \widehat{AHB} = \widehat{\phi} < 180°$, we get $EE' = xV(a+b)/(Vb - xa)$. Similarly we get $DD' = xV(a+b)/(Va - xb)$. Also from Figures 2a and 2b we have $EE'/x = EC/MC$ and $MC = \left(x^2 + b^2 - 2xb\cos\widehat{\phi}\right)^{1/2}$, hence $EC = \left(EE'/x)(x^2 + b^2 - 2xb\cos\widehat{\phi}\right)^{1/2}$. We substitute EE' and get

$$EC = \left((a+b)V(x^2 + b^2 - 2xb\cos\widehat{\phi})^{1/2}\right)/(Vb - xa), \tag{16}$$

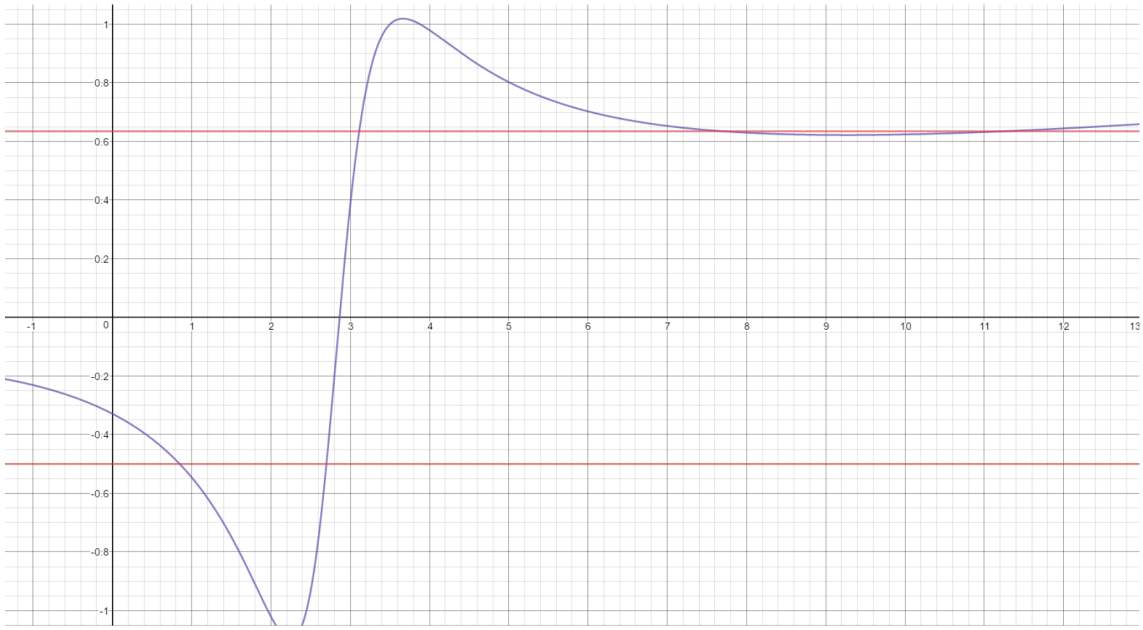

Figure 6: Graph of Eq. (19): y-axis with the values of $\cos\widehat{\phi}$, x-axis with values of x; the parameters are relevant for the Figures 8b and 8c.

$$BD = \left((a+b)V(x^2+a^2+2xa\cos\widehat{\phi})^{1/2}\right)/(Va-xb). \tag{17}$$

From these two equations follows for $BD = CE$

$$\begin{aligned} f(x) &= x^3(a-b)+2x^2(a^2+b^2-ab)\cos\widehat{\phi}+x[(a-b)(a^2+b^2-V^2)-4abV\cos\widehat{\phi}] \\ &\quad -2abV(a-b-V\cos\widehat{\phi})=0. \end{aligned} \tag{18}$$

In the case $\widehat{\phi} = 90°$ and $a \neq b$ the above equation becomes Eq. (9). As in Eq. (12), this equation controls the conditions which allow equal cevians for a non-isosceles $\triangle ABC$, where these cevians intersect exactly on AH ($MH = x$). We solve this equation for $\cos(\widehat{\phi})$ and obtain

$$\cos\widehat{\phi} = F_2(x)/G_2(x), \tag{19}$$

where

$$F_2(x) = \left[x^3 - x(V^2-(a^2+b^2)) - 2abV\right](a-b) \tag{20}$$

and

$$G_2(x) = 2\left[-(a^2+b^2-ab)x^2+2abVx-abV^2\right]. \tag{21}$$

A typical example of the graph of Eq. (19) is given in Figure 6. Equation (20) equals Eq. (9) multiplied with the constant $a-b$ (which is negative because $a<b$, as assumed from the beginning). The discriminant for Eq. (15) is $\Delta = -4V^2ab(b-a)^2 < 0$. Therefore $G_2(x) \neq 0$ for all values of x.

From the above the Eq. (19) can be deduced, which is another form of Eq. (18). It shows a continuous function of x and allows to calculate for which value of $\widehat{\phi}$ we have equal cevians for any given x, or vice versa, as it can be seen in Figure 6. The root r_1 of (19) is such that either, in the case $\widehat{\phi} < 90°$, holds $0 < x_1 \leq r_1$ or, in the case $\widehat{\phi} > 90°$, holds $0 < r_1 \leq x_1$, where x_1 is the positive root of Eq. (9), which is an obvious fact, as seen in Figure 6. Obviously, since Eq. (9) always has a positive root (as analysed in Theorem 2), the same holds for Eq. (19) in case of $\widehat{\phi} < 90°$. Moreover, as analysed in Case 4 of the present section and seen in Figure 6, there is a possibility to have another one (double) or two different roots in the same case $\widehat{\phi}$. In the case $\widehat{\phi} > 90°$ it is possible to have two positive roots $r_2 < r_1$ of (19) (as in Figure 6). Then the smaller root induces external cevians BD and CE in P1 (Case 3, as analysed below in the same section), such as in Figure 3b. This is also obvious from Figures 2a and 2b, where by keeping $\triangle ABC$ and $\widehat{\phi}$ constant and moving M in P1 along the extension of AH, as long as BD is in P2 and CE is in P1 (necessary conditions: $\widehat{ABM} > (180° - \widehat{ABM})$ for BD in P2 and $\widehat{ACM} < (180° - \widehat{BAC})$ for CE in P1). If MH increases BD decreases (within the range $[\infty, 0]$) and CE increases (within the range $[0, \infty]$). So, always only one solution of Case 2 exists for any value of a, b, V, and $\widehat{\phi}$.

Theorem 4. *Let $\triangle ABC$ be a triangle with an internal cevian AH and two external cevians, BD in* P2 *and CE in* P1*, intersecting at point M on AH in* P1*. Let $BH = a$, $HC = b$, $AH = V$, $MH = x$, and angle $0 < \widehat{AHB} = \widehat{\phi} < 180°$. Then the following condition controls the equality of cevians BD and CE for a non-isosceles triangle $\triangle ABC$:*

For any value of a, b and V, there is always one solution $x = r_1$ of (19) for which the two cevians BD and CE are equal, where either $0 < x_1 < r_1$ in the case $\widehat{\phi} < 90°$ or $0 < r_1 < x_1$ in the case $\widehat{\phi} > 90°$. Here x_1 is the positive root of Eq. (9).

Case 3, *BD and CE in* P1.

In Figures 3a and 3b we have the case of $\triangle ABC$ with external cevians BD and CE in P1, $\widehat{\phi} < 90°$ and $\widehat{\phi} > 90°$, respectively (necessary conditions for Case 3 are $\widehat{ABM} < (180° - \widehat{BAC})$ for BD in P1 and $\widehat{ACM} < (180° - \widehat{BAC})$ for CE in P1). From Appendix 3, which holds for the general case $0 < \widehat{AHB} = \widehat{\phi} < 180°$, and from the Figures 3a and 3b (from which we get that $MC = (x^2 + b^2 - 2xb\cos\widehat{\phi})^{1/2}$ and $BM = (x^2 + a^2 + 2xa\cos\widehat{\phi})^{1/2})$ we notice that exactly the same equations are valid as those in the previous Case 2. So, Eqs. (18), (19), (20), and (21) hold in this case as well. From Appendix 5 we get $BD < CE$ for the case $\widehat{\phi} < 90°$; so there is no solution for Eqs. (18) and (19).

In the previous Case 2 we have noticed that Eq. (19) can have up to two solutions $(0 < r_2 < r_1)$ for $\widehat{\phi} > 90°$ (as in Figure 6). As shown in Theorem 4 of Case 2, there is always one solution of Case 2 for which the two cevians BD and CE are equal, for any value of a, b, V, and $\widehat{\phi}$. So, the other solution, if it exists, has to be for Case 3 (Case 4 cannot exist, as shown in Appendix 6). The smaller solution r_2 is related to Case 3 and $\widehat{\phi} > 90°$, such as in Figure 3b. This is because of the necessary conditions for Case 2 and those for Case 3, according to which MH of Case 3 is always smaller than MH of Case 2, for any value of a, b, V, and $\widehat{\phi}$.

Theorem 5. *Let $\triangle ABC$ be a triangle with an internal cevian AH and two external cevians, BD and CE in* P1 *intersecting at point M on AH in* P1*. Let $BH = a$, $HC = b$, $AH = V$, $MH = x$, and the angle $0 < \widehat{AHB} = \widehat{\phi} < 180°$. There are two conditions related to the equality of cevians BD and CE, provided that the $\triangle ABC$ is non-isosceles:*

1. In the case $\widehat{\phi} < 90°$ there is always $BD < CE$.

2. In the case $\widehat{\phi} > 90°$ there is a possibility for only one solution $x = r_2$ of (19) belonging to Case 3, for which the two cevians BD and CE are equal for any value of a, b and V, where $0 < r_2 < x_1$ with x_1 being the positive root of Eq. (9). The other solution r_1 of (19) $(0 < r_2 < r_1 < x_1)$ belongs to Case 2 and follows Theorem 4.

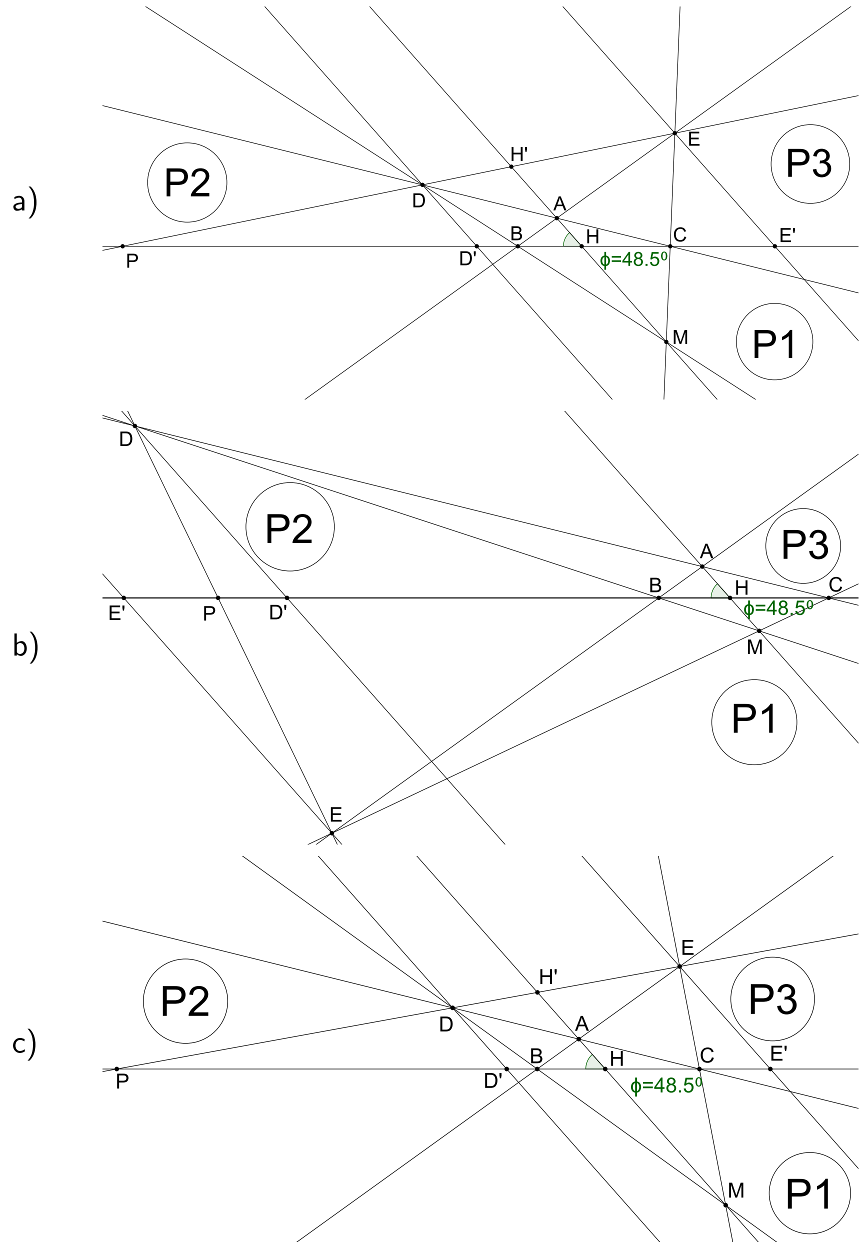

Figure 7: Cevian BD in P2 and point M in P1, $\widehat{\phi} < 90°$,
a) Cevian CE in P3, Case 4, solution r_2 of Eq. (19), when $r_3 < r_2 < r_1$;
b) Cevian CE in P1, Case 2, solution r_3 of (19), c) Cevian CE in P3, Case 4, solution r_1 of (19).

Case 4, *BD in* P2 *and CE in* P3.

Figures 4a, 4b, 7a, and 8a show the case of $\triangle ABC$ with external cevians BD in P2 and CE in P3, $\widehat{\phi} < 90°$ and $\widehat{\phi} > 90°$. In Figs. 4a, 4b and 7a the cevians intersect at M in P1 on the extension of AH (necessary conditions for this type of Case 4 are $\widehat{ABM} > (180° - \widehat{BAC})$ for BD in P1 and $\widehat{ACM} > (180° - \widehat{BAC})$ for CE in P3). In Figure 8a the point M lies on the extension of AH opposite to P1 (which is the necessary condition for this type of Case 4). For the general case $0 < \widehat{\phi} < 180°$, from Appendix 3 (which is valid for Figures 4a, 4b and 7a), from Appendix 2 (which is valid for Figure 8a) and from the Figures 4a, 4b, 7a and 8a (from these figures we get $MC = (x^2 + b^2 - 2xb\cos\widehat{\phi})^{1/2}$ and $BM = (x^2 + a^2 + 2xa\cos\widehat{\phi})^{1/2}$) we

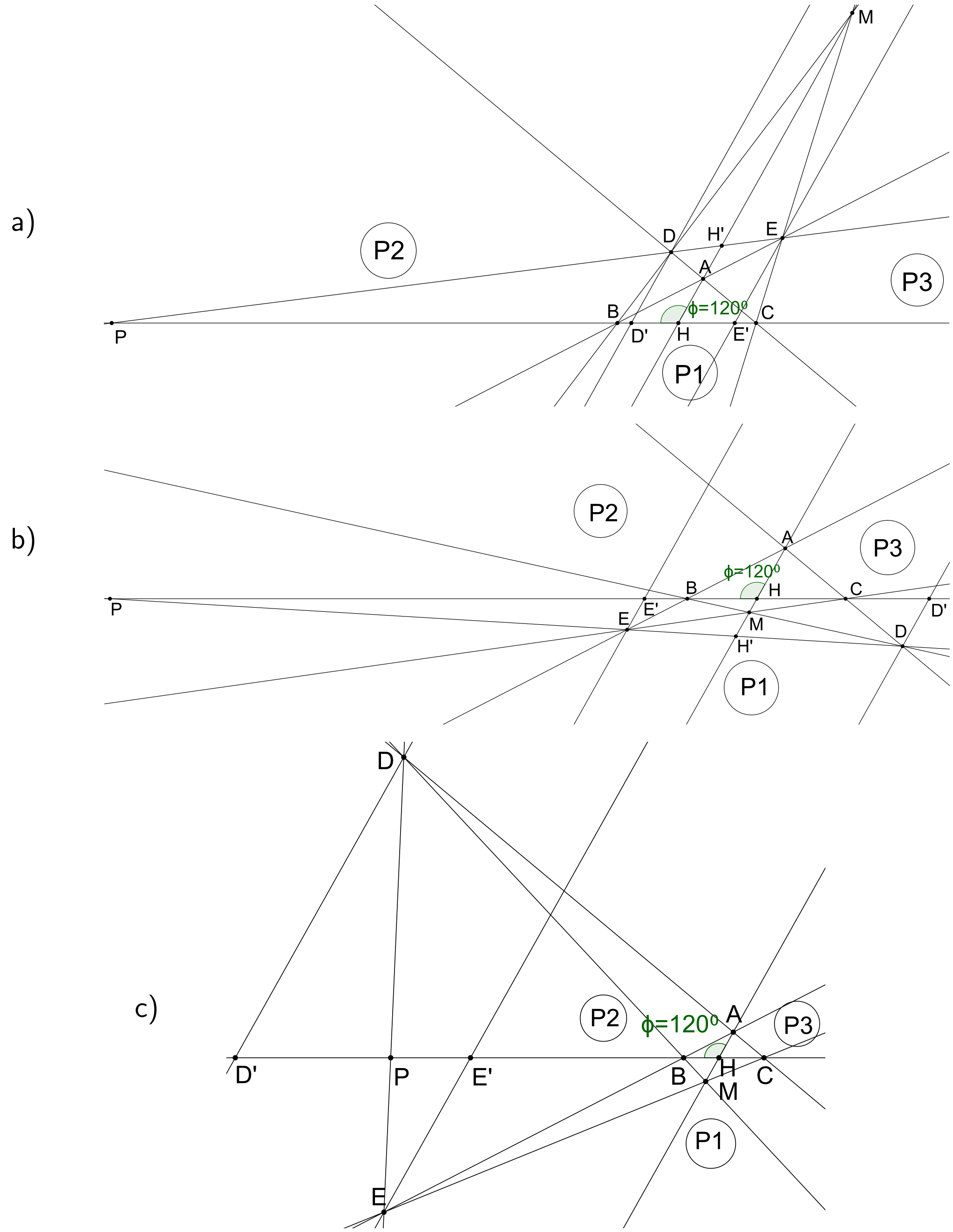

Figure 8: a) Cevians BD in P2, CE in P3 and point M opposite P1, $\widehat{\phi} > 90°$, Case 4, solution r_1 of (13) (Sections 4 and 5, $r_1 > V$); b) Cevians BD, CE and point M in P1, $\widehat{\phi} < 90°$, Case 3, solution r_2 of (19) where $r_2 < r_1$; c) Cevians BD in P2, CE and point M in P1, $\widehat{\phi} < 90°$, Case 2, solution r_1 of (19).

notice that exactly the same equations are valid as those in the previous Cases 2 and 3. So, in this case Eqs. (18), (19), (20), and (21) are valid as well.

If $\widehat{\phi} > 90°$, such as in Figure 4b (where the cevians intersect at M on the extension of AH in P1), there is no solution of (19) for Case 4, because $BD < CE$ as shown in Appendix 6. There is always a solution of Eq. (19) (Figure 6) belonging to Case 2 (Theorem 4) and possibly a solution belonging to Case 3, as mentioned above and shown in Figures 8c and 8b. Also, as described in Section 4 and Case 1, where $\widehat{\phi} > 90°$, it is possible that one of the solutions of (13) induces external cevians BD in P2 and CE in P3 (belonging to Case 4), intersecting at M on the extension of AH opposite to P1, such as in Figure 8a. The Figures 8a, 8b and 8c use the same triangle $\triangle ABC$ (Table 2).

If $\widehat{\phi} < 90°$, as mentioned in Case 3 above and shown in Theorem 4 of Case 2, there is always one solution of (19) belonging to Case 2 for which the two cevians BD and CE are equal for any value of a, b, V, and $\widehat{\phi}$. So, the other solution, if it exists, can only belong to Case 4. By studying Eq. (19), we notice that if we set the first derivative to zero, this is a quartic equation with respect to x (specifically $F_2'(x)G_2(x) - F_2(x)G_2'(x) = 0$), which either has no real solution or 2 or 4 real solutions. Also, (19) goes to $-\infty$ for $x \to -\infty$, and it goes to ∞ for $x \to \infty$. So, for $0 < \widehat{\phi} < 90°$ in the case of equal cevians the following facts are valid: for each value of $\cos\widehat{\phi}$ the Eq. (19) gives us one solution belonging to Case 2 for r_3 and possibly two solutions of Case 4 (no solution of (19) for Case 3 and $\widehat{\phi} < 90°$) for r_2 and r_1, where $r_3 < r_2 \leq r_1$ as shown in Figures 6, 7b, 7a, and 7c. The Figures 7a, 7b and 7c use the same $\triangle ABC$ (Table 2). Like in previous cases, we can have a double solution when $r_2 = r_1$ (which can be found by using tools such as GraphSketch or Desmos).

Theorem 6. *Let $\triangle ABC$ be a triangle with an internal cevian AH and two external cevians BD in* P2 *and CE in* P3*, intersecting at point M on an extension of AH, either in* P1 *or in the opposite direction. Let $BH = a$, $HC = b$, $AH = V$, $MH = x$, and the angle $0 < \widehat{AHB} = \widehat{\phi} < 180°$. The following conditions control the equality of cevians BD and CE (if $\triangle ABC$ is non-isosceles):*

1. In the case $\widehat{\phi} > 90°$, for M on the extension of AH in P1 *there is always $BD < CE$. If point M is in the opposite direction of* P1 *then there is a possibility of one solution based on Eq. (13) (also mentioned in Theorem 3).*

2. In the case $\widehat{\phi} < 90°$ there is always one solution r_3 of Eq. (19) $(r_3 > 0)$ belonging to Case 2 and following Theorem 4, plus the possibility of two solutions $r_3 < r_2 \leq r_1$ of (19) belonging to Case 4, for any value of a, b and V.

Case 5, *BD in* P1 *and CE in* P3.

In Appendix 7 it is shown that in the case $BH < HC$ and $0 < \widehat{\phi} < 180°$ it is impossible to have BD in P1 and CE in P3.

6. Equality of angle-bisectors as cevians

*Direct Proof of the **Steiner-Lehmus (S-L) theorem**.*

In Figure 9a we have the case of $\triangle ABC$ with external angle-bisectors BD_1 in P2 and CE_1 in P1 and internal angle-bisectors BD_0 and CE_0 and AH. Since AH is an internal angle-bisector, we have: $AB/AC = a/b$, hence $AB^2/AC^2 = a^2/b^2$ and

$$\left(a^2 + V^2 - 2aV\cos\widehat{\phi}\right)/(b^2 + V^2 + 2bV\cos\widehat{\phi}) = a^2/b^2,$$

consequently

$$\left(1 + (V/a)^2 - 2(V/a)\cos\widehat{\phi}\right) / \left(1 + (V/b)^2 + 2(V/b)\cos\widehat{\phi}\right) = 1$$

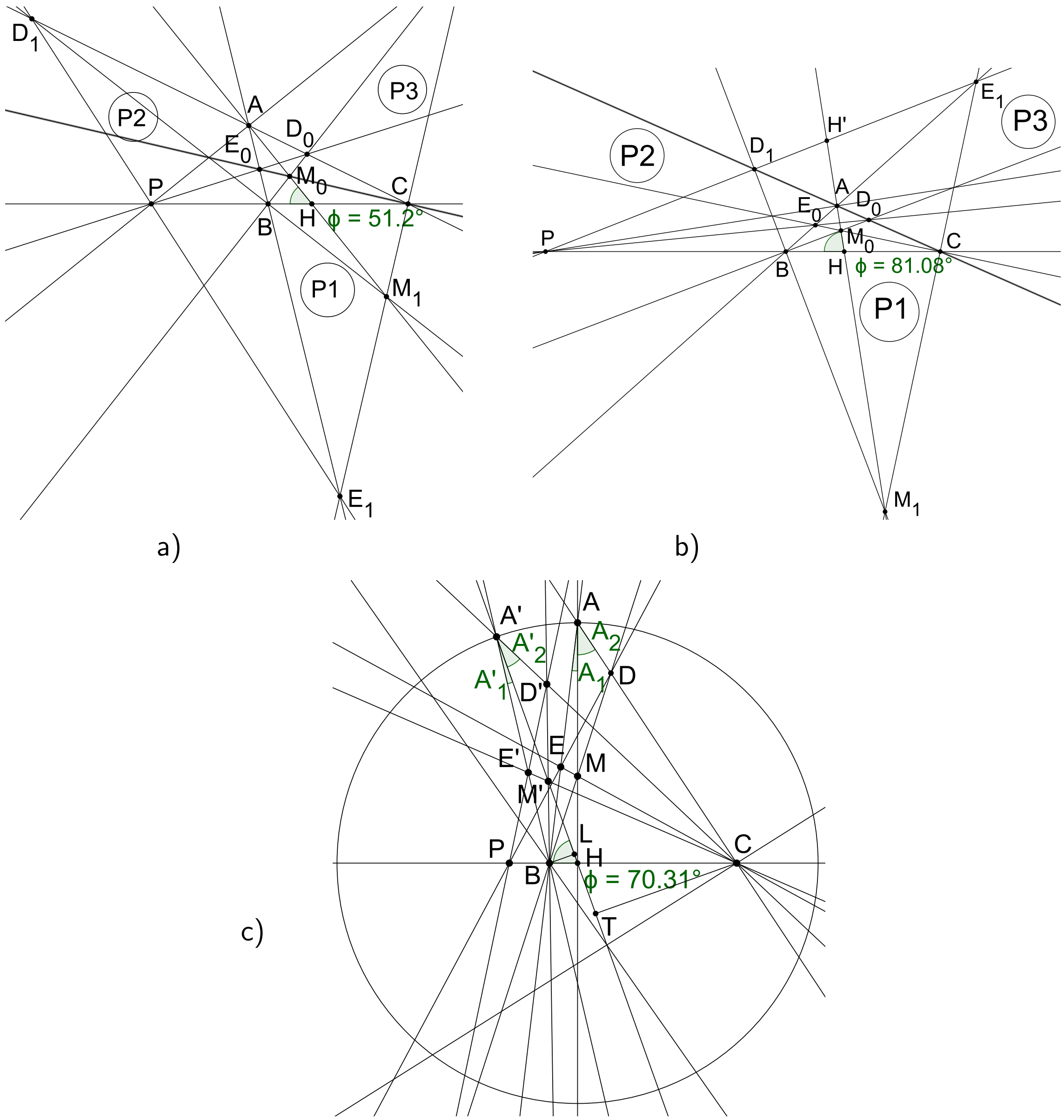

Figure 9: a, b) Internal and external angle-bisectors of a triangle,
M_0 and M_1 (in P1) intersection points of internal and external angle-bisectors,
c) Cevians BD', CE', point M' in $\triangle A'BC$, $\widehat{\phi} < 90°$ and $\widehat{BA'C} \leq 45°$.

and therefore $(V/a) - (V/b) - 2\cos\widehat{\phi} = 0$. Thus we obtain

$$\cos\widehat{\phi} = \frac{V(b-a)}{2ab}\,. \tag{22}$$

We notice that, according to (22), if $\widehat{\phi} = 90°$ then $a = b$. Also, if we substitute the value of $\cos\widehat{\phi}$ from (22) in Eq. (12) (internal equal cevians, Case 1, $\widehat{\phi} \leq 90°$) we get the following equation:

$$\begin{aligned} f(x) \;=\; & x^3(a-b)2ab - x^2 2(a^2+b^2-ab)V(b-a) + x\left[(a-b)(a^2+b^2-V^2)2ab\right. \\ & \left. - 4abV^2(b-a)\right] + 2abV\left((a-b)2ab - V^2(b-a)\right) = 0, \end{aligned}$$

and further

$$f(x) = (a-b)\left(x^3ab + x^2(a^2+b^2-ab)V + xab(a^2+b^2+V^2) + abV(2ab+V^2)\right) = 0.$$

For a, b, V, and $0 < x < V$ we get

$$x^3ab + x^2(a^2+b^2-ab)V + xab(a^2+b^2+V^2) + abV(2ab+V^2) > 0,$$

which is part of the equation above. So, this equation holds (i.e., both Eqs. (22) and (12) are valid) if $a = b$. As a result, equal internal angle-bisectors exist only in the case of an isosceles triangle ($a = b$), which is the Steiner-Lehmus theorem, proven with what can be considered as a direct proof.

Addition to the ***S-L theorem****.*
Since both Eqs. (12) and (22) are valid only when $a = b$ and $0 < x < V$, it is obvious that $BE \neq CE$ for $0 < x < V$ and $a \neq b$, if M is on AH itself (not on its extensions). Similar results based on different approaches appear in [8], [9] and [10].

Proof of ***S-L theorem*** *for external angle-bisectors.*
Equation (22) implies that there is always only one value of $\widehat{\phi}$ for each combination of a, b, V (as long as $-1 < \cos\widehat{\phi} < 1$). Also for any given triangle, its internal and external angle-bisectors are always uniquely defined. We also know from Theorems 2 and 4, related to Case 2, that there is always one solution of (19) for which the two cevians BD in P2 and CE in P1 are equal for any value of a, b, V, and $0 < \widehat{\phi} < 180°$. For some values of these parameters we can have equal external angle-bisectors BD_1 and CE_1 belonging to Case 2, such as displayed in Figure 9a.
For $\widehat{\phi} < 90°$ there is no solution in Case 3, as Theorem 5 states, and this includes external angle-bisectors. In Case 4 and external angle-bisectors, such as in Figure 9b, we have the following: $\widehat{D_1H'M_1} < 90°$, and from $\triangle M_1D_1E_1$ $\widehat{D_1M_1H'} < \widehat{H'M_1E_1}$ and $\widehat{M_1D_1E_1} > \widehat{D_1E_1M_1}$, therefore $D_1H' < H'E_1$ (this holds also because $(P,H;C,B) = -1$, $(P,H';E_1,D_1) = -1$). The triangle $\triangle M_1D_1E_1$ of Figure 9b is equivalent to the triangle $\triangle ABC$ of Figure 1a (in the same way as $\triangle AED$ with $\triangle ABC$ in Appendix 5). Therefore, according to Appendix 2, where $EB < DC$ in Figure 1a, we have $BD_1 < CE_1$ in Figure 9b. This proves that there are no equal external angle-bisectors in Case 4.

Addition to the ***S-L theorem*** *for external angle-bisectors.*
Given that AH is an angle-bisector of $\triangle ABC$, for the relevant cevians BD and CE in P1 (intersecting at M on the extension of AH) holds always $BD < CE$ for Case 3, as Theorem 5 states. For Case 4 and M on the extension of AH in P1, as in the previous paragraph, the triangle $\triangle MDE$ is equivalent to the triangle $\triangle ABC$ of Figure 1a. Therefore, according to Appendix 2, where $EB < DC$ in Figure 1a, we have $BD < CE$. So, there is no equality of cevians for any point M on the extension of the angle-bisector AH in P1. Similarly we treat Case 4, when M is on the extension of the cevian-bisector AH but not in P1 (similar results in [10] and [1]).

7. Equicevian points in internal and external cevians cases

Case 1, *BD and CE in triangle ABC.*
In Figures 1a, 1c and 1b we have the cases of $\triangle ABC$ with internal cevians AH, BD and CE. Let us suppose $a + b = 1$ in order to ease the calculations.

We start with the case of Figure 1c ($\widehat{\phi} = 90°$) and let M ($x = MH$) be an equicevian point, which means $BD = CE = AH = V$. Then we get from Eqs. (1) and (2)

$$\begin{aligned} x^2 + a^2 &= (Va + xb)^2, \text{ hence } & x^2 + a^2 &= V^2a^2 + x^2b^2 + 2Vabx \text{ and} \\ x^2 + b^2 &= (Vb + xa)^2, \text{ hence } & x^2 + b^2 &= V^2b^2 + x^2a^2 + 2Vabx. \end{aligned}$$

Consequently, $b^2 - a^2 = V^2(b^2 - a^2) + x^2(a^2 - b^2)$ and therefore $x = (V^2 - 1)^{1/2}$. So, considering (3), we get

$$S = f\left(x = (V^2 - 1)^{1/2}\right) = (V^2 - 1)^{3/2} - (V^2 - 1)^{1/2}(V^2 - (a^2 + b^2)) + 2abV = 0,$$

hence

$$\begin{aligned} S &= (V^2 - 1)^{1/2}(V^2 - 1 - V^2 + a^2 + b^2) + 2abV = 0, \\ S &= (V^2 - 1)^{1/2}(a^2 + b^2 - 1) + 2abV = 0 \end{aligned}$$

and $(V^2 - 1)^{1/2} < V$ for $V > 1$, thus

$$S = (V^2 - 1)^{1/2}(a^2 + b^2 - 1) + 2abV > (V^2 - 1)^{1/2}(a^2 + b^2 - 1 + 2ab) = 0,$$

because we have assumed that $a + b = 1$, thus $S > 0$ when $V > 1$.

If $V \leq 1 = a+b < (a^{2/3}+b^{2/3})^{3/2}$ there is no solution of (3), a fact stated also in Theorem 1. So, no equicevian point with internal cevians is possible when $\widehat{\phi} = 90°$. As M moves along AH and M_h is the orthocenter of $\triangle ABC$, we have the following:

Remark 1. If $0 < x \leq M_hH$ then $BD(x)$ decreases when x increases, and $BD < BC$.

Remark 2. If $M_hH < x < V$ then $BD(x)$ increases when x increases.

Having already obtained that there is no solution of (3) which gives us $BD = CE = V$, we get from Remarks 1 and 2

Remark 3. For all x-values (or M points) satisfying (3) the inequality $BD < V$ is valid.

Let us assume that $\widehat{\phi} < 90°$, as in Figure 1a. Then we obtain from Eqs. (2) and (8)

Remark 4. $BD(\widehat{\phi} < 90°) < BD(\widehat{\phi} = 90°)$ if a, b, V, and x are the same for the two cases of $\widehat{\phi}$.

Also from Theorem 3 we get

Remark 5. $0 < x_2 \leq r_2 < r_1 \leq x_1 < V$, where r_1 and r_2 are solutions of Eq. (12), and x_1 and x_2 are solutions of Eq. (3).

From Remarks 3, 4 and 5 we get the following theorem:

Theorem 7. *Let M be a point in the plane of $\triangle ABC$ and let AH, BD and CE be three internal cevians through M, inside the triangle. Let $BH = a$, $HC = b$, $AH = V$, $MH = x$, and the angle $0 < \widehat{AHB} = \widehat{\phi} \leq 90°$. Then for the point M two equal cevians are possible ($BD = CE$, as in Theorem 3), but M can never be an equicevian point.*

In Figure 1b with $\widehat{\phi} > 90°$ we get from (10) and (11) and for $BD = CE = V$

$$\begin{aligned} x^2 + a^2 - 2ax\cos\widehat{\phi} &= V^2a^2 + x^2b^2 + 2Vabx \text{ and} \\ x^2 + b^2 + 2bx\cos\widehat{\phi} &= V^2b^2 + x^2a^2 + 2Vab. \end{aligned}$$

Together with $a + b = 1$ we obtain

$$x^2(b - a) + 2x\cos\widehat{\phi} + (b - a)(1 - V^2) = 0.$$

This equation together with (12) determines the equicevian points in this case. This case has been thoroughly analysed in [3].

Case 2, BD *in* P2*,* CE *in* P1, and **Case 4**, BD *in* P2 *and* CE *in* P3.
As can easily be seen in Figures 2a, 2b, 2c, 4a, 4b, 4c, and 7a, by taking a parallel to AH passing through B and intersecting AC in D'', we always have $BD > BD'' > AH$. So it is impossible in these cases to have an equicevian point. In Figure 8a the parallel to AH will have to pass through point D and will intersect BC in D', where we always have $BD > DD' > AH$. So it is impossible in this case to have an equicevian point as well.

Case 3, BD *and* CE *in* P1.
In this case, as shown and analysed thoroughly in [3], we can have equicevian points according to Theorem 5 when $\widehat{\phi} > 90°$.

8. Calculation of the limit of the angle $\widehat{BA'C}$

In [2] is stated and proved that $\widehat{BAC} \leq 45°$ under the conditions of Case 1 and $\widehat{\phi} = 90°$ (Section 2). In the present work we prove below that $\widehat{BA'C} \leq 45°$ under the conditions of Section 4, Case 1, and $\widehat{\phi} < 90°$ (shown in Figure 9c as $\triangle A'BC$).

From $\triangle ABC$ and $\triangle A'BC$ at Figure 9c, we have

$$\begin{aligned}
\tan \widehat{A_1} &= \tan(\widehat{BAH}) &= BH/AH &= a/V, \\
\tan \widehat{A_2} &= \tan(\widehat{HAC}) &= CH/AH &= b/V, \\
\tan \widehat{A'_1} &= \tan(\widehat{BA'H}) &= BL/A'L &= a\sin\widehat{\phi}/(V - a\cos\widehat{\phi}), \\
\tan \widehat{A'_2} &= \tan(\widehat{HA'C}) &= CT/A'T &= b\sin\widehat{\phi}/(V + b\cos\widehat{\phi}),
\end{aligned}$$

where $A'H = AH = V$, $\widehat{BHA'} = \widehat{\phi} < 90°$, $\widehat{BHA} = 90°$, $BL \perp A'H$, and $CT \perp A'H$. From the above we get

$$\tan(\widehat{A'_1} + \widehat{A'_2}) = (\tan\widehat{A'_1} + \tan\widehat{A'_2})/(1 - \tan\widehat{A'_1}\tan\widehat{A'_2}).$$

So, after the necessary calculations,

$$\tan(\widehat{A'_1} + \widehat{A'_2}) = \sin\widehat{\phi}\, V(a+b)/\left((V - a\cos\widehat{\phi})(V + b\cos\widehat{\phi}) - ab(\sin\widehat{\phi})^2\right).$$

Similarly, we get

$$\tan(\widehat{A_1} + \widehat{A_2}) = V(a+b)/(V^2 - ab).$$

We also have the following remarks:

Remark 6. $\sin\widehat{\phi}\, V(a+b) < V(a+b)$.

Remark 7. $(V - a\cos\widehat{\phi})(V + b\cos\widehat{\phi}) - ab(\sin\widehat{\phi})^2 > V^2 - ab$, since

$$V^2 + bV\cos\widehat{\phi} - aV\cos\widehat{\phi} - ab(\cos\widehat{\phi})^2 - ab(\sin\widehat{\phi})^2 > V^2 - ab,$$

and therefore $V(b-a)\cos\widehat{\phi} > 0$, which holds true.

Remark 8. $\widehat{BAC} = \widehat{A_1 + A_2} \leq 45°$, as proven in [11].

From the two equations for $\tan(\widehat{A_1} + \widehat{A_2})$ and $\tan(\widehat{A'_1} + \widehat{A'_2})$ and the three remarks we get $\widehat{BA'C} = \widehat{A'_1 + A'_2} \leq 45°$.

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Appendix 1

Given a triangle $\triangle ABC$, its height AH and two internal cevians BD and CE intersecting at a point M on AH, prove by Euclidean Geometry that when $BD = CE$ then $AB = AC$, i.e., the triangle is isosceles.

The answer to this problem is analysed in Section 2 and summarized in Theorem 1, where it becomes obvious that the statement to be proven as requested above is conditional. The correct formulation of the problem should be as follows:

Given a triangle $\triangle ABC$, its height AH and two internal cevians BD and CE intersecting at point M on AH, determine all the necessary conditions in order to have $BD = CE$ and provide the relevant proofs.

It is interesting to note that until recently I haven't found in a book or paper anything even remotely related to the description of the problem, which actually increased my drive to research and develop the present work. As I finally realized, the reason why this problem does not exist as such, is that it does not have a straightforward answer, as shown in the definition of Theorem 1. However, there is nothing new under the sun, since — as I found out quite recently — in [2] a very good analysis of the problem is given, which leads to the same results as the Theorems 1 and Theorem 2 of the present work.

Appendix 2

In Figures 1a, 1b, 1c (with internal cevians), and 8a (with external cevians) we have the case where BD and CE exist either internally or externally, in which case they intersect at M on the extension of AH opposite to P1 (Figure 8a). Since $EE' \parallel AH$, $AH = V$ and $MH = x$, we have $EE'/x = (a+b-BE')/b$ and $EE'/V = BE'/a$, hence $BE' = aEE'/V$. So, we get $EE'b = xa + xb - xaEE'/V$ and further $EE' = xV(a+b)/(Vb+xa)$. Similarly, we obtain $DD' = xV(a+b)/(Va+xb)$.

In the figures mentioned above, the points P, B, H, C form a harmonic set (see [3]), so $(P,H;C,B) = -1$ and in the same way $(P,H';D,E) = -1$. Also from these figures we get $EE'/AH = EB/AB$ and $DD'/AH = DC/AC$. From these relations and from the formulae for EE' and DD' we get

$$(Va+xb)/(Vb+xa) = (EB/DC)(AC/AB).$$

Since $AC > AB$ (because $a < b$ and $\widehat{\phi} \leq 90°$ for Figures 1a and 1c) and $(Va+xb) < (Vb+xa)$ (because $x < V$), we deduce $EB < DC$ when $0 < \widehat{\phi} \leq 90°$, as in the Figures 1a and 1c.

Appendix 3

In Figures 2a, 2b, 2c, 3a, 3b, and 3c we have a triangle $\triangle ABC$ with $0 < \widehat{\phi} \leq 180°$ and external cevians where BD exists either in P2 or P1 and CE exists in P1. Since $EE' \parallel AH$, $AH = V$ and $MH = x$, we have $EE'/x = (a+b+BE')/b$ and $EE'/V = BE'/a$, hence $BE' = aEE'/V$. So we conclude

$$EE'b = xa + xb + xaEE'/V, \quad \text{hence} \quad EE' = xV(a+b)/(Vb-xa).$$

Similarly, $DD' = xV(a+b)/(Va-xb)$.

In Figures 4a, 4b and 4c we have $0 < \widehat{\phi} \leq 180°$ and external cevians BD in P2 and CE in P3. Since $EE' \parallel AH$, $AH = V$ and $MH = x$, we have $EE'/x = CE'/b$, hence $CE' = bEE'/x$ and $EE'/V = (a+b+CE')/a$. Thus we get

$$EE'a = V(a+b) + bVEE'/x, \quad \text{hence} \quad EE' = xV(a+b)/(ax-Vb).$$

Similarly $DD' = xV(a+b)/(bx-Va)$.

Since in all cases (as seen in Sections 2 to 5), in order to tackle the issue of square roots, we raise the equations, which use the above calculations of EE' and DD', to the power 2 and we always consider the absolute values of $(Vb-ax)$ and $(Va-bx)$ in all sections of this work.

Appendix 4

From

$$f(x_d) = [(V^2-(a^2+b^2))/3]^{3/2} - [(V^2-(a^2+b^2))/3]^{1/2}[V^2-(a^2+b^2)] + 2abV = 0$$

follows

$$f(x_d) = f(V) = (V^2-(a^2+b^2))^{3/2}((1/3)^{3/2}-(1/3)^{1/2}) + 2abV = 0.$$

Since $(1/3)^{3/2} - (1/3)^{1/2} = (1/3)^{1/2}(1/3-1) = -2(1/3)^{3/2}$,

$$f(V) = (V^2-(a^2+b^2))^{3/2}(-2(1/3)^{3/2}) + 2abV = 0$$

and finally

$$V^2 - 3V^{2/3}(ab)^{2/3} - (a^2+b^2) = 0.$$

If we assume $V = Y^{3/2}$ then we have $Y^3 - 3Y(ab)^{2/3} - (a^2 + b^2) = 0$. This cubic equation is solvable and according to [14, p. 32] we get $a_1 = 0$, $a_2 = -3(ab)^{2/3}$, $a_3 = -(a^2 + b^2)$, $Q = a_2/3 = -(ab)^{2/3}$, $R = -a_3/2 = (a^2 + b^2)/2$, and $D = Q^3 + R^2 = (a^2 - b^2)^2/4 > 0$. Since $D > 0$, there is only one real solution which is $Y = S + T - a_1/3$, where

$$\begin{aligned} S &= (R + (Q^3 + R^2)^{1/2})^{1/3} = ((a^2 + b^2)/2 + (a^2 - b^2)/2)^{1/3} = a^{2/3}, \\ T &= (R - (Q^3 + R^2)^{1/2})^{1/3} = ((a^2 + b^2)/2 - (a^2 - b^2)/2)^{1/3} = b^{2/3}, \end{aligned}$$

and $Y = a^{2/3} + b^{2/3}$. Since we have $V = Y^{3/2}$, we get

$$f(x_d) = f(V) = 0 \quad \text{for} \quad V = V_d = (a^{2/3} + b^{2/3})^{3/2}.$$

We get also

$$x_d = [(V^2 - (a^2 + b^2))/3]^{1/2} = [((a^{2/3} + b^{2/3})^3 - (a^2 + b^2))/3]^{1/2},$$

hence $x_d = (ab)^{1/3}(a^{2/3} + b^{2/3})^{1/2}$.

From the above and from Eqs. (1) and (2) we easily obtain

$$BD_d = CE_d = (a + b)(a^{2/3} + b^{2/3})/(a^{4/3} + b^{4/3} + (ab)^{2/3})^{1/2}.$$

Appendix 5

In Figures 3a, 3b, and 3c we have the case of $\triangle ABC$ with $0 < \widehat{\phi} < 180°$ and external cevians BD and CE in P1. We prove below for the case $0 < \widehat{\phi} \leq 90°$ (Figures 3a and 3c) that always $BD < CE$:

The triangle $\triangle AED$ of the Figures 3a and 3c is equivalent to the $\triangle ABC$ of the Figures 1a and 1c in terms of having the following two harmonic sets: $(P, H; C, B) = -1$ and $P, H'; D, E) = -1$ in the same way as shown in Appendix 2. More specifically, the following elements are respectively equivalent: the sides AE, AD and ED, angle $\widehat{AH'E}$ and the points B, C, P of $\triangle AED$ with the sides AB, AC and BC, angle $\widehat{\phi}$ and points E, D, P of $\triangle ABC$. Bearing in mind that $\widehat{AH'E} < \widehat{AHB} = \widehat{\phi}$, we can conclude that $BE < CD$ for the Figures 3a and 3c based on the above mentioned equivalence, as proven in Appendix 2 for the Figures 1a and 1c.

From the cosine law for $\triangle BCE$ follows

$$CE = \left[(BE)^2 + (BC)^2 - 2(BE)(BC)\cos\widehat{EBC}\right]^{1/2}.$$

Similarly at $\triangle BCD$ we have

$$BD = \left[(CD)^2 + (BC)^2 - 2(CD)(BC)\cos\widehat{BCD}\right]^{1/2}.$$

Furthermore hold $90° < \widehat{BCD} < \widehat{EBC}$ and $BE < CD$. Thus follows that we always get $BD < CE$, when $0 < \phi < 90°$, and the cevians BD and CE are external.

Appendix 6

In the Figures 4a, 4b, and 4c we have the case of $\triangle ABC$ with $0 < \widehat{\phi} < 180°$ and external cevians BD in P2 and CE in P3, and their point M of intersection exists in P1. We prove below that for the case $180° > \widehat{\phi} \geq 90°$ (Figures 4b and 4c) we always have $BD < CE$.

The triangle $\triangle MDE$ of these figures is equivalent to $\triangle ABC$ of Figures 1a, 1b and 1c in terms of having the following two harmonic sets: $(P, H'; D, E) = -1$ and $P, H; C, B) = -1$ in the same way as shown in Appendix 2. More specifically, the following elements are respectively equivalent: the sides MD, ME and DC, angle $\widehat{DH'M}$ and points B, C, P of $\triangle MDE$ with the sides AB, AC and BC, angle $\widehat{\phi}$ and points E, D, P of $\triangle ABC$. We note that $\widehat{DH'M} < \widehat{BHM} \leq 90°$ for the Figures 4b and 4c. In Appendix 2 we showed that $EB < DC$. Therefore, due to the equivalence with the present case, it follows $BD < CE$, when $\widehat{\phi} \geq 90°$ (Figures 4b and 4c).

Table 2: Data of figures

Figure	a	b	V	$\widehat{\phi} - degr.$	$CASE$
1a	2.24	7.76	19.92	80	1
1b	2.24	7.76	10.66	100.2	1
1c	2.24	7.76	11.2	90	1
2a	2.24	7.76	7.57	61.25	2
2b	2.24	7.76	7.57	100.5	2
2c	2.24	7.76	7.57	90	2
3a	2.24	7.76	7.57	61.25	*3
3b	2.24	7.76	7.57	124.6	3
3c	2.24	7.76	7.57	90	*3
4a	2.24	7.76	3.6	76.2	*4
4b	2.24	7.76	3.09	109	*4
4c	2.24	7.76	3.93	90	*4
7a	3.35	4.65	1.97	48.51	4
7b	3.35	4.65	1.97	48.51	2
7c	3.35	4.65	1.97	48.51	4
8a	3.52	4.48	2.92	120	4
8b	3.52	4.48	2.92	120	3
8c	3.52	4.48	2.92	120	2
9a	3.13	6.87	7.19	51.2	bisectors 2
9b	3.78	6.22	3	81.1	bisectors *4
9c	1.2	6.8	10.28	70.3	1

* where $BD \neq CE$

Appendix 7

In Figure 3a we observe that when $HM/AH < BH/HC$ then BD and the extension of AC intersect in P1, and when $HM/AH = BH/HC$ then $BD \parallel AC$. Since $BH < HC$ we have $BH/HC < HC/BH$. So, given that $HM/AH < BH/HC$, we get $HM/AH < HC/BH$, which guarantees that CE and the extension of AB intersect in P1 (when $HM/AH = HC/BH$ then $CE \parallel AB$). The above proves that Case 5 with BD in P1 and CE in P3 is impossible when $BH < HC$ and $0 < \widehat{\phi} < 180°$.

Note that the data of all figures are summarized in Table 2.

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