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Svetlana Ribbons with Intersecting Axes in a Hyperbolic Plane of Positive Curvature

Lyudmila Romakina

Department of Geometry, Saratov State University
410012 Astrakhanskaya 83, Saratov, Russia
email: romakinaln@mail.ru

This paper is devoted to the memory of my beloved daughter
Morozkina Svetlana (Aug. 08, 1991 – Aug. 31, 2015)

Abstract. In the Cayley-Klein model, a Lobachevskiĭ plane Λ^2 is realized in the projective plane P_2 in the interior of an oval curve. A hyperbolic plane $\widehat{H}$ of positive curvature is realized in the ideal domain of the Lobachevskiĭ plane. We study here trajectories of the midpoint of a segment with its endpoints running along two orthogonal lines intersecting in $\widehat{H}$. Such trajectories are called Svetlana Ribbons. We prove that Cassini Ovals of the Euclidean plane E_2 can be images of Svetlana Ribbons with intersecting axes.

Key Words: Lobachevskiĭ plane Λ^2, hyperbolic plane $\widehat{H}$ of positive curvature, Svetlana Ribbon, Cassini Oval

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1. Introduction

1.1. The hyperbolic plane $\widehat{H}$ of positive curvature

A projective Cayley-Klein model of a *Lobachevskiĭ plane* Λ^2 is the internal domain of an oval curve γ (or conic, in other terminology, see [4, 5, 6]) of the projective plane P_2. A *hyperbolic plane* $\widehat{H}$ *of positive curvature* can be realized in the ideal domain of a Lobachevskiĭ plane. The planes Λ^2 and $\widehat{H}$ are connected components of the *extended hyperbolic plane* H^2 [25, 16]. The oval curve γ is called the *absolute* of the planes H^2, $\widehat{H}$, and Λ^2. The group G of projective automorphisms of the oval curve γ is the fundamental group of transformations for H^2, $\widehat{H}$, and the Lobachevskiĭ plane Λ^2. The lines of the planes $\widehat{H}$ and H^2 belong to three types. The main objects on lines of all types are discussed in [16].

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1.2. Problem statement

One of the motions considered in kinematics is the *Cardan motion* which is defined as the motion of a plane Γ_1 with respect to a coinciding plane Γ such that two points A and B of Γ_1 move along two orthogonal lines l, m of Γ (see, for instance, [2], [6, Section 2.3] or [12]). In Euclidean geometry, an arbitrary point of the plane Γ_1 traces in general an ellipse during a Cardan motion (see [6, Theorem 2.3.1]). In particular, the midpoint of the moving segment AB describes a circle. In [3] it is proved that, in general, in an elliptic plane the path of an arbitrary point of the generating line AB during the Cardan motion is a quartic curve. In this work the paths of the midpoint of the generating segment AB during Cardan motions are investigated in $\widehat{H}$. Let us give a rigorous definitions of the research objects.

Let AB be an elliptic or hyperbolic segment of constant length δ (or a parabolic segment) in the plane $\widehat{H}$. The endpoints of the segment AB move along two orthogonal lines l and m such that $A \in l$ and $B \in m$. The midpoint S (quasimidpoint S_0) (see [16, § 4.2]) of the segment AB describes a curve during such a motion. This curve is called *Svetlana Ribbon* (*Svetlana Quasiribbon*) with *axes* l, m and *base segment* AB (or *base*, for short). A Svetlana Ribbon and a Svetlana Quasiribbon with a common base are called *conjugate*.

We denote Svetlana Ribbons by Sr. Depending on the types of the axes and the base, we obtain different types of Svetlana Ribbons. For their designation we use the symbols H, E, or P depending on whether the axes of the Svetlana Ribbon belong to a hyperbolic, elliptic, or parabolic pencil, respectively. Furthermore, we use the symbols e, h, or p when the base of the Svetlana Ribbon is an elliptic, hyperbolic, or parabolic segment, respectively. Hence, the types of Svetlana Ribbons in the plane $\widehat{H}$ according to the types of the axes and the base are as follows: $Sr(Hp)$, $Sr(He)$, $Sr(Hh)$, $Sr(Pp)$, $Sr(Pe)$, $Sr(Ph)$, $Sr(Ep)$, $Sr(Ee)$, and $Sr(Eh)$.

In this article we investigate the Svetlana Ribbons with intersecting axes in the plane $\widehat{H}$. We find canonical equations of these curves and we prove that Euclidean Cassini Ovals can be the images of Svetlana Ribbons with intersecting axes in the Euclidean plane E_2. Using Bottema's technique in [3], we derive the equations of the path for any point on the generating line AB during the Cardan motion.

1.3. Objectives

1. *Development of the plane $\widehat{H}$ geometry.*

The research of remarkable curves in the Euclidean plane is one of the classical fields of Euclidean geometry. In the geometry of the plane $\widehat{H}$ some metric properties of curves of degree two are known (see [8, 10, 11, 13, 16, 19, 18, 22]). In this article some families of remarkable curves of degree four in the plane $\widehat{H}$ are investigated.

2. *Uncovering some relations between different geometrical systems.*

With the example of Svetlana Ribbons we show that the same object of the projective plane P_2 can determine different remarkable curves in the planes E_2 and $\widehat{H}$. The Lemniscate of Bernoulli in the plane E_2 is projectively equivalent to the Svetlana Ribbon of type $Sr(Hp)$ in the plane $\widehat{H}$. The families of connected (non-connected) Cassini Ovals of the plane E_2 and of Svetlana Ribbons of type $Sr(He)$ ($Sr(Hh)$) of the plane $\widehat{H}$ are projectively equivalent, too. These facts remind of the idea of F. Klein about the relativity of metric properties of figures (see [7]) and can serve as an important argument in our discussion on the geometry in our real physical space (see [14]).

3. *Visualization of objects of the plane $\widehat{H}$.*

In order to visualize objects of hyperbolic planes, it is necessary to develop methods to construct images of these objects in a Euclidean plane. Some matters concerning the images of figures of the plane $\widehat{H}$ are solved in [1, 15, 21, 22]. Here we construct the image of Svetlana Ribbons in a Euclidean plane. To this end, we consider the Euclidean plane extended by the infinitely far line as a projective plane. The removal of the line at infinity is analytically characterized by the transition from homogeneous projective coordinates to inhomogeneous affine coordinates (see formulae (3.1) and (4.7)). This allows us to use means of the computer visualization of Euclidean objects for displaying objects of hyperbolic planes.

2. Main notions and metric formulae

2.1. Metric formulae of the $\widehat{H}$ geometry

A *canonical frame of the first type* in the plane H^2 (or $\widehat{H}$) is a projective frame $R^* = \{A_1, A_2, A_3; E\}$, whose vertices form an autopolar triangle of first degree with respect to the absolute curve γ, and the unit point E lies on tangents to the curve γ drawn from the vertices A_1 and A_2. The family of canonical frames of the first type in the plane H^2 depends on three parameters [16, § 4.1.1].

In any canonical frame R^* of the first type the absolute curve γ is given by the equation

$$x_1^2 + x_2^2 - x_3^2 = 0. \tag{2.1}$$

If points A and B of an elliptic or hyperbolic line have coordinates (a_p) and (b_p), $p = 1, 2, 3$, then the distance $|AB|$ between them in the frame R^* can be expressed by the formulae

$$\begin{aligned} \cos\frac{|AB|}{\rho} &= \pm\frac{a_1b_1 + a_2b_2 - a_3b_3}{\sqrt{a_1^2 + a_2^2 - a_3^2}\sqrt{b_1^2 + b_2^2 - b_3^2}} \quad \text{or} \\ \cosh\frac{|AB|}{\rho} &= \pm\frac{a_1b_1 + a_2b_2 - a_3b_3}{\sqrt{a_1^2 + a_2^2 - a_3^2}\sqrt{b_1^2 + b_2^2 - b_3^2}}, \end{aligned} \tag{2.2}$$

respectively, where $\rho \in \mathbb{R}_+$ is the *curvature radius* of the plane $\widehat{H}$.

The length of an elliptic (hyperbolic) line equals $\pi\rho$ ($i\pi\rho$) [16, §§ 4.4.1, 4.4.3]. The length of a short (long) elliptic segment is less (more) than $\pi\rho/2$. Two orthogonal points on an elliptic line determine two right segments of length $\pi\rho/2$. The orthogonality condition $A \perp B$ in the frame R^* has the form

$$a_1b_1 + a_2b_2 - a_3b_3 = 0. \tag{2.3}$$

The real coordinates (a_p), $p = 1, 2, 3$, of proper points in $\widehat{H}$ satisfy the inequality

$$a_1^2 + a_2^2 - a_3^2 > 0. \tag{2.4}$$

The coordinates (u_p) of a parabolic line in $\widehat{H}$ satisfy

$$u_1^2 + u_2^2 - u_3^2 = 0. \tag{2.5}$$

2.2. Elliptic cycles in the plane $\widehat{H}$

In [22] conics of the plane $\widehat{H}$ are classified. It is proved, that the basic geometric covariants and the property of a curve to be convex determine 15 types of conics in $\widehat{H}$. The proper conics of four types (hypercycles, horocycles, elliptic cycles, and hyperbolic cycles) are motion trajectories of points in the plane $\widehat{H}$ (see [17, 22]). We called them *cycles*. Cycles of the plane $\widehat{H}$ can be used for constructing partitions of this plane (see [15, 23, 24]).

In this article a new metric property of elliptic and hyperbolic cycles of the plane $\widehat{H}$ is proved: *an elliptic (hyperbolic) cycle of the plane $\widehat{H}$ can be the trajectory of some point during the Cardan motion with a right generating segment where one endpoint coincides with the centre of the given cycle*. In particular, the elliptic cycle of radius $\pi\rho/4$ is the Svetlana Ribbon of type $Sr(He)$ with a right elliptic base segment.

In [17] and [22] elliptic cycles are defined via their position with respect to the absolute of the plane $\widehat{H}$. An *elliptic cycle* is an oval curve that touches the absolute at two real points. Each point of the absolute (elliptic cycle), except the two tangency points, lies in the exterior of the elliptic cycle (absolute, respectively). An elliptic cycle has two connected branches and its interior is a connected domain in $\widehat{H}$.

In [22] an elliptic cycle is defined metrically as follows. Let C be a proper point in $\widehat{H}$. The set α_{e} of points in $\widehat{H}$ such that their elliptic distance to a given point C is a real number r, where $r \in (0; \pi\rho/2)$, is called an *elliptic cycle* with *centre* C and *radius* r. The polar line of the cycle's centre with respect to the absolute is called the *base* of the cycle.

Each elliptic cycle in $\widehat{H}$ has the metric property of being an equidistant curve to a line. The elliptic cycle of radius r is the set of points in $\widehat{H}$ which lie at distance $h = \pi\rho/2 - r$ to its base.

Using the relations (2.2), we can derive the equation of the elliptic cycle α_{e} with centre A_2 and radius r in the frame $R^* = \{A_1, A_2, A_3; E\}$ as

$$x_1^2 - x_2^2 \tan^2 \frac{r}{\rho} - x_3^2 = 0. \tag{2.6}$$

2.3. Cassini Ovals in the Euclidean plane

Let F_1 and F_2 be two fixed points in the plane E_2 and let d be a constant. Then a *Cassini Oval* with *focal points* F_1 and F_2 can be defined as the set (or locus) of points X such that the product of the distances from X to F_1 and F_2 is d^2 (see [6, Section 3.2], [9], or [26, Chapter VI, § 10]).

In the Cartesian coordinate system Oxy with the focal points $F_1(c, 0)$ and $F_2(-c, 0)$ the Cassini Oval has the equation [26, Chapter VI, § 10, (1)]

$$\left(x^2 + y^2\right)^2 - 2c^2 \left(x^2 - y^2\right) + c^4 - d^4 = 0. \tag{2.7}$$

In the case $c < d$ the Cassini Oval described by (2.7) is a simple closed curve. Under $c = d$ the Cassini Oval is a *Lemniscate of Bernoulli*. In the case $c > d$ the Cassini Oval consists of two simple closed curves.

3. A canonical equation of Svetlana Ribbons

Let l and m be orthogonal lines intersecting in the hyperbolic plane $\widehat{H}$ of curvature radius ρ, $\rho \in \mathbb{R}_+$. Then the lines l and m belong to different non-parabolic types. Let l be a hyperbolic line and m be elliptic. Let A and B be proper points of the plane $\widehat{H}$ such that $A \in l$ and $B \in m$. If AB is a non-parabolic line, let also be $|AB| = \delta$. By definition (see [16, § 4.2.2]), the whole segment AB belongs to the plane $\widehat{H}$.

To study the Svetlana Ribbons with the base segment AB and the axes l and m, we introduce a canonical frame $R^* = \{A_1, A_2, A_3; E\}$ of the first type, putting the coordinate lines A_1A_3 and A_1A_2 to the lines l and m, respectively. In the frame R^* the points A, B, and the line AB can be set by the coordinates: $A(1 : 0 : a)$, $B(b : 1 : 0)$, $AB(-a : ab : 1)$, where $a, b \in \mathbb{R}$ and $|a| < 1$ due to condition (2.4) for the point A.

Let $S(x_1 : x_2 : x_3)$ be the midpoint of the segment AB in the frame R^*. The points A and B move along the lines crossing at the point A_1. Therefore the trajectory of the midpoint S (quasimidpoint S_0) of the segment AB does not cross the polar line A_2A_3 of point A_1 with respect to the absolute. Since the line A_2A_3 has the equation $x_1 = 0$, the first coordinate x_1 of the point S is other than zero. Hence, the coordinates of S can be written in the form $(1 : x : y)$, where

$$x = \frac{x_2}{x_1}, \qquad y = \frac{x_3}{x_1}. \tag{3.1}$$

If AB is a non-parabolic segment we denote the coordinates of its quasimidpoint S_0 by $(1 : \overline{x} : \overline{y})$.

By definition of the midpoint and quasimidpoint of a non-parabolic segment, we have $S_0 \perp S$ and the cross ratio $(ABSS_0) = -1$ (see [16, §§ 4.2.2, 4.2.3]). In the case of a parabolic segment AB we denote the coordinates of the absolute point K on the line AB by $(1 : \overline{x} : \overline{y})$. The point K lies on the polar line of S with respect to the absolute. Consequently, $S \perp K$.

The conditions $S \perp S_0$ (or $S \perp K$) and $(ABSS_0) = -1$ in coordinate form (see (2.3)) yield the relations

$$1 + x\overline{x} - y\overline{y} = 0, \qquad x\overline{y} + y\overline{x} = 0. \tag{3.2}$$

Since the points S and S_0 (or S and K) lie on the line AB, we have

$$-a + abx + y = 0, \qquad -a + ab\overline{x} + \overline{y} = 0. \tag{3.3}$$

From (3.2) and (3.3) we obtain

$$\overline{x} = -\frac{x}{x^2 + y^2}, \qquad \overline{y} = \frac{y}{x^2 + y^2}. \tag{3.4}$$

The equalities (3.3) and (3.4) yield

$$a = \frac{2y}{x^2 + y^2 + 1}, \qquad b = -\frac{x^2 + y^2 - 1}{2x}. \tag{3.5}$$

If AB is an elliptic (hyperbolic) segment of length δ we use the notation

$$\cos\frac{\delta}{\rho} = \lambda^{-1} \qquad \left(\cosh\frac{\delta}{\rho} = \lambda^{-1}\right), \tag{3.6}$$

where $|\lambda| > 1$ $(0 < \lambda < 1$, respectively). Then, via the first (second) formula from (2.2), we obtain for the elliptic (hyperbolic) segment AB

$$\lambda^2 b^2 = (1 - a^2)(b^2 + 1), \quad |\lambda| > 1 \quad (0 < \lambda < 1). \tag{3.7}$$

For a short or long elliptic segment AB (see [16, § 4.2.2]) we have $\lambda > 1$ or $\lambda < -1$, respectively. For a right elliptic segment AB we have $\lambda = \infty$.

In the case of a parabolic segment AB the coordinates $(-a : ab : 1)$ of the line AB, by condition (2.5), satisfy

$$a^2 + a^2b^2 - 1 = 0. \tag{3.8}$$

This equation is equivalent to (3.7) with $\lambda = \pm 1$. Consequently, the canonical equation of the Svetlana Ribbon of type $Sr(Hp)$ can be derived as a special type of the canonical equation of the Svetlana Ribbon with intersecting axes.

After eliminating the parameters a and b from the expressions (3.5) and (3.7) (or (3.5) and (3.8)), we obtain

$$\frac{\lambda^2\left(x^2+y^2-1\right)^2}{4x^2} = \left(1-\frac{4y^2}{\left(x^2+y^2+1\right)^2}\right)\left(\frac{\left(x^2+y^2-1\right)^2}{4x^2}+1\right), \tag{3.9}$$

which can be rewritten as

$$\left[(1+\lambda)\left(x^2+y^2\right)^2+2\left(x^2-y^2\right)+1-\lambda\right]\left[(1-\lambda)\left(x^2+y^2\right)^2+2\left(x^2-y^2\right)+1+\lambda\right]=0. \tag{3.10}$$

During the transformation from (3.9) to (3.10) we applied squaring. Therefore (3.10) can have extraneous roots. Let us analyse the obtained result.

The Equation (3.10) determines a reducible algebraic curve σ which splits into two quartic curves σ_1 and σ_2 given by the following equations:

$$\sigma_1\colon\quad (1+\lambda)\left(x^2+y^2\right)^2+2\left(x^2-y^2\right)+1-\lambda=0, \tag{3.11}$$

$$\sigma_2\colon\quad (1-\lambda)\left(x^2+y^2\right)^2+2\left(x^2-y^2\right)+1+\lambda=0. \tag{3.12}$$

Coming back to the coordinates $(x_1 : x_2 : x_3)$ via the formulae (3.1), we obtain

$$\sigma_1\colon\quad (1+\lambda)\left(x_2^2+x_3^2\right)^2+2\left(x_2^2-x_3^2\right)x_1^2+(1-\lambda)x_1^4=0, \tag{3.13}$$

$$\sigma_2\colon\quad (1-\lambda)\left(x_2^2+x_3^2\right)^2+2\left(x_2^2-x_3^2\right)x_1^2+(1+\lambda)x_1^4=0. \tag{3.14}$$

The curves σ_1 (3.13) and σ_2 (3.14) meet the absolute curve γ (2.1) in the common points $E_{13}(1:0:1)$ and $E_{31}(1:0:-1)$. The curves have no other common real points. Each of the curves σ_1 and σ_2 is symmetric with respect to the lines A_1A_2 and A_1A_3. Hence, point A_1 is a symmetry centre of both curves.

The Equations (3.13) and (3.14) differ only in the sign of the parameter λ. Therefore, for the analysis of the results, it is enough to consider one of the options $\lambda > 0$ or $\lambda < 0$. We assume $\lambda > 0$ and analyse below all possibilities.

1. $\lambda > 1$, the segment AB is *short elliptic*.
We choose one of the possible locations of the elliptic segment AB, placing the points A and B on the line A_1A_2. Then $A = A_1(1:0:0)$, and point B has the coordinates $\left(1:\sqrt{\lambda^2-1}:0\right)$. The point $Q_1\left(\sqrt{\lambda+1}:\sqrt{\lambda-1}:0\right)$ $\left(Q_2\left(-\sqrt{\lambda-1}:\sqrt{\lambda+1}:0\right)\right)$ is one of the two real points of intersection of the line A_1A_2 and the curve σ_1 (3.13) (σ_2 (3.14)). The points Q_1 and Q_2 are orthogonal on the line AB (see (2.3)). Since $A \perp A_2$, point A_2 does not belong to the short segment AB. Using the coordinates of the points A, B, Q_1, and A_2, we obtain the cross ratio

$$(ABQ_1A_2) = \frac{\begin{vmatrix} 1 & 0 \\ \sqrt{\lambda+1} & \sqrt{\lambda-1} \end{vmatrix}\begin{vmatrix} 1 & \sqrt{\lambda^2-1} \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & \sqrt{\lambda^2-1} \\ \sqrt{\lambda+1} & \sqrt{\lambda-1} \end{vmatrix}} = -\frac{1}{\lambda} < 0.$$

Hence, point Q_1 belongs to the segment AB. Since

$$(ABQ_1Q_2) = \frac{\begin{vmatrix} 1 & 0 \\ \sqrt{\lambda+1} & \sqrt{\lambda-1} \end{vmatrix}\begin{vmatrix} 1 & \sqrt{\lambda^2-1} \\ -\sqrt{\lambda-1} & \sqrt{\lambda+1} \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ -\sqrt{\lambda-1} & \sqrt{\lambda+1} \end{vmatrix}\begin{vmatrix} 1 & \sqrt{\lambda^2-1} \\ \sqrt{\lambda+1} & \sqrt{\lambda-1} \end{vmatrix}} = -1,$$

the point Q_1 (Q_2) is the midpoint (quasimidpoint) of the short segment AB.
Thus, under the condition $\lambda > 1$, the curve σ_1 (3.13) (σ_2 (3.14)) is the Svetlana Ribbon (Svetlana Quasiribbon) of a short elliptic segment.

2. $0 < \lambda < 1$, the segment AB is *hyperbolic*.
We choose one of the possible locations of this segment, placing points A and B on the line A_1A_3. Then $B = A_1(1:0:0)$, and the point A has the coordinates $\left(1:0:\sqrt{1-\lambda^2}\right)$. The point $N_1\left(\sqrt{1+\lambda}:0:\sqrt{1-\lambda}\right)$ $\left(N_2\left(\sqrt{1-\lambda}:0:\sqrt{1+\lambda}\right)\right)$ is one of the four points of intersection of the line A_1A_3 and the curve σ_1 (3.13) (σ_2 (3.14)). Under the condition $0 < \lambda < 1$ the points N_1 and N_2 are real and orthogonal. Since the point A_3 lies in the Lobachevskiĭ plane, it does not belong to the segment AB of the plane $\widehat{H}$. Using the coordinates of the points A, B, N_1, and A_3, we obtain:

$$(ABN_1A_3) = \frac{\begin{vmatrix} 1 & \sqrt{1-\lambda^2} \\ \sqrt{1+\lambda} & \sqrt{1-\lambda} \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \sqrt{1-\lambda^2} \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ \sqrt{\lambda+1} & \sqrt{\lambda-1} \end{vmatrix}} = -\lambda < 0.$$

Hence, point N_1 belongs to the segment AB. Since

$$(ABN_1N_2) = \frac{\begin{vmatrix} 1 & \sqrt{1-\lambda^2} \\ \sqrt{1+\lambda} & \sqrt{1-\lambda} \end{vmatrix} \begin{vmatrix} 1 & 0 \\ \sqrt{1-\lambda} & \sqrt{1+\lambda} \end{vmatrix}}{\begin{vmatrix} 1 & \sqrt{1-\lambda^2} \\ \sqrt{1-\lambda} & \sqrt{1+\lambda} \end{vmatrix} \begin{vmatrix} 1 & 0 \\ \sqrt{\lambda+1} & \sqrt{\lambda-1} \end{vmatrix}} = -1,$$

the point N_1 (N_2) is the midpoint (quasimidpoint) of the segment AB.
Thus, under the condition $0 < \lambda < 1$, the curve σ_1 (3.13) (σ_2 (3.14)) is a Svetlana Ribbon (Svetlana Quasiribbon) of a hyperbolic segment.

3. $\lambda = 1$, the segment AB is *parabolic*.
In this case the curve σ_2 (3.14) coincides with the absolute curve γ (2.1). A direct verification shows that the curve σ_1 (3.13) is the Svetlana Ribbon of each parabolic segment with its endpoints on the lines l and m.

So, Eq. (3.13) (or (3.14)) describes the Svetlana Ribbons (Svetlana Quasiribbons, respectively) with intersecting axes. We call it the *canonical equation* of these curves.

4. Images of Svetlana Ribbons in the Euclidean plane

In order to obtain images of the curves σ_1 (3.13) and σ_2 (3.14) in an Euclidean plane E_2, we consider the coordinate line A_2A_3 of the frame R^* in the plane P_2 as the infinitely far line t_∞ of this plane. We introduce a Cartesian coordinate system Oxy in E_2 by specifying the origin O at A_1 and the axes Ox and Oy at the lines A_1A_2 and A_1A_3, respectively. The orientations of the coordinate axes are fixed by the location of the unit point E of the frame R^*. We choose those rays on the lines A_1A_2, A_1A_3 as being positive which are edges of the triangle $A_1A_2A_3$ containing the point E. In the frame R^* the system Oxy is determined by the coordinates $O = A_1(1:0:0)$, $t_\infty = A_2A_3(1:0:0)$, $Ox = m = A_1A_2(0:0:1)$, and $Oy = l = A_1A_3(0:1:0)$.

The transformation from the projective coordinates $(x_1 : x_2 : x_3)$ in the frame R^* of P_2 to the Cartesian coordinates $(x; y)$ in the system Oxy of E_2 is carried out by the formulae (3.1). The absolute curve γ (2.1) in the system Oxy has the equation

$$x^2 - y^2 + 1 = 0. \tag{4.1}$$

The Eqs. (3.11) and (3.12) of the curves σ_1 and σ_2 determine in the coordinate system Oxy the Cassini Ovals (see (2.7)). Let us figure out the Euclidean relation between the curves σ_1 (3.13) and σ_2 (3.14).

Assume that an oval curve α is given in the frame R^* (in the system Oxy) by the equation

$$x_1^2 - x_2^2 - x_3^2 = 0 \qquad \left(x^2 + y^2 = 1\right). \tag{4.2}$$

In the plane $\widehat{H}$ the curve α is an elliptic cycle with centre A_2 and radius $\pi\rho/4$ (see Eq. (2.6)). The line A_1A_3 is the base of this elliptic cycle α. In the plane E_2 the curve α is a unit circle with centre O. In the system Oxy the inversion $(x, y) \mapsto (x', y')$ of the plane E_2 with respect to α satisfies

$$x' = \frac{x}{x^2 + y^2}\,, \qquad y' = \frac{y}{x^2 + y^2}\,. \tag{4.3}$$

By applying these formulas to Eq. (3.11), we prove that in E_2 the curves σ_1 (3.11) and σ_2 (3.12) correspond each other in the inversion with respect to the curve α.

The centre of the cycle α in $\widehat{H}$ is the point A_2. Therefore the inversion[1] of $\widehat{H}$ with respect to the curve α and the inversion of E_2 with respect to this curve are different transformations.

In Sections 4.1 – 4.3 we consider all types of Svetlana Ribbons with intersecting axes in accordance with admissible values λ.

4.1. Svetlana Ribbons of type *Sr*(*Hp*)

Assume that the base segment AB of the Svetlana Ribbon ω lies on a parabolic line. Then this Svetlana Ribbon can be described by Eq. (3.11) with $\lambda = 1$. Thus, the canonical equation of a Svetlana Ribbon of type $Sr(Hp)$ has the form

$$\left(x^2 + y^2\right)^2 + x^2 - y^2 = 0. \tag{4.4}$$

In the plane E_2 Eq. (4.4) determines a Lemniscate of Bernoulli with the focal points $F_1\left(0, 1/\sqrt{2}\right)$ and $F_2\left(0, -1/\sqrt{2}\right)$ (see Eq. (2.7)). This fact leads to the following theorem about Svetlana Ribbons of type $Sr(Hp)$.

Theorem 1. *Svetlana Ribbons of type $Sr(Hp)$ are rational algebraic curves of degree four. These curves contain two absolute points and consist of two branches which are crossed in the plane $\widehat{H}$. The Lemniscate of Bernoulli is an example of an image of a Svetlana Ribbon of type $Sr(Hp)$ in an Euclidean plane E_2.*

Of course, the Euclidean metric properties of the curve (4.4) are no longer valid in $\widehat{H}$.

In Figure 1a an Euclidean image of the Svetlana Ribbon (4.4) of the plane $\widehat{H}$ is shown. The change of scale at the transition from Figure 1a to Figure 1b allows us to represent the Svetlana Ribbon of type $Sr(Hp)$ in the plane $\widehat{H}$. In Figure 1a the absolute line t_∞ of the Euclidean plane E_2 is infinitely far, but the absolute curve γ of the plane $\widehat{H}$ is finite. In Figure 1b the line t_∞ as well as the curve γ are infinitely far.

[1] Such transformation in $\widehat{H}$ can be defined similar to an inversion with respect to a hypercycle (see [20]).

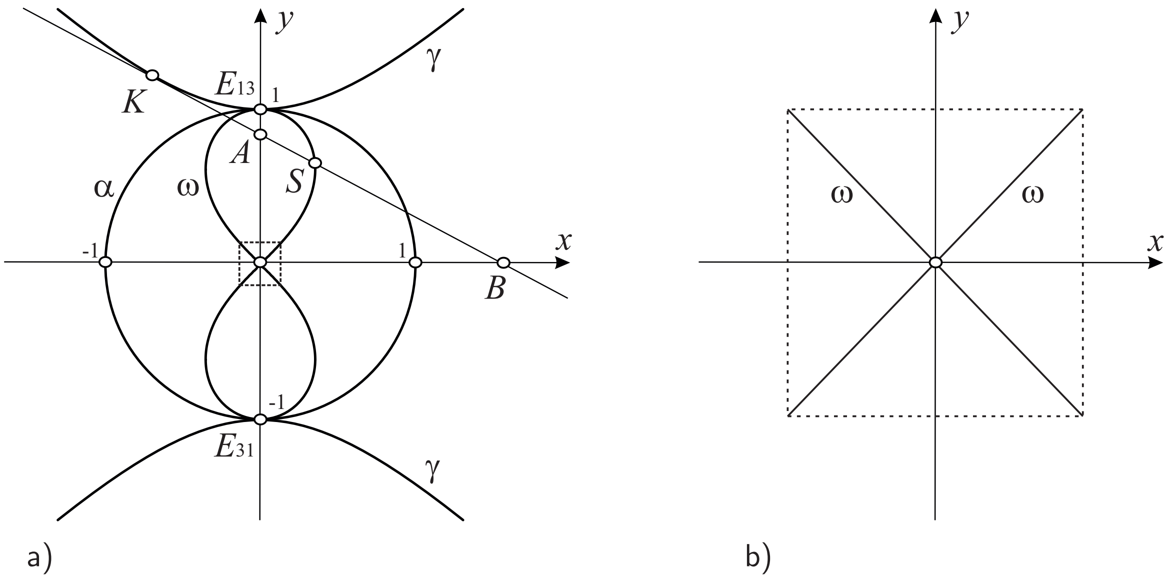


Figure 1: a) The Svetlana Ribbon ω of type $Sr(Hp)$, the absolute curve γ, the midpoint S of the segment AB on the parabolic line with the absolute point K, the common points E_{13} and E_{31} of the curves γ and ω, and the elliptic cycle α of the plane $\widehat{H}$.
b) A fragment of the Svetlana Ribbon ω of type $Sr(Hp)$.

The Equation (3.12) of the curve σ_2 under the condition $\lambda = 1$ determines the absolute γ (4.1) of the plane $\widehat{H}$. The curve γ is described by the absolute point of the parabolic line AB. This absolute point can be formally considered as the quasimidpoint of the segment AB. Therefore the absolute curve can be formally considered as the Svetlana Quasiribbon with a parabolic base and intersecting axes.

4.2. Svetlana Ribbons of type $Sr(He)$

Now we assume that the base segment AB of the Svetlana Ribbon lies on an elliptic line. In this case the parameter λ in (3.6) satisfies the inequality $|\lambda| > 1$. Let the segments between the points A and B be unequal. Then we denote the short and the long segment between these points by ν_1 and ν_2, respectively. Since these lengths are connected by the condition $|\nu_1| + |\nu_2| = \pi\rho$, the values λ_1, λ_2 from (3.6) for these segments satisfy the equality $\lambda_2 = -\lambda_1$, where $\lambda_1 > 1$. Obviously, the midpoint of the segment ν_1 (ν_2) is the quasimidpoint of the segment ν_2 (ν_1). Therefore, according to the reasonings from Section 3, Eq. (3.13) ((3.14)) with $|\lambda| > 1$ describes the Svetlana Ribbons of type $Sr(He)$ with a short (long) elliptic base.

In the Euclidean plane E_2 the Equation (3.11) ((3.12)) determines a connected Cassini Oval with the focal axis Oy (Ox). In $\widehat{H}$ the points E_{13} and E_{31} of the curve σ_1 (3.11) (σ_2 (3.12)) belong to the absolute. Consequently, in the plane $\widehat{H}$ the Svetlana Ribbon of type $Sr(He)$ consists of two connected branches which tend in two directions to infinity.

Below we investigate special cases of the location of the points A and B on an elliptic line.

I. The points A and B are *orthogonal*. Then $|\nu_1| = |\nu_2| = \pi\rho/2$ and $\lambda = \infty$. Since

$$\lim_{\lambda\to\infty} \frac{2}{1+\lambda} = \lim_{\lambda\to\infty} \frac{2}{1-\lambda} = 0, \qquad \lim_{\lambda\to\infty} \frac{1-\lambda}{1+\lambda} = \lim_{\lambda\to\infty} \frac{1+\lambda}{1-\lambda} = -1,$$

at the limit $\lambda \to \infty$ the Equations (3.11) and (3.12) take the form

$$\left(x^2 + y^2\right)^2 - 1 = 0. \tag{4.5}$$

The curve (4.5) consists of the elliptic cycle α (4.2) and a zero curve β with the equation $x^2 + y^2 + 1 = 0$ in the system Oxy. The curve β has not real points. Consequently, the Svetlana Ribbon of type $Sr(He)$ with the right elliptic base AB is the elliptic cycle with base l and radius $\pi\rho/4$.

Notice that under the condition $A \perp B$ the endpoint B of the segment AB is the fixed point A_2. Hence, the Cardan motion degenerates into a rotation about the point A_2. The endpoint A of the segment AB can be any point of the line l in $\widehat{H}$.

Under the condition $\lambda \to \infty$ the Svetlana Ribbons σ_1 (3.11) and σ_2 (3.12) advances the cycle α (4.2) internally and externally, respectively.

II. The segment ν_1 with endpoints A and B is equal to *one-third of an elliptic line*[2]; then $|\nu_1| = \pi\rho/3$ and $\lambda = 2$. The midpoint (quasimidpoint) of the segment ν_1 describes the Svetlana Ribbon given by the equation

$$3\left(x^2 + y^2\right)^2 + 2\left(x^2 - y^2\right) - 1 = 0 \qquad \left(\left(x^2 + y^2\right)^2 - 2\left(x^2 - y^2\right) - 3 = 0\right). \tag{4.6}$$

The curves with the Equations (4.6) are boundaries for families of Svetlana Ribbons of type $Sr(He)$ with a common geometric property. To prove this supposition, we use well-known properties of Cassini Ovals in the plane E_2 (see [26]). Let us consider all possibilities depending on values λ, where $|\lambda| > 1$.

1. If $1 < \lambda < 2$, then for the base length δ of each Svetlana Ribbon σ_1 (3.11), by the condition (3.6), we have $\delta < \pi\rho/3$. In this case each Svetlana Ribbon σ_1 of type $Sr(He)$ lies between the first curve from (4.6) and the Svetlana Ribbon ω of type $Sr(Hp)$. Moreover, for each Svetlana Ribbon σ_1 there is a line which is orthogonal to the axis Ox and has four common real points with σ_1. Each line, orthogonal to the line Oy, crosses the Svetlana Ribbon σ_1 in at most two real points.
2. If $\lambda > 2$, then $\pi\rho/3 < \delta < \pi\rho/2$. In this case each Svetlana Ribbon σ_1 (3.11) of type $Sr(He)$ lies between the curve α (4.2) and the first curve from (4.6). Therewith each line has not more than two common real points with each Svetlana Ribbon σ_1.
3. If $-2 < \lambda < -1$, then $\delta > 2\pi\rho/3$. In this case each Svetlana Ribbon σ_1 (3.11) of type $Sr(He)$ lies in the interior of the second curve with the Equation (4.6). Moreover, for each Svetlana Ribbon σ_1 (3.11) there is a line which is orthogonal to the axis Oy and has four common real points with σ_1. Each line orthogonal to Ox crosses the Svetlana Ribbon σ_1 in at most two real points.
4. If $\lambda < -2$, then $\pi\rho/2 < \delta < 2\pi\rho/3$. In this case each Svetlana Ribbon σ_1 (3.11) of type $Sr(He)$ lies between the second curve (4.6) and α (4.2). Therewith each line has at most two common real points with the curve σ_1.

Thus, the following theorem about Svetlana Ribbons of type $Sr(He)$ is proved.

[2] Such segment has a special meaning in the geometry of the plane $\widehat{H}$. For example, the elliptic segment of length $\pi\rho/3$ subtends a horocycle chord of length ρ [17, § 2.4.3], [18]. Moreover, such a segment determines a special case of fan triangulations of the plane $\widehat{H}$ [15]. A way of constructing one third of an elliptic line is shown in [15, Theorems 8, 10].

Theorem 2. *The Svetlana Ribbons of type $Sr(He)$ are algebraic curves of degree four. These curves contain two absolute points and consist of two non-intersecting branches. Depending on the base length δ, the Svetlana Ribbons of type $Sr(He)$ with the hyperbolic (elliptic) axis l (m) have the following properties.*

- *In the case $\delta < \pi\rho/2$ ($\delta > \pi\rho/2$) connected Cassini Ovals with the focal axis l (m) show up as Euclidean images of such Svetlana Ribbons.*
- *In the case $\delta = \pi\rho/2$ the Svetlana Ribbon is reducible; it consists of two curves of degree two. One of them has no real points, the other curve is an elliptic cycle with the base l and radius $\pi\rho/4$.*
- *In the case $\delta < \pi\rho/3$ there are lines orthogonal to the line m and crossing the Svetlana Ribbon in four real points. Each line orthogonal to l crosses the Svetlana Ribbon in at most two real points.*
- *In the case $\pi\rho/3 \le \delta \le 2\pi\rho/3$ no line crosses the Svetlana Ribbon in more than two real points.*
- *In the case $\delta > 2\pi\rho/3$ there are lines orthogonal to l and crossing the Svetlana Ribbon in four real points. Each line orthogonal to m crosses the Svetlana Ribbon in at most two real points.*

In order to display the studied objects, we represent them in the plane E_2. In Figure 2, the Svetlana Ribbon σ_1 (3.11) and the Svetlana Quasiribbon σ_2 (3.12) of type $Sr(He)$ with $\lambda = 7$ are presented. The line at infinity for Figure 2a coincides with the line A_2A_3, for Figure 2b with the line A_1A_2. In the first case the Equations (3.11) and (3.12) are transformed by (3.1), in the second case by the formulae

$$x = \frac{x_2}{x_3}, \qquad y = \frac{x_1}{x_3}. \tag{4.7}$$

After this transformation the origin O of the system Oxy coincides with the point A_3, and

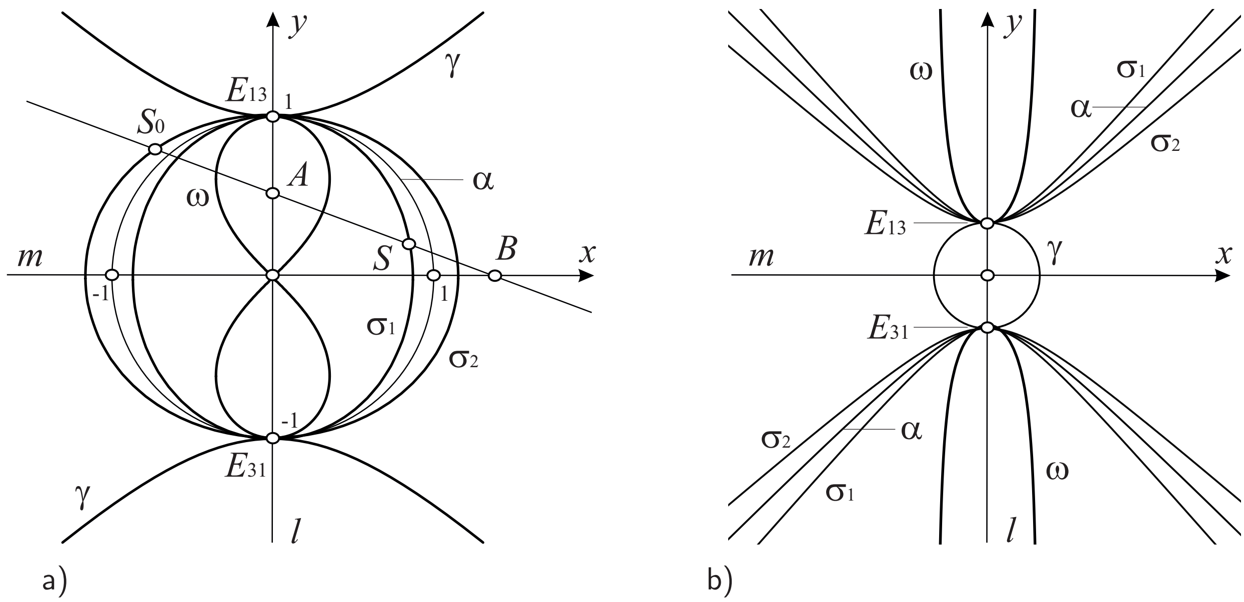
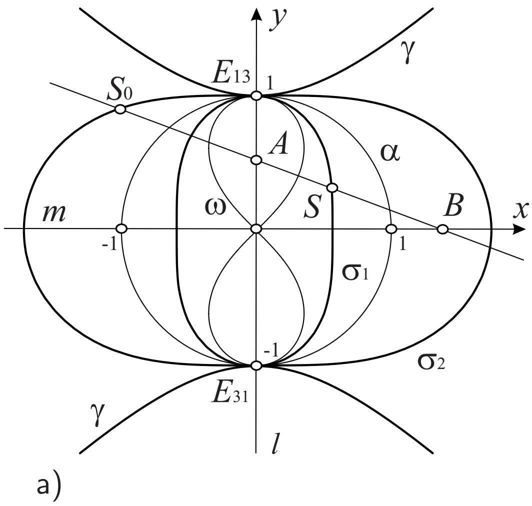


Figure 2: The Svetlana Ribbon σ_1 (3.11) (or (4.8)) and the Svetlana Quasiribbon σ_2 (3.12) (or (4.9)) of type $Sr(He)$ with $\lambda = 7$, the Svetlana Ribbon ω (4.4) of type $Sr(Hp)$, the absolute oval curve γ (2.1), and the elliptic cycle α (4.2) in the plane E_2 with the infinitely far line in a) as A_2A_3, in b) as A_1A_2.



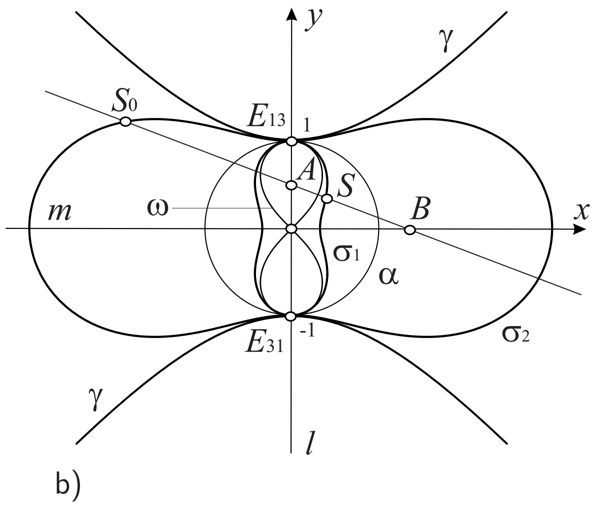


Figure 3: The Svetlana Ribbon σ_1 (3.11) and the Svetlana Quasiribbon σ_2 (3.12) of type $Sr(He)$, in a) with $\lambda = 2$, in b) with $\lambda = 5/4$, the Svetlana Ribbon ω (4.4) of type $Sr(Hp)$, the absolute oval curve γ (2.1), and the elliptic cycle α (4.2) in the plane E_2 with the infinitely far line A_2A_3.

the coordinate axis Ox (Oy) is placed on the line A_2A_3 (A_1A_3). By virtue of (4.7), the curves σ_1 (3.13) and σ_2 (3.14) satisfy in (x, y)-coordinates the equations

$$\sigma_1: \quad (1+\lambda)\left(x^2+1\right)^2 + 2\left(x^2-1\right)y^2 + (1-\lambda)y^4 = 0, \tag{4.8}$$

$$\sigma_2: \quad (1-\lambda)\left(x^2+1\right)^2 + 2\left(x^2-1\right)y^2 + (1+\lambda)y^4 = 0. \tag{4.9}$$

The Svetlana Ribbon σ_1 (3.11) and the Svetlana Quasiribbon σ_2 (3.12) of type $Sr(He)$ are shown in Figures 3a and 3b with $\lambda = 2$ and $\lambda = 5/4$, respectively. The line A_2A_3 is assumed as line at infinity.

4.3. Svetlana Ribbons of type $Sr(Hh)$

Assume that the base segment AB of any Svetlana Ribbon lies on a hyperbolic line. Then $0 < \lambda < 1$. Since the segment AB belongs to the plane $\widehat{H}$, its midpoint S (quasimidpoint S_0) lies in the plane $\widehat{H}$ (Lobachevskiĭ plane Λ^2). Therefore also the entire Svetlana Ribbon (Svetlana Quasiribbon) with the base AB belongs to $\widehat{H}$ (Λ^2).

After the transformation by (3.1), Eq. (3.11) determines in the system Oxy a curve which does not have common real points with the axis Ox. This curve is symmetric with respect to both coordinate axes and crosses the axis Oy in four real points. Therefore, the image of the Svetlana Ribbon σ_1 (3.13) of type $Sr(Hh)$ in the plane E_2 is a curve consisting of two connected Cassini Ovals. Thus, the following theorem on Svetlana Ribbons of type $Sr(Hh)$ is proved.

Theorem 3. *The Svetlana Ribbons of type Sr(Hh) are algebraic curves of degree four. These curves contain two absolute points and consist of two non-intersecting branches. Non-connected Cassini Ovals with the focal axis l can be Euclidean images of Svetlana Ribbons of type Sr(Hh) with the hyperbolic axis l.*

In Figure 4 the Svetlana Ribbons σ_1, $\overline{\sigma}_1$ and the Svetlana Quasiribbons σ_2, $\overline{\sigma}_2$ of type $Sr(Hh)$ are displayed. In Figure 4a the line A_2A_3 has been chosen as line at infinity, in

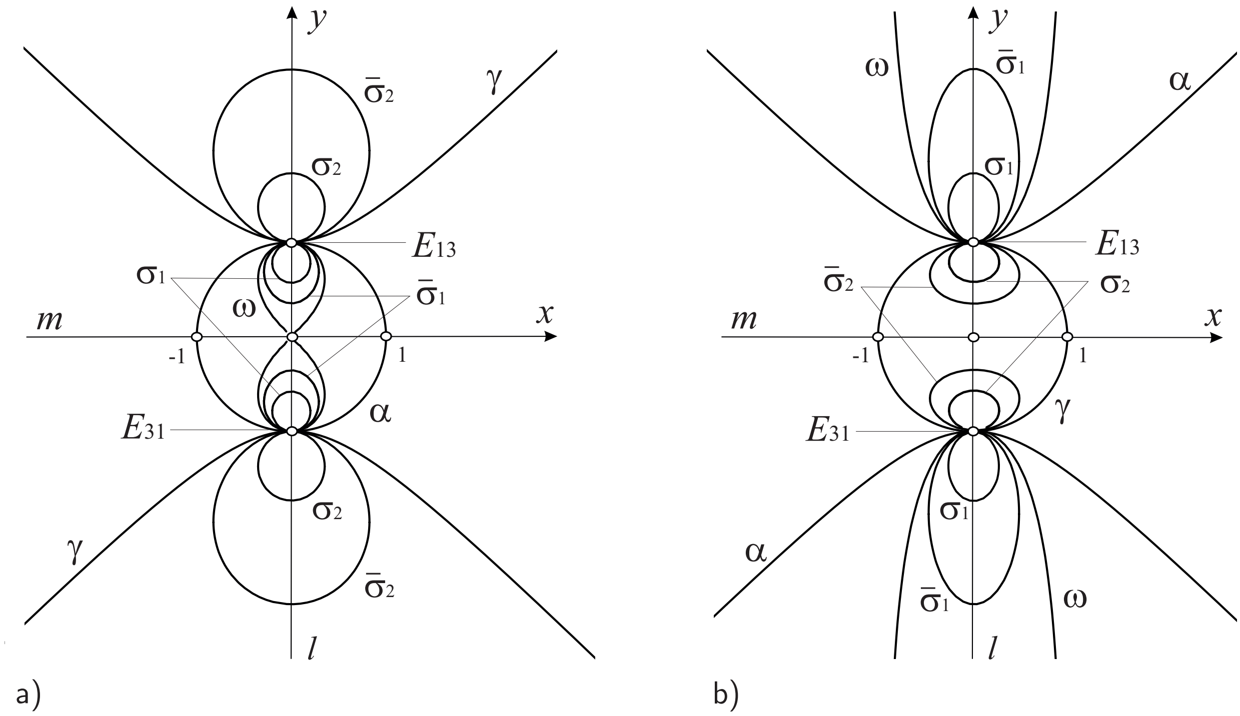


Figure 4: The Svetlana Ribbons σ_1 with $\lambda = 1/2$, $\overline{\sigma}_1$ with $\lambda = 7/9$ and the Svetlana Quasiribbons σ_2 with $\lambda = 1/2$, $\overline{\sigma}_2$ with $\lambda = 7/9$ of type $Sr(Hh)$, the Svetlana Ribbon ω (4.4) of type $Sr(Hp)$, the absolute oval curve γ (2.1), the elliptic cycle α (4.2) in the plane E_2 with the infinitely far line in a) as A_2A_3, in b) as A_1A_2.

Figure 4b the line A_1A_3. The curves σ_1 ($\overline{\sigma}_1$) and σ_2 ($\overline{\sigma}_2$) shown in Figure 4a are given by (3.11) and (3.12), respectively, with $\lambda = 1/2$ ($\lambda = 7/9$). The curves of Figure 4b are given by (4.8) and (4.9), respectively, again with $\lambda = 1/2$ ($\lambda = 7/9$).

5. Paths of an arbitrary point on the moving line during the Cardan motion

The approach of O. Bottema in [3] allows us to obtain also in the planes $\widehat{H}$ and Λ^2 the equations of the paths of an arbitrary point on the generating line during a Cardan motion. We present here briefly the main phases of the derivation of such equations for the Cardan motion with a non-parabolic generating line and orthogonal axes intersecting in the plane $\widehat{H}$. We still assume that in $\widehat{H}$ the elliptic line m and the hyperbolic line l are orthogonal. The elliptic or hyperbolic segment AB of constant length δ moves in the plane $\widehat{H}$ such that $A \in l$ and $B \in m$.

If the points A and B are orthogonal on the hyperbolic line then point A lies in the Lobachevskiĭ plane. This does not correspond to the conditions of our task. Therefore we consider the case of orthogonality of A and B only for an elliptic line AB. In this case the point B is fixed at the pole of the line l with respect to the absolute. The Cardan motion degenerates into the rotation about the point B. The path of an arbitrary point X of the plane $\widehat{H}$ under the rotation about B can be as follows (see [24]): an elliptic or hyperbolic cycle with centre B when the line BX is elliptic or hyperbolic, respectively, or a parabolic line BX.

From now on we assume that the points A and B are not orthogonal. In the frame R^*, as defined in Section 3, we set the points A and B by the coordinates $(1:0:a)$ and $(1:b:0)$, respectively. Via the second (first) formula of (2.2) we obtain the following relations between the parameters a and b for the hyperbolic (elliptic) line AB:

$$\left(1-a^2\right)\left(1+b^2\right)\cosh^2\frac{\delta}{\rho}=1 \qquad \left(\left(1-a^2\right)\left(1+b^2\right)\cos^2\frac{\delta}{\rho}=1\right).$$

As a consequence, we introduce an auxiliary parameter p such that

$$p=\left(1-a^2\right)\cosh\frac{\delta}{\rho},\quad p^{-1}=\left(1+b^2\right)\cosh\frac{\delta}{\rho} \quad \left(p=\left(1-a^2\right)\cos\frac{\delta}{\rho},\quad p^{-1}=\left(1+b^2\right)\cos\frac{\delta}{\rho}\right).$$

These expressions yield

$$a^2=\frac{\cosh\frac{\delta}{\rho}-p}{\cosh\frac{\delta}{\rho}},\quad b^2=\frac{1-p\cosh\frac{\delta}{\rho}}{p\cosh\frac{\delta}{\rho}} \qquad \left(a^2=\frac{\cos\frac{\delta}{\rho}-p}{\cos\frac{\delta}{\rho}},\quad b^2=\frac{1-p\cos\frac{\delta}{\rho}}{p\cos\frac{\delta}{\rho}}\right). \tag{5.1}$$

The coordinates $(1+t:bt:a)$ with $t\in\mathbb{R}$ define points $M(t)$ on the line AB with $A=M(0)$ and $B=M(\infty)$. Let us choose a new parametrization of the line AB such that

$$t=t(\mu)=p\,\frac{\tanh\frac{\delta}{2\rho}+\mu}{\tanh\frac{\delta}{2\rho}-\mu} \qquad \left(t=t(\mu)=p\,\frac{\tan\frac{\delta}{2\rho}+\mu}{\tan\frac{\delta}{2\rho}-\mu}\right). \tag{5.2}$$

In this parametrization the coordinates (x_1,x_2,x_3) of the current point M on the hyperbolic (elliptic) line AB satisfy the following equalities:

$$\begin{aligned}
x_1^2 &= \left[\mu-\tanh\frac{\delta}{2\rho}-p\left(\mu+\tanh\frac{\delta}{2\rho}\right)\right]^2\cosh\frac{\delta}{\rho}\\
\left(x_1^2\right. &= \left.\left[\mu-\tan\frac{\delta}{2\rho}-p\left(\mu+\tan\frac{\delta}{2\rho}\right)\right]^2\cos\frac{\delta}{\rho}\right),\\
x_2^2 &= p\left(1-p\cosh\frac{\delta}{\rho}\right)\left(\mu+\tanh\frac{\delta}{2\rho}\right)^2 \qquad \left(x_2^2 = p\left(1-p\cos\frac{\delta}{\rho}\right)\left(\mu+\tan\frac{\delta}{2\rho}\right)^2\right),\\
x_3^2 &= \left(\cosh\frac{\delta}{\rho}-p\right)\left(\mu-\tanh\frac{\delta}{2\rho}\right)^2 \qquad \left(x_3^2 = \left(\cos\frac{\delta}{\rho}-p\right)\left(\mu-\tan\frac{\delta}{2\rho}\right)^2\right).
\end{aligned} \tag{5.3}$$

After calculating the expression $x_1^1+x_2^2-x_3^2$ from (5.3), we obtain

$$p=\frac{x_1^2+x_2^2-x_3^2}{4\left(1-\mu^2\right)\sinh^2\frac{\delta}{2\rho}} \qquad \left(p=\frac{x_1^2+x_2^2-x_3^2}{4\left(1+\mu^2\right)\sin^2\frac{\delta}{2\rho}}\right). \tag{5.4}$$

According to (5.3) we have

$$\begin{aligned}
\Theta\equiv x_2^2\left(\mu-\tanh\frac{\delta}{2\rho}\right)^2-x_3^2\left(\mu+\tanh\frac{\delta}{2\rho}\right)^2 &= \left[2p-(p^2+1)\cosh\frac{\delta}{\rho}\right]\left(\mu^2-\tanh^2\frac{\delta}{2\rho}\right)^2\\
\left(\Theta\equiv x_2^2\left(\mu-\tan\frac{\delta}{2\rho}\right)^2-x_3^2\left(\mu+\tan\frac{\delta}{2\rho}\right)^2\right. &= \left.\left[2p-(p^2+1)\cos\frac{\delta}{\rho}\right]\left(\mu^2-\tan^2\frac{\delta}{2\rho}\right)^2\right),\\
\Omega\equiv x_2^2x_3^2=p\left[(1+p^2)\cosh\frac{\delta}{\rho}\right. &- \left.p\left(1+\cosh^2\frac{\delta}{\rho}\right)\right]\left(\mu^2-\tanh^2\frac{\delta}{2\rho}\right)^2\\
\left(\Omega\equiv x_2^2x_3^2=p\left[(1+p^2)\cos\frac{\delta}{\rho}\right.\right. &- \left.\left.p\left(1+\cos^2\frac{\delta}{\rho}\right)\right]\left(\mu^2-\tan^2\frac{\delta}{2\rho}\right)^2\right).
\end{aligned} \tag{5.5}$$

Consequently,

$$p\,\Theta+\Omega=-p^2\left(\mu^2-\tanh^2\frac{\delta}{2\rho}\right)^2\sinh^2\frac{\delta}{\rho} \quad \left(p\,\Theta+\Omega=p^2\left(\mu^2-\tan^2\frac{\delta}{2\rho}\right)^2\sin^2\frac{\delta}{\rho}\right). \tag{5.6}$$

Using the expressions from (5.5), we can eliminate the parameter p from (5.4) and (5.6). As a result we obtain below the equation of the path for an arbitrary point on the hyperbolic and elliptic generating line during a Cardan motion:

$$\begin{aligned}(x_1^2+x_2^2-x_3^2)^2\Big(\mu^2-\tanh^2\frac{\delta}{2\rho}\Big)^2\cosh^2\frac{\delta}{2\rho}+(x_1^2+x_2^2-x_3^2)(1-\mu^2)\\ \times\Big[x_2^2\Big(\mu-\tanh\frac{\delta}{2\rho}\Big)^2-x_3^2\Big(\mu+\tanh\frac{\delta}{2\rho}\Big)^2\Big]+4x_2^2x_3^2(1-\mu^2)^2\sinh^2\frac{\delta}{2\rho}=0,\end{aligned}\tag{5.7}$$

$$\begin{aligned}(x_1^2+x_2^2-x_3^2)^2\Big(\mu^2-\tan^2\frac{\delta}{2\rho}\Big)^2\cos^2\frac{\delta}{2\rho}-(x_1^2+x_2^2-x_3^2)(1+\mu^2)\\ \times\Big[x_2^2\Big(\mu-\tan\frac{\delta}{2\rho}\Big)^2-x_3^2\Big(\mu+\tan\frac{\delta}{2\rho}\Big)^2\Big]-4x_2^2x_3^2(1+\mu^2)^2\sin^2\frac{\delta}{2\rho}=0.\end{aligned}\tag{5.8}$$

In the case $\mu=0$ the Equations (5.7) and (5.8) are equivalent to Eq. (3.13) with the notations from (3.6) and they describe the Svetlana Ribbons. Hence, a vanishing parameter μ corresponds to the midpoint S of the segment AB. Via (5.2) we find the geometrical meaning of the parameter μ for the current point M on the curve (5.7) ((5.8)):

$$\mu=\frac{1-(ABSM)}{1+(ABSM)}\tanh\frac{\delta}{2\rho}=\tanh\frac{|SM|}{\rho}\qquad\Big(\mu=\frac{1-(ABSM)}{1+(ABSM)}\tan\frac{\delta}{2\rho}=\tan\frac{|SM|}{\rho}\Big).$$

The quartic curves (5.7) and (5.8) have three centers A_1, A_2, and A_3. They meet the absolute at the points E_{13} and E_{31} and have no other real points on the absolute.

Thus, the trajectory of an arbitrary point of a non-parabolic generating line during the Cardan motion with intersecting axes in the plane $\widehat{H}$ is a quartic curve with three centers and two times tangent to the absolute at the points on the hyperbolic axis of the motion.

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References

[1] L. Bessonov, L. Romakina: *The image of simple carpets on simple 4-contour of the hyperbolic plane of positive curvature* [in Russian]. Mech. Mat. **16**, 6–9 (2014).

[2] O. Bottema: *On Cardan positions for the plane motion of a rigid body.* Proceedings Koninklijke Nederlandse Akademie van Wetenschappen **52**, 643–651 (1949).

[3] O. Bottema: *Cardan motion in elliptic geometry.* Can. J. Math. **XXVII** (1), 37–43 (1975).

[4] H. Busemann, P. Kelly: *Projective Geometry and Projective Metrics.* Academic Press Inc., New York. 1953.

[5] N. Efimov: *Higher Geometry* [in Russian]. MAIK, Nauka/Interperiodika, FIZMATLIT, 2004.

[6] G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Conics. From the ancient Greeks to 21st century developments.* Springer Spektrum, Springer-Verlag Berlin Heidelberg 2016.

[7] F. Klein: *Vergleichende Betrachtungen über neuere geometrische Forschungen (Programm zum Eintritt in die philosophische Facultät und den Senat der Universität zu Erlangen).* A. Deichert, Erlangen 1872.

[8] F. Klein: *Vorlesungen über Nicht-Euclidische Geometrie*. Verlag von Julius Springer, 1928.

[9] D. Lawden: *Families of ovals and their orthogonal trajectories*. Math. Gaz. **83(498)**, 410–420 (1999).

[10] S. Pevzner: *A detailed classification of second-order irreducible curves in the Lobatchevsky plane by means of the focal-director invariants* [in Russian]. Izv. Vyssh. Uchebn. Zaved. Mat. **6**, 85–90 (1962).

[11] S. Pevzner: *Properties of second-order curves in the Lobatchevsky plane which are dual under the focal directrix* [in Russian]. Izv. Vyssh. Uchebn. Zaved. Mat. **5**, 39–50 (1961).

[12] A. Prokhorov (ed.): *Physical encyclopedia, Vol. 1* [in Russian]. Soviet encyclopedia, Moscow 1988.

[13] L. Romakina: *Cycles on the Hyperbolic Plane of Positive Curvature*. J. Mat. Sciences **212** (5), 605–621 (2016). Translated from Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **415**, 137–162 (2013).

[14] L. Romakina: *Development of ideas about geometry of the space surrounding us* [in Russian]. In: Effective researches of modernity, Scientific articles collection of the 10th Internat. Scientific Conference of Eurasian Scientific Association, Moscow, Oct. 2015, **10**, 18–21 (2015).

[15] L. Romakina: *Fan triangulations of a hyperbolic plane of positive curvature*. Siberian Adv. Math. **24** (3), 204–221 (2014). Translated from Mat. Tr. **16** (2), 142–168 (2013).

[16] L. Romakina: *Geometry of the hyperbolic plane of positive curvature, P. 1: Trigonometry* [in Russian]. Publishing House of the Saratov University, 2013.

[17] L. Romakina: *Geometry of the hyperbolic plane of positive curvature, P. 2: Transformations and simple partitions* [in Russian]. Publishing House of the Saratov University, 2013.

[18] L. Romakina: *Horocycle Arch Length on Hyperbolic Plane of Positive Curvature* [in Russian]. Mech. Mat. **16**, 58–61 (2014).

[19] L. Romakina: *Hyperbolic ortrises of the hyperbolic plane of positive curvature* [in Russian]. Mech. Mat. **14**, 62–66 (2012).

[20] L. Romakina: *Inversion with respect to a hypercycle of a hyperbolic plane of positive curvature*. J. Geom. **107** (1), 137–149 (2016).

[21] L. Romakina: *On the area of a trihedral on a hyperbolic plane of positive curvature*. Siberian Adv. Math. **25** (2), 138–153 (2015), Translated from Mat. Tr. **17** (2), 184–206 (2014).

[22] L. Romakina: *Oval lines of the hyperbolic plane of positive curvature* [in Russian]. Izv. Saratov Univ. (N.S.), Ser. Math. Mech. Inform. **12** (3), 37–44 (2012).

[23] L. Romakina: *Partitions of a hyperbolic plane of positive curvature by correct horocyclic n-trapezes* [in Russian]. Chebyshevskiĭ Sb. **16** (3), 376–416 (2015).

[24] L. Romakina: *Simple partitions of a hyperbolic plane of positive curvature*. Sb. Math. **203** (9), 1310–1341 (2012), Translated from Mat. Sb. **203** (9), 83–116 (2012).

[25] B. Rosenfeld: *Non-Euclidean spaces* [in Russian]. Nauka, Moscow 1969.

[26] A. Savelov: *Flat curves: Systematization, properties, application (reference guide)* [in Russian]. State publishing house of physical and mathematical literature, 1960.

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