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Projective Proof and Generalization of Chasles's Theorem Along Non-Torsal Lines

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Abstract. We prove Chasles's theorem for non-developable ruled surfaces, originally published in 1839, and generalize it to higher-dimensional projective extensions of the real space for the first polars along non-torsal lines. We avoid the use of differential geometry and re-prove the theorem strictly projectively, using only incidence properties for surfaces of higher degree.

Key Words: Chasles's theorem, ruled surface, projective methods, polar surface

MSC: 51A05, 51N15, 51N35, 14J26

1. Introduction

In 1839, the French mathematician Michel Floréal Chasles formulated in [2, p. 53] the following theorem about non-developable ruled surfaces:

Theorem 1 (Chasles)**.** *Four tangent planes along a non-torsal line of a ruled surface have the same cross-ratio as their contact points.*

The theorem has become of great use in descriptive geometry (see [5, p. 648–650]). It provides a simple construction of the tangent plane in a given point or of the tangent surface along a given ruling on a non-developable ruled surface (Figure 1).

Apart from its projective nature, the theorem is usually studied in differential geometry. Chasles used infinitesimal calculus for the definition of a non-developable ruled surface ("surface gauche") in his proof, and therefore he crossed the border to projective incidence geometry. Interested readers can find more about the errors in Chasles's original proof of his theorem in [9]. A proof based on the differential approach can be also found in the recent article [6, p. 509, 510] by Önder and Uğurlu and with more insight in projective differential geometry in [4, p. 48] by Hlavatý, and in [7, p. 228–229].

In this paper, we generalize the theorem to algebraic surfaces, and we re-prove it strictly projectively. The algebraic language enables a straightforward generalization to higher dimensions. We use the standard axiomatic system of the projective extension of the real space with elements at infinity. A complete list of axioms of $\mathbb{RP}^3$ is given in [8, pp. 16, 18, 24] and in [3, p. 15]. An extension to higher-dimensional spaces is described also in [8, pp. 2, 3, 29–33].

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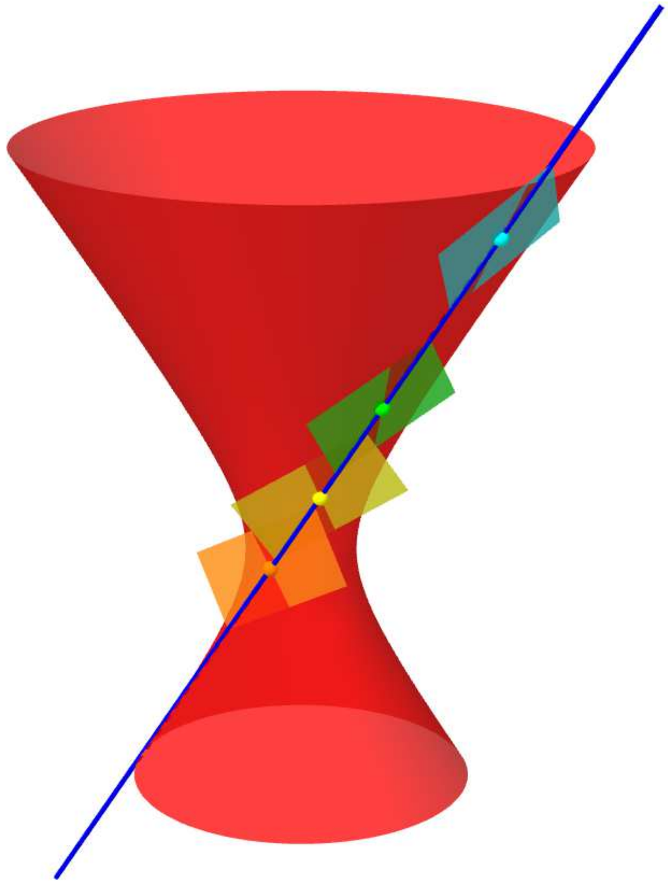

Figure 1: Tangent planes through a ruling on a hyperboloid

2. Chasles's theorem along non-torsal lines

We will give an algebraic proof in $\mathbb{RP}^m$ for $m \geq 3$. Consecutive steps of the proof are visualized on the Clebsch surface in $\mathbb{RP}^3$, which is an algebraic surface of 3rd degree (Figure 2). We first define necessary terms and show that all polars of a pole on a surface with respect to this surface are tangent surfaces at this point. Then we prove the existence of a projectivity between the pencil of first polars and the range of its poles on a non-torsal line of a surface. At the end, we will also give a projective proof of Chasles's theorem for non-developable ruled surfaces.

A range of points or a pencil of lines, hyperplanes or conics etc. are *one-dimensional forms*. Each element x of such a form is a linear combination of two distinct elements p and q of this form and can be uniquely represented by a pair of real numbers $(\rho k_1,\ \rho k_2) \neq (0,0)$ for arbitrary $\rho \neq 0$, and vice-versa. The pair $(\rho k_1,\ \rho k_2)$ is called *projective coordinates* of the element x in the one-dimensional form. Furtheron, an element x will be denoted by $x(\rho k_1,\ \rho k_2)$.

A triple of elements $0(0,1)$, $e(1,1)$ and $\infty(1,0)$ of a one-dimensional form is called its *projective coordinate system* S. A *projectivity* between two one-dimensional forms with projective coordinate systems S and S', respectively, is given by a linear map between the projective coordinates $x(k_1,k_2)$ and $x'(k'_1,k'_2)$ of corresponding elements, i.e.,

$$\begin{pmatrix} k'_1 \\ k'_2 \end{pmatrix} = \xi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix},$$

where $\xi \neq 0$ is a function of k_1, k_2 and $\alpha\delta - \beta\gamma \neq 0$ for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Let $\mathcal{S}$ be a surface of degree n described by the algebraic form $\boldsymbol{F}_n(X)$ of degree n induced by the multilinear form $\boldsymbol{F}(\underbrace{X, X, \ldots, X}_{n \text{ times}})$. A point P lies on $\mathcal{S}$ if $\boldsymbol{F}(P) = 0$. This point is said to be *singular* if

$$\frac{\partial \boldsymbol{F}_n(P)}{\partial p_i} = 0 \text{ for all } i = 0, \ldots, m\,.$$

Otherwise, we call P a *regular* point of the surface $\mathcal{S}$. The polar hypersurface of a regular point $P(p_1, p_2, \ldots, p_m, p_0)$

$$\pi_1(\mathcal{S}, P)\colon\ \boldsymbol{F}(P, \underbrace{X, \ldots, X}_{(n-1) \text{ times}}) = \sum_{i_1 = 0, \ldots, m} p_{i_1} \frac{\partial \boldsymbol{F}_n(X)}{\partial x_{i_1}} = 0$$

is said to be its *first polar*. In the formula, we use only the formal derivation of a polynomial.

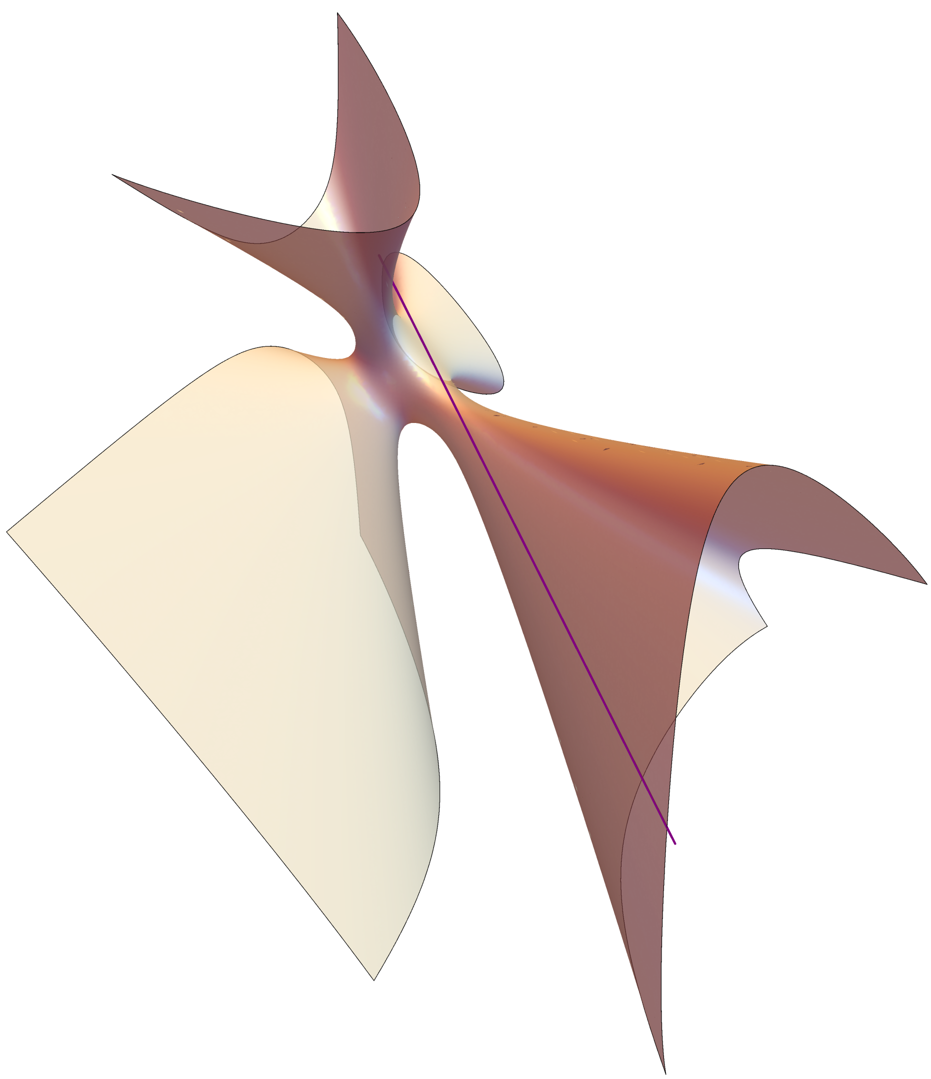

Figure 2: Clebsch surface and its non-torsal line

Applying the same process on π_1 will give us the *2nd polar* of P with respect to $\mathcal{S}$

$$\pi_2(\mathcal{S},P)\colon\ \boldsymbol{F}(P,P,\underbrace{X,\dots,X}_{(n-2)\ \text{times}}) = \sum_{i_2} p_{i_2}\,\frac{\sum_{i_1} p_{i_1}\frac{\partial \boldsymbol{F}_n(X)}{\partial x_{i_1}}}{\partial x_{i_2}} = 0\ \text{ for all }\ i_1,i_2 = 0,\dots,m.$$

The *r-th polar* of P with respect to $\mathcal{S}$ can be, after some arithmetic modifications, written as

$$\pi_r(\mathcal{S},P)\ :\ \boldsymbol{F}(\underbrace{P,\dots,P}_{r\ \text{times}}\underbrace{X,\dots,X}_{(n-r)\ \text{times}}) = \sum \frac{r!}{r_1!r_2!\dots r_m!r_0!}\,p_1^{r_1}\dots p_m^{r_m}p_0^{r_0}\,\frac{\partial^r \boldsymbol{F}_n(X)}{\partial x_1^{r_1}\dots\partial x_m^{r_m}\partial x_0^{r_0}} = 0,$$

where we sum over all $(m+1)$-tuples $(r_1,\dots,r_m,r_0)$ such that $r_j \geq 0$ and $\sum_{j=0}^{m} r_j = r$. For our purposes the *last polar* $(r = n-1)$ will be of major interest. As we are operating on symmetric multilinear forms, we can dually formulate[1] the last polar hereby as

$$\pi_{n-1}(\mathcal{S},P)\colon\ \boldsymbol{F}(\underbrace{P,\dots,P}_{(n-1)\ \text{times}},X) = \sum_{i=0,\dots,m} x_i\,\frac{\partial \boldsymbol{F}_n(P)}{\partial p_i} = 0.$$

The last polar is a linear form; so it is a polar hyperplane.

Example 1. For visualizations (cf. Figures 2–4) the following equations of the Clebsch surface $\mathcal{S}_{CL}$ are used:

$$\begin{aligned} \text{in } \mathbb{E}^3\colon \qquad & (x^3+y^3+z^3+1)-(x+y+z+1)^3 &=& \ 0,\\ \text{in } \mathbb{RP}^3\colon\ \ & (x_1^3+x_2^3+x_3^3+x_0^3)-(x_1+x_2+x_3+x_0)^3 &=& \ 0. \end{aligned}$$

The points $P=(1,-1,0,1)$, $Q=(4,-4,0,1)$, $W=(3,-3,0,1)$, and $R=(7,-7,0,1)$ on the line p have the first polars

$$\begin{aligned}
\pi_1(\mathcal{S}_{CL},P)\colon\quad & \boldsymbol{F}(P,X,X) = -6x_0x_1-6x_0x_2-6x_1x_2-6x_2^2-6x_0x_3-6x_1x_3-6x_2x_3-3x_3^2=0,\\
\pi_1(\mathcal{S}_{CL},Q)\colon\quad & \boldsymbol{F}(Q,X,X) = -6x_0x_1+9x_1^2-6x_0x_2-6x_1x_2-15x_2^2-6x_0x_3-6x_1x_3-6x_2x_3-3x_3^2=0,\\
\pi_1(\mathcal{S}_{CL},W)\colon\quad & \boldsymbol{F}(W,X,X) = -6x_0x_1+6x_1^2-6x_0x_2-6x_1x_2-12x_2^2-6x_0x_3-6x_1x_3-6x_2x_3-3x_3^2=0,\\
\pi_1(\mathcal{S}_{CL},R)\colon\quad & \boldsymbol{F}(R,X,X) = -6x_0x_1+18x_1^2-6x_0x_2-6x_1x_2-24x_2^2-6x_0x_3-6x_1x_3-6x_2x_3-3x_3^2=0.
\end{aligned}$$

Let t be a line through the points P and Q. Each point on t has homogenous coordinates k_1Q+k_2P for $(k_1,k_2)\neq(0,0)$. The intersection points of t and $\mathcal{S}$ are the roots of the polynomial after the Taylor expansion

$$\begin{aligned}\boldsymbol{F}_n(k_1Q+k_2P) \;=\; & k_2^n\boldsymbol{F}_n(P) + \frac{k_2^{n-1}k_1}{1!}\,\boldsymbol{F}(\underbrace{P,\dots,P}_{(n-1)\ \text{times}},Q) + \cdots\\ & + \frac{k_2k_1^{n-1}}{(n-1)!}\,\boldsymbol{F}(P,\underbrace{Q,\dots,Q}_{(n-1)\ \text{times}}) + \frac{k_1^n}{n!}\,\boldsymbol{F}_n(Q) = 0.\end{aligned} \tag{1}$$

If the projective coordinates (k_1',k_2') of a point T on t are a double root of the polynomial (1) we call T a *double point of intersection* of t and $\mathcal{S}$. A line, for which a regular point of a surface is at least a double point of intersection, is called a *tangent line*, and the point is its *tangent point*. If each line of a plane through the same point is a tangent line of a surface, the plane is said to be a *tangent plane* at this point.

[1] A combinatorial proof can be found in [1, Section 31, p. 70–74].

Lemma 2. *The last polar of a regular point with respect to a surface is the tangent plane of the surface at this point.*

Proof. Let T be a regular point on the surface $\mathcal{S}\colon\ \boldsymbol{F}_n(T)=0$ and Q be a point not on $\mathcal{S}$, but on $\pi_{n-1}(\mathcal{S},T)$, i.e.,

$$\boldsymbol{F}(\underbrace{T,\dots,T}_{(n-1)\ \text{times}},Q)=0.$$

Then the substitution of the line $\overline{TQ}$ into (1) implies $k_2=0$ as a root with a multiplicity of at least 2. Hence, point T is at least a double point of this intersection. □

Lemma 3. *Each polar of a regular point P on a surface with respect to this surface contains P, and P is the tangent point of the common tangent plane of all polars.*

Proof. The equation of a surface $\mathcal{S}$ can be rearranged with respect to the last coordinate

$$\boldsymbol{F}_n(X)=x_0^n u_0+x_0^{n-1}u_1+\dots+x_0u_{n-1}+u_n=0,$$

where the u_i are forms of degree i in $x_1,\dots,x_m$. Assume $O_0=(0,\dots,0,1)$ to be a regular point of $\mathcal{S}$.[2] Substituting O_0 into X will vanish all forms $u_1,\dots,u_n$. It holds that

$$\boldsymbol{F}_n(O_0)=u_0=0.$$

Therefore the equation of $\mathcal{S}$ (through O_0) simplifies to

$$\boldsymbol{F}_n(X)=x_0^{n-1}u_1+\dots+x_0^{r+1}u_{n-r-1}+x_0^r u_{n-r}+\dots+x_0u_{n-1}+u_n=0$$

and

$$\begin{aligned}\pi_r(\mathcal{S},O_0)\colon\quad &(n-1)\dots(n-r)x_0^{n-1-r}u_1+(n-2)\dots(n-r-1)x_0^{n-r-2}u_2\\ &+\dots+(r+1)\dots2\,x_0u_{n-r-1}+r(r-1)\dots1\,u_{n-r}=0.\end{aligned}$$

The point O_0 certainly satisfies the equation of this polar. The last polar of O_0 on $\mathcal{S}$ is the polar hyperplane

$$\pi_{n-1}(\mathcal{S},O_0)\colon\ (n-1)!\,u_1=0,$$

and it is obviously the last polar for each of the previous polars. □

If the tangent planes to a surface are the same for all points on a line of the surface then this line is said to be a *torsal line*. A line that is not torsal is called *non-torsal*.

Theorem 4. *The pencil of first polars through a non-torsal line of an algebraic surface in $\mathbb{RP}^m$ for $m\geq 3$ is in projectivity with the range of their contact points.*

Proof. Let $\mathcal{S}$ be an algebraic surface of degree $n\geq 2$ and p be a non-torsal line of this surface. Let P and Q be distinct regular points on p with different tangent planes. The tangent plane to $\mathcal{S}$ at P contains all tangent lines through P, therefore it also contains the line p and the point Q. Applying Lemmas 2 and 3, each polar of the points P and Q induced by $\mathcal{S}$ must also contain p (Figure 3). Furthermore, the line p is the axis of the pencil of r-th polars of the points in the range k_1Q+k_2P.

[2] There is always a projective transformation, which maps the considered point to O_0.

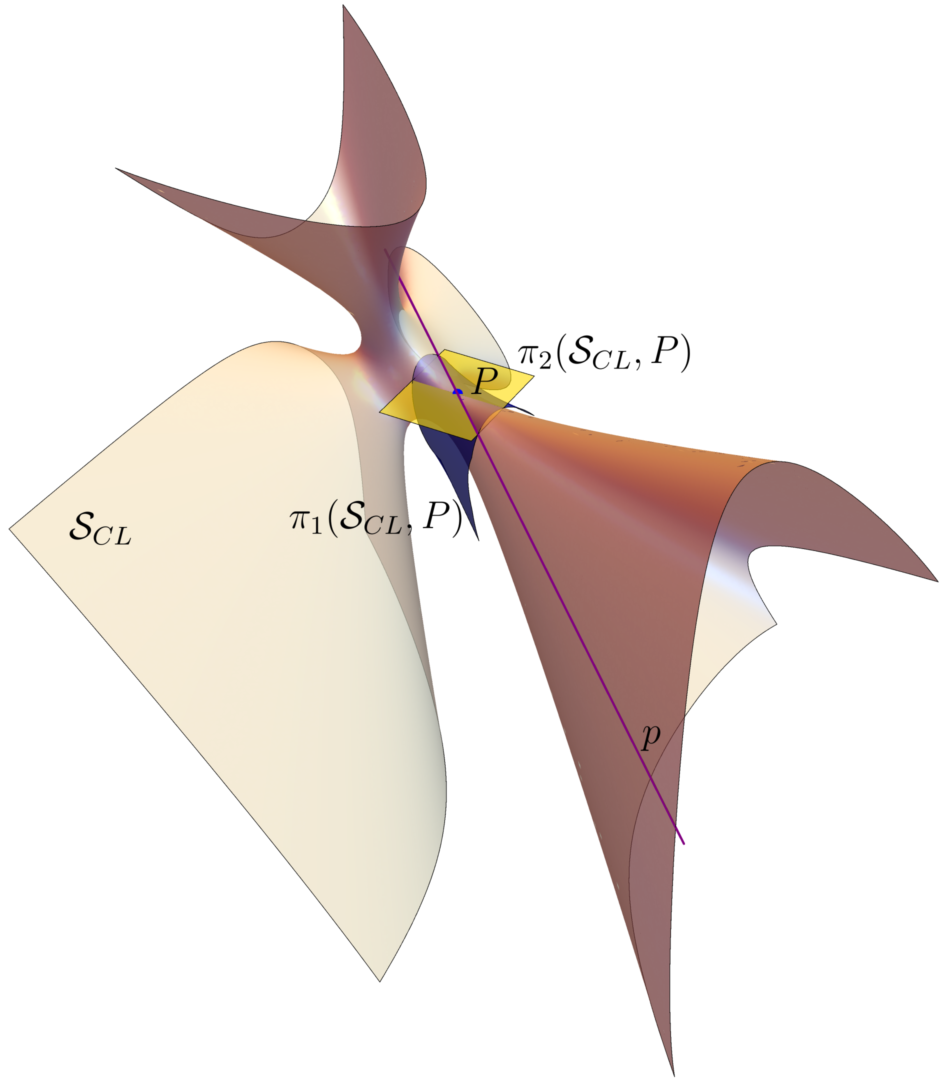

Figure 3: First and last polar at the point P

The last step is to show that there is a projectivity between the range of points on p and their first polars. Let W be a point in the range $k_1Q + k_2P$ with the projective coordinates $(k_1,\, k_2)$. The first polar of W with respect to $\mathcal{S}$ is

$$\begin{aligned}\pi_1(\mathcal{S}, W)\colon\ \boldsymbol{F}(W, \underbrace{X, \ldots, X}_{(n-1)\ \text{times}}) &= \boldsymbol{F}(k_1Q + k_2P, \underbrace{X, \ldots, X}_{(n-1)\ \text{times}}) \\ &= k_1\boldsymbol{F}(Q, \underbrace{X, \ldots, X}_{(n-1)\ \text{times}}) + k_2\boldsymbol{F}(P, \underbrace{X, \ldots, X}_{(n-1)\ \text{times}}) = 0.\end{aligned} \tag{2}$$

It is a surface of degree $(n-1)$ in the one-dimensional form (pencil of surfaces) with the same projective coordinates $(k_1,\, k_2)$ as the point W in its range. Therefore there is a projectivity between the range of points on a non-torsal line and the pencil of first polars of these points with respect to $\mathcal{S}$ (Figure 4). □

Example 1, ctd. Let us verify the validity of the previous theorem on Example 1:
The point $W = (3, -3, 0, 1)$ in the range $k_1P + k_2Q$ for $P = (1, -1, 0, 1)$ and $Q = (4, -4, 0, 1)$

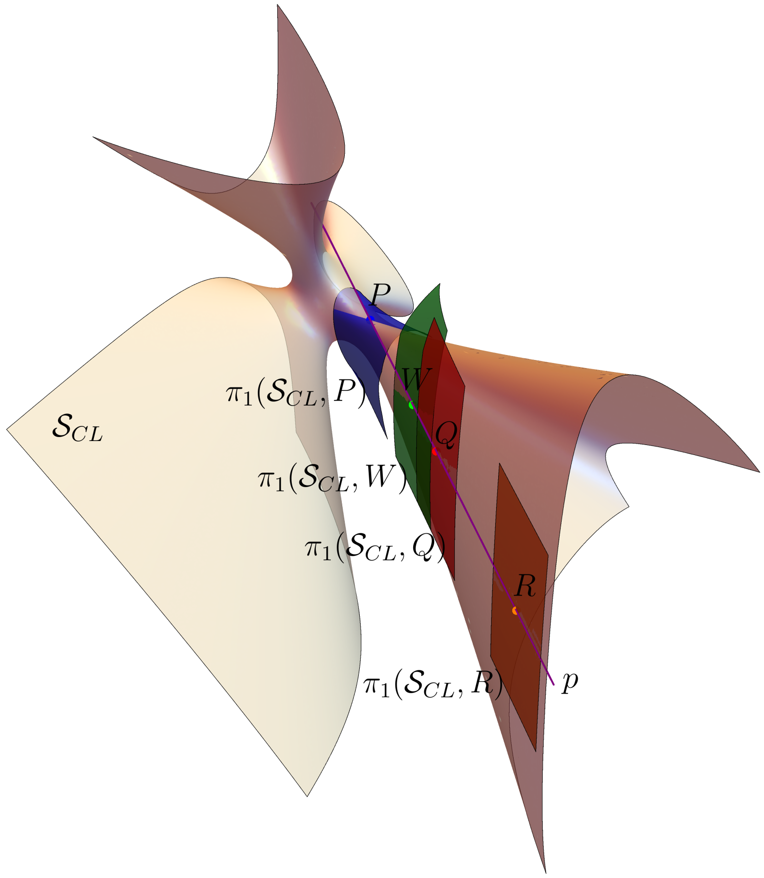

Figure 4: First polars of four distinct points on a non-torsal line

has the projective coordinates $(k_1,\, k_2) = \left(\frac{1}{3},\ \frac{2}{3}\right)$. The same holds for the first polars:

$$\begin{aligned}
& k_1\boldsymbol{F}(P, X, X) + k_2\boldsymbol{F}(Q, X, X) = \\
= \quad & \tfrac{1}{3}\left(-6x_0x_1 - 6x_0x_2 - 6x_1x_2 - 6x_2^2 - 6x_0x_3 - 6x_1x_3 - 6x_2x_3 - 3x_3^2\right) \\
& + \tfrac{2}{3}\left(-6x_0x_1 + 9x_1^2 - 6x_0x_2 - 6x_1x_2 - 15x_2^2 - 6x_0x_3 - 6x_1x_3 - 6x_2x_3 - 3x_3^2\right) \\
= \quad & -6x_0x_1 + 6x_1^2 - 6x_0x_2 - 6x_1x_2 - 12x_2^2 - 6x_0x_3 - 6x_1x_3 - 6x_2x_3 - 3x_3^2 = \boldsymbol{F}(W, X, X).
\end{aligned}$$

The point $R = l_1P + l_2Q$ and its first polar have projective coordinates $(l_1,\, l_2) = (-1, 2)$ in their one-dimensional forms.

A *ruled surface* is defined algebraically in [1, p. 510] as a surface with the property that through each of its points passes a line of the surface. A ruled surface is *developable* if each line is torsal. Otherwise, the surface is *non-developable*. Through each regular point of a non-developable ruled surface of degree $n > 2$ passes exactly one ruling[3].

Example 2. For visualizations (cf. Figure 5) the following equations of the Plücker conoid $\mathcal{S}_{PL}$ are used:

$$\begin{aligned}
\text{in } \mathbb{E}^3\colon \quad & z(x^2 + y^2) - 4(x^2 - y^2) &= 0, \\
\text{in } \mathbb{RP}^3\colon \quad & x_1^2x_3 - 4x_1^2x_0 + x_2^2x_3 + 4x_2^2x_0 &= 0.
\end{aligned}$$

[3]The statement can be found in general literature on geometry of ruled surfaces and we will not prove it in this paper (see [1, 7]).

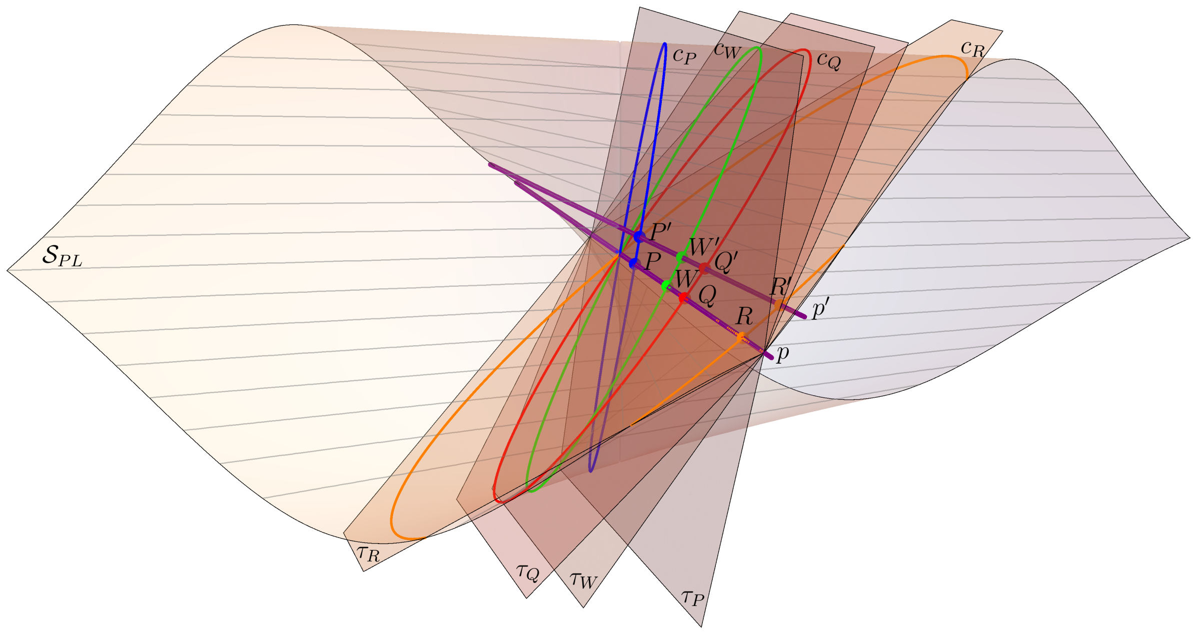

Figure 5: A pencil of tangent planes at a range of points on a ruling

The first and the 2nd (= last) satisfy, respectively,

$$\pi_1(\mathcal{S}_{PL}, P): \quad x_1^2 p_3 - 4x_1^2 p_0 + 2x_1 x_3 p_1 - 8x_1 x_0 p_1 + x_2^2 p_3 + 4x_2^2 p_0 + 2x_2 x_3 p_2 + 8x_2 x_0 p_2 = 0,$$
$$\pi_2(\mathcal{S}_{PL}, P): \quad 2x_1 p_1 p_3 - 8x_1 p_1 p_0 + 2x_2 p_2 p_3 + 8x_2 p_2 p_0 + x_3 p_1^2 + x_3 p_2^2 - 4x_0 p_1^2 + 4x_0 p_2^2 = 0.$$

Theorem 5 (Chasles's theorem)**.** *The pencil of tangent planes at the points of a non-torsal line of a ruled surface in* $\mathbb{RP}^m$ *for* $m \geq 3$ *is in projectivity with the range of their contact points.*

Proof. For surfaces of second degree the theorem is a straightforward consequence of Theorem 4. Let $\mathcal{S}$ be a surface of degree $n > 2$ and P and Q be regular points which lie on a ruling p. At each regular point on p there is only one tangent plane of $\mathcal{S}$ — its last polar. Dually, each tangent plane through p touches $\mathcal{S}$ in one contact point. Let τ_P and τ_Q be the tangent planes of $\mathcal{S}$ at P and Q, respectively. Furthermore let τ_W be a plane in the pencil $k_1\tau_P + k_2\tau_Q$ with the projective coordinates (k_1, k_2) and axis p. We will prove, that $W = k_1 P + k_2 Q$.
Let p' be another ruling of $\mathcal{S}$, such that p' intersects the pencil of planes $\tau_W = k_1\tau_P + k_2\tau_Q$ in regular points $W' = k_1 P' + k_2 Q'$. The planes τ_P, τ_Q and τ_W cut $\mathcal{S}$ in p and in plane curves c_P, c_Q and c_W of degree $n-1$, respectively. P', Q', W' are intersections of c_P, c_Q and c_W with p'. Since $n > 2$, no other ruling passes through the points P', Q' or W'. This holds for any choice of $p' \neq p$. Each algebraic curve of degreee n is determined by $\dfrac{n(n+3)}{2}$ points. If we construct $\dfrac{n(n+3)}{2}$ rulings, we obtain for each point on c_W that $c_W = k_1 c_P + k_2 c_Q$. The points P, Q and W are intersections of c_P, c_Q and c_W with p, therefore also $W = k_1 P + k_2 Q$. $\square$

3. Conclusion

Chasles's theorem is, by its nature, a great result in projective geometry. However, proofs of the theorem have always slipped into the use of more powerful methods of infinite closeness in the 19th century, or projective differential geometry and differential geometry nowadays. In this paper the theorem was proven purely projectively and generalized to non-torsal lines of algebraic surfaces in the projective extension of the real space. The use of the language of algebraic projective geometry has brought Chasles's theorem back to its place in projective geometry.

There is plenty of future work to do in the purification of classical projective geometry. This also leads to a question: "Is there a general method to avoid the use of non-projective methods for theorems of projective geometry originally proven with the use of infinite closeness ?" The importance of purification is given not only in algebraic but especially in synthetic geometry, as the computer 3D graphics has opened for us a gate to visualizing and manipulating objects, which are almost impossible to be drawn by hand. Projective geometry is therefore a convenient tool for computers, since one obtains precise results, not based on numerical methods.

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