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Addendum to Concurrency and Collinearity in Hexagons

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Abstract. In [1] a remarkable trigonometric equation, tied to various possible concurrencies and collinearities associated to a hexagonal path, was unveiled. In this sequel we relate this equation to cross-ratios of collinear points, and consequently get a trigonometric form for Brianchon's theorem. We also show how limiting cases of our theorems in [1] yield new proofs for two classical theorems of Ceva and Menelaus.

Key Words: Hexagon, Cross-Ratio, Sine-Cross Ratio Theorem, Brianchon's Theorem, Ceva's Theorem, Menelaus' Theorem

MSC 2010: 51M04, 97G60, 51A05, 51A45

1. The Sine-Cross Ratio Theorem

Let $A_1A_2A_3A_4A_5A_6$ be a hexagonal path in general position (no two lines through its vertices are parallel) in some Euclidean plane. Fixing a core triangle in the hexagon, say $\triangle A_1A_3A_5$, nine (measures of) oriented angles were associated in [1] about its vertices, $\alpha = m(\widehat{A_3A_1A_5})$, $\beta = m(\widehat{A_5A_3A_1})$, $\gamma = m(\widehat{A_1A_5A_3})$, $\alpha^- = m(\widehat{A_2A_1A_3})$, $\alpha^+ = m(\widehat{A_5A_1A_6})$, $\beta^- = m(\widehat{A_4A_3A_5})$, $\beta^+ = m(\widehat{A_1A_3A_2})$, $\gamma^- = m(\widehat{A_6A_5A_1})$, and $\gamma^+ = m(\widehat{A_3A_5A_4})$ (cf. Figure 1, the counterclockwise-oriented angles are positive and the clockwise, negative). Then the main results of [1] were that the trigonometric equation

$$\begin{aligned} &\sin(\alpha+\alpha^+)\sin(\beta+\beta^+)\sin(\gamma+\gamma^+)\sin\alpha^-\sin\beta^-\sin\gamma^- \\ &= \sin(\alpha+\alpha^-)\sin(\beta+\beta^-)\sin(\gamma+\gamma^-)\sin\alpha^+\sin\beta^+\sin\gamma^+ \end{aligned} \tag{1}$$

is satisfied if and only if

– the three main diagonals in the hexagon, $\overleftrightarrow{A_1A_4}$, $\overleftrightarrow{A_2A_5}$, and $\overleftrightarrow{A_3A_6}$, are concurrent *(Sine-Concurrency Theorem)*, or if and only if

– the three intersecting points of the pairs of lines, $\overleftrightarrow{A_1A_2}$ and $\overleftrightarrow{A_4A_5}$, $\overleftrightarrow{A_2A_3}$ and $\overleftrightarrow{A_5A_6}$, and $\overleftrightarrow{A_3A_1}$ and $\overleftrightarrow{A_6A_4}$, are collinear *(Sine-Collinearity Theorem)*.

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We prove here one more equivalence to (1) in the form of the following

Sine-Cross Ratio Theorem. *With the above notations in the hexagon $A_1A_2A_3A_4A_5A_6$, let E_1, E_2, E_3, and E_4, be the intersection points of $\overleftrightarrow{A_1A_2}$, $\overleftrightarrow{A_2A_3}$, $\overleftrightarrow{A_3A_4}$, and $\overleftrightarrow{A_4A_5}$ with $\overleftrightarrow{A_5A_6}$, respectively, and similarly let F_1, F_2, F_3, and F_4, be the intersection points of those same four lines with $\overleftrightarrow{A_6A_1}$ (cf. Figure 2). Then the trigonometric equation* (1) *holds true if and only if*

$$[E_1, E_2, E_3, E_4] = [F_1, F_2, F_3, F_4], \tag{2}$$

where $[E_1, E_2, E_3, E_4]$ and $[F_1, F_2, F_3, F_4]$ stand for the cross ratios of those respective points.

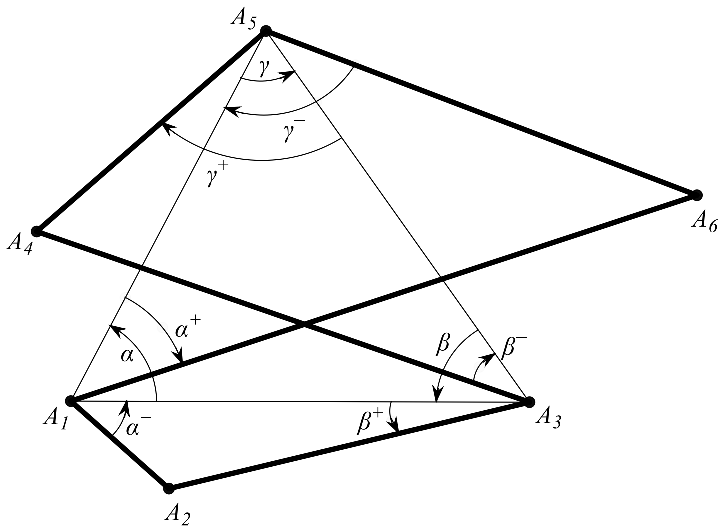

Figure 1: A hexagonal path $A_1A_2A_3A_4A_5A_6$ and the nine relevant angles, α, $\alpha^{\pm}$, β, $\beta^{\pm}$, and γ, $\gamma^{\pm}$.

Recall that the cross ratio [2, 3] of four (distinct) *collinear* points C_1, C_2, C_3, and C_4 in some Euclidean plane, denoted $[C_1, C_2, C_3, C_4]$, is the real number

$$[C_1, C_2, C_3, C_4] := \frac{\vec{C_1C_3}}{\vec{C_3C_2}} \Big/ \frac{\vec{C_1C_4}}{\vec{C_4C_2}}, \tag{3}$$

where for two points A and B we denote by $\vec{AB}$ the vector with origin A and end B (different from $\overrightarrow{AB}$, by which we denote the ray originating at A through the point B). In general two vectors cannot be divided, except when they are proportional, as in (3), in which case by their ratio we mean the proportionality constant. When the Euclidean plane is identified with the complex number system $\mathbb{C}$, as we will do in the proof of the theorem, denoting by $c_j \in \mathbb{C}$ the affix of C_j, $j = 1, \ldots, 4$, gives

$$[C_1, C_2, C_3, C_4] = [c_1, c_2, c_3, c_4] = \frac{(c_3 - c_1)(c_2 - c_4)}{(c_2 - c_3)(c_4 - c_1)}. \tag{4}$$

Proof of the Sine-Cross Ratio Theorem. The proof follows the basic steps used in [1] while dealing with the theorem's 'older cousins', the Sine-Concurrency and Sine-Collinearity Theorems. We refer to [1] for full details.

Identifying the Euclidean plane with the complex number system $\mathbb{C}$ there is no loss of generality in assuming that the circumcenter of $\triangle A_1A_3A_5$ has affix 0, and the affixes p_1, p_2, and p_3 of A_1, A_3, and A_5 respectively, are such that $|p_1| = |p_2| = |p_3| = 1$. Let q_1, q_2, and q_3 be the affixes of the other three vertices of the hexagon, respectively A_4, A_6, and A_2.

If $e_j \in \mathbb{C}$, respectively f_j, is the affix of E_j, respectively F_j, $j = 1, \dots, 4$ (see Figure 2), then

$$
\begin{aligned}
&E_1 = \overleftrightarrow{A_1A_2} \cap \overleftrightarrow{A_5A_6} \quad \text{and} \quad e_1 = \overleftrightarrow{p_1q_3} \cap \overleftrightarrow{p_3q_2} = -\frac{\det\begin{bmatrix} p_1 - q_3 & p_1\overline{q}_3 - \overline{p}_1q_3 \\ p_3 - q_2 & p_3\overline{q}_2 - \overline{p}_3q_2 \end{bmatrix}}{\det\begin{bmatrix} p_1 - q_3 & \overline{p}_1 - \overline{q}_3 \\ p_3 - q_2 & \overline{p}_3 - \overline{q}_2 \end{bmatrix}} \\
&E_2 = \overleftrightarrow{A_2A_3} \cap \overleftrightarrow{A_5A_6} \quad \text{and} \quad e_2 = \overleftrightarrow{p_2q_3} \cap \overleftrightarrow{p_3q_2} = -\frac{\det\begin{bmatrix} p_2 - q_3 & p_2\overline{q}_3 - \overline{p}_2q_3 \\ p_3 - q_2 & p_3\overline{q}_2 - \overline{p}_3q_2 \end{bmatrix}}{\det\begin{bmatrix} p_2 - q_3 & \overline{p}_2 - \overline{q}_3 \\ p_3 - q_2 & \overline{p}_3 - \overline{q}_2 \end{bmatrix}} \\
&E_3 = \overleftrightarrow{A_3A_4} \cap \overleftrightarrow{A_5A_6} \quad \text{and} \quad e_3 = \overleftrightarrow{p_2q_1} \cap \overleftrightarrow{p_3q_2} = -\frac{\det\begin{bmatrix} p_2 - q_1 & p_2\overline{q}_1 - \overline{p}_2q_1 \\ p_3 - q_2 & p_3\overline{q}_2 - \overline{p}_3q_2 \end{bmatrix}}{\det\begin{bmatrix} p_2 - q_1 & \overline{p}_2 - \overline{q}_1 \\ p_3 - q_2 & \overline{p}_3 - \overline{q}_2 \end{bmatrix}} \\
&E_4 = \overleftrightarrow{A_4A_5} \cap \overleftrightarrow{A_5A_6} = A_5 \quad \text{and} \quad e_4 = p_3,
\end{aligned}
\tag{5}
$$

and also

$$
\begin{aligned}
&F_1 = \overleftrightarrow{A_1A_2} \cap \overleftrightarrow{A_6A_1} = A_1 \quad \text{and} \quad f_1 = p_1 \\
&F_2 = \overleftrightarrow{A_2A_3} \cap \overleftrightarrow{A_6A_1} \quad \text{and} \quad f_2 = \overleftrightarrow{p_2q_3} \cap \overleftrightarrow{p_1q_2} = -\frac{\det\begin{bmatrix} p_2 - q_3 & p_2\overline{q}_3 - \overline{p}_2q_3 \\ p_1 - q_2 & p_1\overline{q}_2 - \overline{p}_1q_2 \end{bmatrix}}{\det\begin{bmatrix} p_2 - q_3 & \overline{p}_2 - \overline{q}_3 \\ p_1 - q_2 & \overline{p}_1 - \overline{q}_2 \end{bmatrix}} \\
&F_3 = \overleftrightarrow{A_3A_4} \cap \overleftrightarrow{A_6A_1} \quad \text{and} \quad f_3 = \overleftrightarrow{p_2q_1} \cap \overleftrightarrow{p_1q_2} = -\frac{\det\begin{bmatrix} p_2 - q_1 & p_2\overline{q}_1 - \overline{p}_2q_1 \\ p_1 - q_2 & p_1\overline{q}_2 - \overline{p}_1q_2 \end{bmatrix}}{\det\begin{bmatrix} p_2 - q_1 & \overline{p}_2 - \overline{q}_1 \\ p_1 - q_2 & \overline{p}_1 - \overline{q}_2 \end{bmatrix}} \\
&F_4 = \overleftrightarrow{A_4A_5} \cap \overleftrightarrow{A_6A_1} \quad \text{and} \quad f_4 = \overleftrightarrow{p_3q_1} \cap \overleftrightarrow{p_1q_2} = -\frac{\det\begin{bmatrix} p_3 - q_1 & p_3\overline{q}_1 - \overline{p}_3q_1 \\ p_1 - q_2 & p_1\overline{q}_2 - \overline{p}_1q_2 \end{bmatrix}}{\det\begin{bmatrix} p_3 - q_1 & \overline{p}_3 - \overline{q}_1 \\ p_1 - q_2 & \overline{p}_1 - \overline{q}_2 \end{bmatrix}}.
\end{aligned}
\tag{6}
$$

The six angle measures $\alpha^\pm$, $\beta^\pm$, and $\gamma^\pm$ can be used now to express q_1, q_2, and q_3 as affine combinations of the primary vertices p_1, p_2 and p_3. It follows that, for indexes modulo 3,

$$
q_k = s_kp_{k+1} + (1 - s_k)p_{k+2}, \quad k = 1, 2, 3\,, \tag{7}
$$

where

$$
s_1 = \frac{e^{2i\gamma^+}\left(1 - e^{2i\beta^-}\right)}{1 - e^{2i(\beta^- + \gamma^+)}}\,, \quad s_2 = \frac{e^{2i\alpha^+}\left(1 - e^{2i\gamma^-}\right)}{1 - e^{2i(\gamma^- + \alpha^+)}}\,, \quad \text{and} \quad s_3 = \frac{e^{2i\beta^+}\left(1 - e^{2i\alpha^-}\right)}{1 - e^{2i(\alpha^- + \beta^+)}}\,. \tag{8}
$$

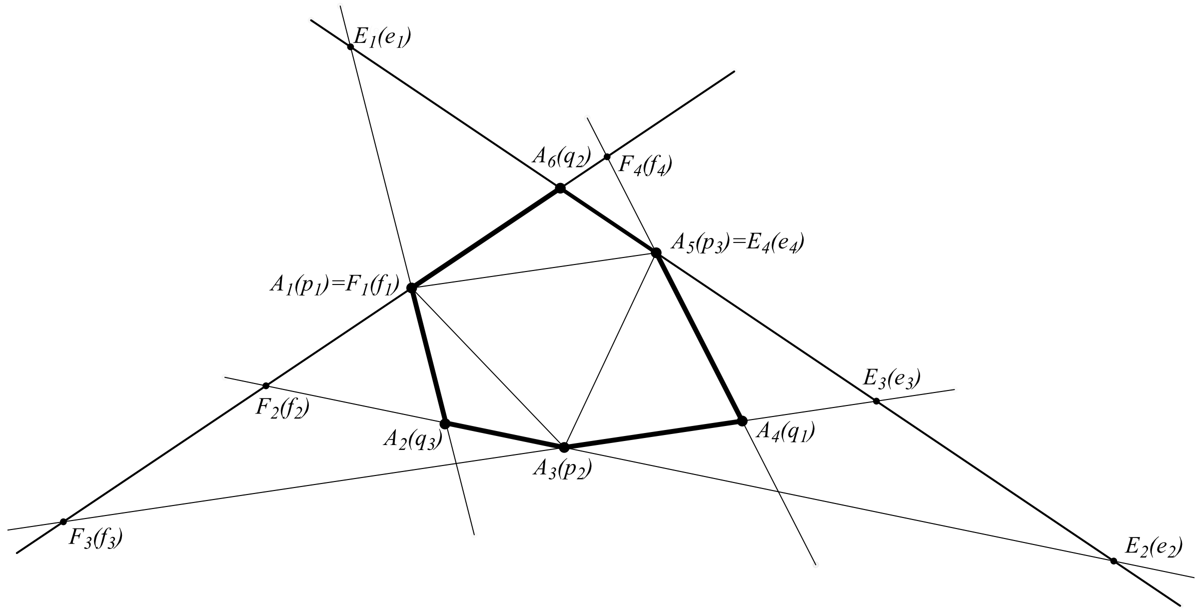

Figure 2: A hexagonal path with cross-ratio points E_1, E_2, E_3, E_4, and F_1, F_2, F_3, F_4.

Since by hypothesis $\overline{p}_k = 1/p_k$, $k = 1, 2, 3$, expressions similar to (7) exist also for $\overline{q}_k$, namely

$$\overline{q}_k = t_k \frac{1}{p_{k+1}} + (1 - t_k)\frac{1}{p_{k+2}}, \quad k = 1, 2, 3, \tag{9}$$

where

$$t_1 = \overline{s}_1 = \frac{s_1}{e^{2i\gamma^+}}, \quad t_2 = \overline{s}_2 = \frac{s_2}{e^{2i\alpha^+}}, \quad \text{and} \quad t_3 = \overline{s}_3 = \frac{s_3}{e^{2i\beta^+}}. \tag{10}$$

Calculating now the cross ratios $[e_1, e_2, e_3, e_4]$ and $[f_1, f_2, f_3, f_4]$ via (4), (5), (6), through (10), appears to be a formidable task, however it is doable. It turns out then if in the expressions of e_j, f_j, $j = 1, \ldots, 4$ given by (5) and (6) we make the substitutions

$$\overline{p}_k \longrightarrow \frac{1}{p_k}, \quad \overline{q}_k \longrightarrow r_k, \quad k = 1, 2, 3,$$

and set further

$$q_k := s_k p_{k+1} + (1 - s_k)p_{k+2} \quad \text{and} \quad r_k := t_k \frac{1}{p_{k+1}} + (1 - t_k)\frac{1}{p_{k+2}}, \quad k = 1, 2, 3,$$

then

$$[e_1, e_2, e_3, e_4] - [f_1, f_2, f_3, f_4] = C(\eta - \xi), \tag{11}$$

where

$$\begin{aligned} \xi &= (t_1p_1 - s_1p_2)(t_2p_2 - s_2p_3)(t_3p_3 - s_3p_1), \\ \eta &= \big((1 - s_2)p_1 - (1 - t_2)p_2\big)\big((1 - s_3)p_2 - (1 - t_3)p_3\big)\big((1 - s_1)p_3 - (1 - t_1)p_1\big), \end{aligned} \tag{12}$$

and C is a factor which does not vanish due to the general position of the hexagon $A_1A_2A_3A_4A_5A_6$. We conclude that $[e_1, e_2, e_3, e_4] = [f_1, f_2, f_3, f_4]$ if and only if $\xi = \eta$. It was proved in [1] that this latter equality is equivalent (via (8) and (10), and when the vertices A_1, A_3, and A_5 are on the unit circle) to (1), which concludes the proof of the theorem. $\square$

2. Further applications to the Sine-Theorems

We conclude this sequel to [1] with few further applications. The first one is devoted to a trigonometric form of Brianchon's theorem, a direct application of our Sine-Cross Ratio Theorem. The last two deal with explorations of hexagon-related concurrency and collinearity via trigonometry, when the hexagon is not in general position, an idea already alluded to in [1].

Brianchon's Theorem – Trigonometric Form. *A hexagonal path $A_1A_2A_3A_4A_5A_6$ is circumscribed about a conic, i.e., the sides $\overleftrightarrow{A_1A_2}$, $\overleftrightarrow{A_2A_3}$, $\overleftrightarrow{A_3A_4}$, $\overleftrightarrow{A_4A_5}$, $\overleftrightarrow{A_5A_6}$, and $\overleftrightarrow{A_6A_1}$ (when viewed as full lines) are all tangent to a conic if and only if equation* (1) *associated to $\triangle A_1A_3A_5$ and its nine angles holds true (cf. Figure 3).*

Proof. A projective characterization of a hexagonal path being circumscribed about a conic is precisely given by the equality of cross ratios (2) (Steiner's projectively dual generation of conics, [3]). Then the claim follows from our Sine-Cross Ratio Theorem. Via the Sine-Concurrency Theorem [1] this is then equivalent to the standard Brianchon theorem, i.e., the main diagonal lines $\overleftrightarrow{A_1A_4}$, $\overleftrightarrow{A_2A_5}$, and $\overleftrightarrow{A_3A_6}$ are concurrent (Figure 3). □

Next we use our sine-theorems and continuity arguments to give trigonometric proofs to the classical theorems of Ceva and Menelaus [2]. They are theorems associated to hexagons which are not in general position.

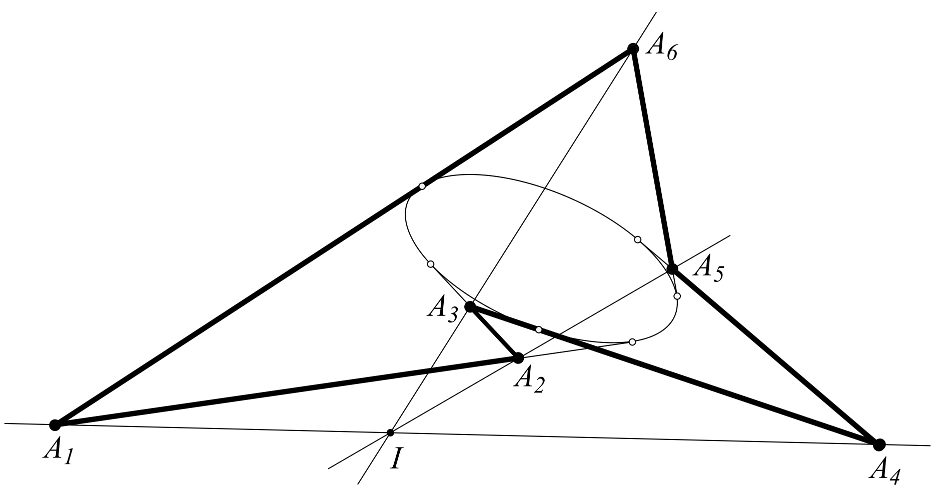

Figure 3: Brianchon's theorem in a non-convex hexagon.

Theorems of Ceva and Menelaus. *In the hexagonal path $A_1A_2A_3A_4A_5A_6$ assume that $A_2 \in \overleftrightarrow{A_1A_3} \setminus \{A_1, A_3\}$, $A_4 \in \overleftrightarrow{A_3A_5} \setminus \{A_3, A_5\}$, and $A_6 \in \overleftrightarrow{A_5A_1} \setminus \{A_5, A_1\}$.*

a) Ceva — If the three lines $\overleftrightarrow{A_1A_4}$, $\overleftrightarrow{A_2A_5}$, and $\overleftrightarrow{A_3A_6}$ are concurrent then

$$\frac{\overrightarrow{A_1A_2}}{\overrightarrow{A_2A_3}}\frac{\overrightarrow{A_3A_4}}{\overrightarrow{A_4A_5}}\frac{\overrightarrow{A_5A_6}}{\overrightarrow{A_6A_1}} = 1. \tag{13}$$

b) Menelaus — If the points A_2, A_4, and A_6 are collinear then

$$\frac{\overrightarrow{A_1A_2}}{\overrightarrow{A_2A_3}}\frac{\overrightarrow{A_3A_4}}{\overrightarrow{A_4A_5}}\frac{\overrightarrow{A_5A_6}}{\overrightarrow{A_6A_1}} = -1. \tag{14}$$

Proof. We forgo the easy analysis showing that in case *a*) only an even number (0 or 2) of vector ratios can be negative, while in case *b*) only an odd number (1 or 3) are so. Therefore, in both cases it suffices to show that

$$\frac{A_1A_2}{A_2A_3}\frac{A_3A_4}{A_4A_5}\frac{A_5A_6}{A_6A_1} = 1. \tag{15}$$

We only want to emphasize the novel idea of using (1) in the proofs, and so work out just two possible hexagon configurations, one in each case.

For case *a*) refer to Figure 4. $A_1A_2A_3A_4A_5A_6$ is the original hexagon and $B_1B_2B_3B_4B_5B_6$ is an approximation of it in general position. Notice that $B_1 = A_1$, $B_3 = A_3$, and $B_5 = A_5$, while $B_2 \neq A_2$, $B_2 \in \overrightarrow{A_5A_2}$, $B_4 \neq A_4$, $B_4 \in \overrightarrow{A_1A_4}$, and $B_6 \neq A_6$, $B_6 \in \overrightarrow{A_3A_6}$ and so the main diagonals of both hexagons intersect at the same point I. The Sine-Concurrency Theorem [1] applies therefore to the hexagonal path $B_1B_2B_3B_4B_5B_6$ and so the following equivalent form of (1) holds true:

$$\frac{\sin(\alpha+\alpha^+)}{\sin(\alpha+\alpha^-)}\frac{\sin(\beta+\beta^+)}{\sin(\beta+\beta^-)}\frac{\sin(\gamma+\gamma^+)}{\sin(\gamma+\gamma^-)}\frac{\sin\alpha^+}{\sin\gamma^-}\frac{\sin\beta^+}{\sin\alpha^-}\frac{\sin\gamma^+}{\sin\beta^-} = 1. \tag{16}$$

A repeated application of the Law of Sines in appropriate triangles gives now

$$\frac{\sin\alpha^+}{\sin\gamma^-} = \frac{B_5B_6}{B_6B_1}, \quad \frac{\sin\beta^+}{\sin\alpha^-} = \frac{B_1B_2}{B_2B_3}, \quad \text{and} \quad \frac{\sin\gamma^+}{\sin\beta^-} = \frac{B_3B_4}{B_4B_5}. \tag{17}$$

Consequently, (16) becomes

$$\frac{\sin(\alpha+\alpha^+)}{\sin(\alpha+\alpha^-)}\frac{\sin(\beta+\beta^+)}{\sin(\beta+\beta^-)}\frac{\sin(\gamma+\gamma^+)}{\sin(\gamma+\gamma^-)}\frac{B_1B_2}{B_2B_3}\frac{B_3B_4}{B_4B_5}\frac{B_5B_6}{B_6B_1} = 1. \tag{18}$$

Letting now $B_2 \to A_2$, $B_4 \to A_4$, and $B_6 \to A_6$ has the effect $\alpha^\pm \to 0$, $\beta^\pm \to 0$, and $\gamma^\pm \to 0$, while α, β and γ remain unchanged. By a continuity argument, in the limit (18) becomes (15), which proves Ceva's Theorem.

b) For Menelaus' Theorem we bypass a limiting argument and instead show how the sine-theorems may sometimes suggest direct trigonometric proofs for hexagons not in general position. To this end refer to Figure 5, where the core triangle is now $\triangle A_2A_4A_6$, instead of the usual $\triangle A_1A_3A_5$. Equivalently, renaming the hexagon $B_1B_2B_3B_4B_5B_6$, where modulo 6, $B_i = A_{i+1}$, $i = 1, 2, \ldots, 6$, allows us to work in the standard notational set-up. Despite the fact that $\triangle B_1B_3B_5$ is degenerate we can still define the familiar nine angles.

Notice that

$$\begin{aligned} &\alpha = \gamma = 0, \quad \beta = \pi, \\ &\alpha^+ = -\alpha^-, \quad \beta^- = -\beta^+, \quad \gamma^+ = \pi - \gamma^-. \end{aligned} \tag{19}$$

The Law of Sines applied in $\triangle A_2A_3A_4$, $\triangle A_4A_5A_6$, and $\triangle A_6A_1A_2$ give respectively

$$\frac{\sin\alpha^-}{\sin\beta^+} = \frac{A_3A_4}{A_2A_3}, \quad \frac{\sin\beta^-}{\sin\gamma^+} = \frac{A_5A_6}{A_4A_5}, \quad \text{and} \quad \frac{\sin\gamma^-}{\sin\alpha^+} = \frac{A_1A_2}{A_6A_1}. \tag{20}$$

Menelaus' Theorem follows now from (19) and (20). □

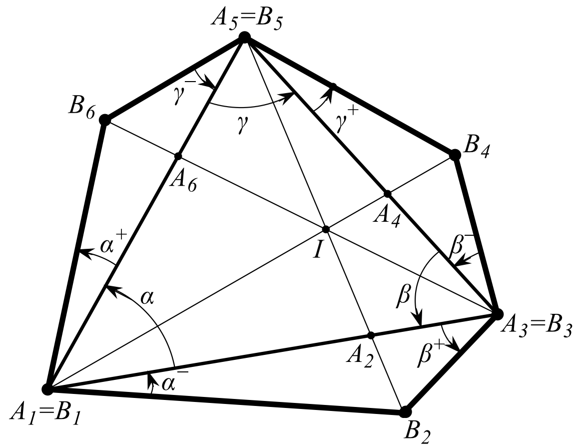

Figure 4: Ceva's theorem in hexagon $A_1A_2A_3A_4A_5A_6$ — trigonometric proof.

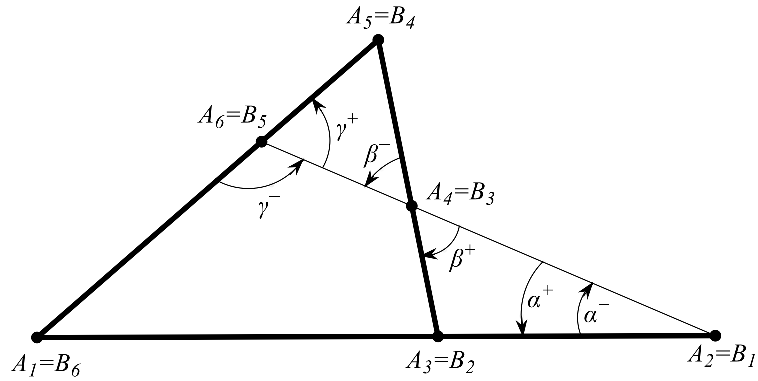

Figure 5: Menelaus' theorem — trigonometric proof.

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