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A Remarkable Quartic Pretzel Curve

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Abstract. The topic of this paper, a quartic curve of a remarkable shape, emerged from solving a locus problem originating from an 18th century Latin book of geometrical exercises using dynamic geometry software. Different mathematical representations and various properties, e.g., of a conchoidal nature or a duality with a sixth degree curve, are the subject of this paper.

Key Words: algebraic curve, geometry teaching, rational parametrization, polar equation, properties of a curve

MSC 2010: 51M04, 51N35

1. Introduction

Presented in this paper is a remarkable quartic curve, the shape of which resembles a pretzel, as shown in Figure 5, that unexpectedly emerged in the contemporary re-solving of one particular locus problem from an 18th century Latin book of geometrical exercises [5].

First, we will introduce the locus problem, the solving of which, with the use of dynamic geometry software, gave the origin of this curve (hereinafter referred to as the *'pretzel curve'*) and we will use its assignment to derive a polar equation of the curve. Then, we will use the acquired result to derive its algebraic equation, which reveals the pretzel curve to be an algebraic curve of the fourth degree. For the overview of its representations to be complete, a way of a rational parametrization of the curve will also be briefly indicated. Finally, we will discuss whether such a curve has ever been introduced prior to this and we will focus on some of its other properties, particularly its conchoidal feature and duality. At all stages of the investigation of the curve we used the free dynamic geometry software GeoGebra [9].

2. Original locus problem

The pretzel curve arose when solving a particular locus problem, namely problem no. 35, given in the Latin book *Exercitationes Geometricae*, authored by Ioannis Holfeld (1747–1814) and published by the Jesuit College of St. Clement in Prague in 1773 [5]. Due to the fact that a detailed analysis of the original assignment and the method of solving this historical

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locus problem is not within the scope of this paper we limit ourselves to mentioning only the essential details here. For further information we refer the reader to [3, 4, 8].

Here we will work with the current interpretation of the problem assignment using directed segments and their ratios. Specifically we will use the notion of the ratio $\mathrm{R}(KLM)$ of three collinear points K, L and M defined by the equality $\overrightarrow{KM} = \mathrm{R}(KLM)\,\overrightarrow{LM}$, where $\overrightarrow{KM}, \overrightarrow{LM}$ are oriented segments. Let us add that a line passing through the two points K and L will be denoted by $\overleftrightarrow{KL}$, an undirected segment connecting points K and L will be called $\overline{KL}$ and its length KL. Then, stated in such contemporary language, the task assigned in problem 35 is as follows.

Given a circle with a center A and a diameter $\overline{MP}$ (see Figure 1). For an arbitrary point B of this circle there is a point C on the ray $\overrightarrow{AB}$ so that

$$\mathrm{R}(MAO) = \mathrm{R}(ACB), \tag{1}$$

where O is the foot of a perpendicular drawn from B to $\overline{MP}$. Find the locus of the points C.

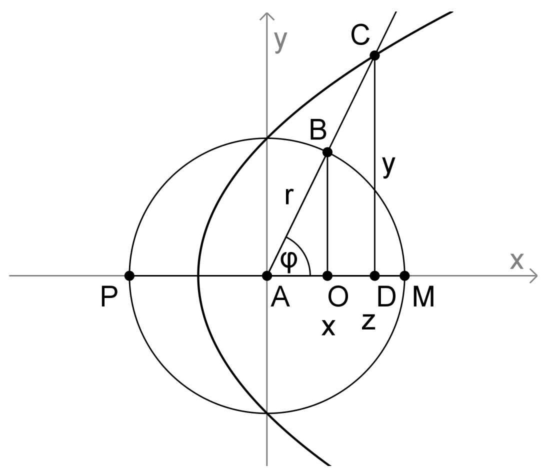

Figure 1: The locus problem

To capture the atmosphere of 18th century geometrical books, it is worth noting that in the original Latin assignment Ioannis Holfeld did not directly write a formula corresponding to (1). Instead, he introduced two alternative partially verbal formulations of its meaning. Typically for that time, the original style of notation does not emphasize the direction of related segments, apparently as a consequence of the mechanical approach to the generation of curves and the strict restraint of solving the problem only from the situation of the figure. As the first formulation, he determined the relation corresponding to (1) as the property of the segment BC to be the fourth proportional of lengths MO, AO and a, where $a = AB$ is the radius of the circle[1]. Alternativelly, he presented the segment BC as the fourth proportional of the versed sine, cosine (both of the radial angle $\angle MAC$) and radius of the given circle. As we will see, the latter notation directly leads to the polar equation of the locus curve.

In the solution provided in [5] Holfeld first assigned variables to the lengths of selected segments as follows: $AD = x$, $DC = y$, $AB = a$, $OM = z$ (see Figure 1). Then, without

[1]For the purpose of the further treatment we denote the radius with 'a', compared to Holfeld's label 'r'.

mentioning the use of them, he applied the similarity of triangles accompanied by the right triangle altitude theorem (also known as 'geometric mean theorem') to derive the locus equation $y^2 = a^2 + 2ax$, i.e., the parabola with the focus $F = [0, 0]$, directrix $d\colon x = -a$ and the focal parameter $p = a$, the plot of which for the particular a is shown in Figure 1.

3. The pretzel curve revelation

For the contemporary solver it is usual to start solving such a locus problem by creating a model in the dynamic geometry software. Besides providing initial information about the shape of the investigated locus, the dynamic features of such software can inspire the solver to a broader application of the problem assignment, which may end, as in the case of our locus problem, with the revelation of a remarkable result.

Various ways exist to construct points C so that they meet the problem assignment. One of the most straightforward of them is based on two parallel projections, as shown in Figure 2, where $\overleftrightarrow{OB} \parallel \overleftrightarrow{MK} \parallel \overleftrightarrow{AL}$ and $\overleftrightarrow{AK} \parallel \overleftrightarrow{LC}$. Moving B on the given circle, triangles $\triangle AKB$ and $\triangle CLB$ remain similar, therefore $\mathrm{R}(MAO) = \mathrm{R}(KLB) = \mathrm{R}(ACB)$.

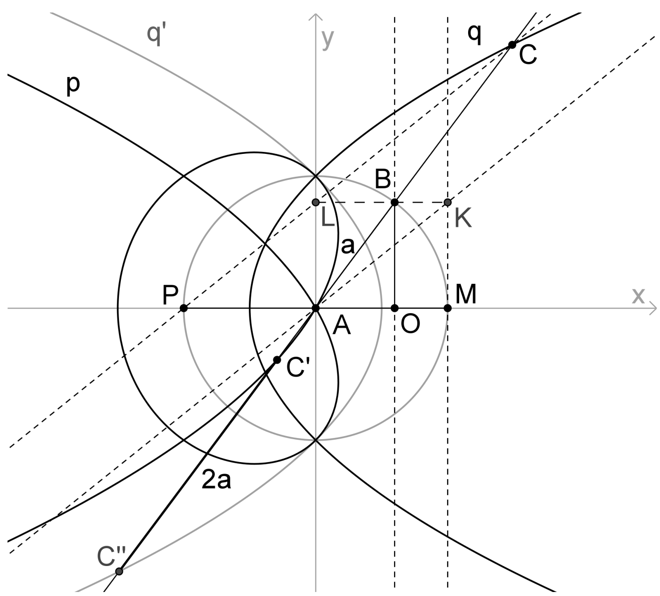

Figure 2: Geometric model in GeoGebra

Applying the *Locus* tool of the *Graphics* view of GeoGebra or entering the command *Locus[C,B]*, a parabola is drawn that corresponds to the above mentioned Holfeld's solution (see the parabola q passing through C in Figure 2). As follows from (1), this solution is directly linked with the orientation of the respective segments. What if we do not take this orientation into account and consider only the lengths of segments mentioned in the assignment? Let us note that this situation of uncertainty of segments' directions inevitably occurs when the problem is solved algebraically, as discussed in [3]. Then we can extend our attention to another possible position of point C, labeled C' in Figure 2, which is actually the reflection of point C along point B. Invoking the command *Locus[C',B]*, or appropriately applying the *Locus* tool within the *Graphics* view, we arrive at the diagram of the pretzel curve (see the curve p passing through C' in Figure 2).

Let us now examine in more detail, again using directed segments, the relationship of the point symmetry between C, point of the parabola q, and C', point of the pretzel curve p. As can be seen from Figure 2, the following relations apply to the respective directed segments $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$, $\overrightarrow{AC'} = \overrightarrow{AB} + \overrightarrow{BC'}$, where $\overrightarrow{BC'} = -\overrightarrow{BC}$. Consequently $\overrightarrow{AC} + \overrightarrow{AC'} = 2\overrightarrow{AB}$. Considering the parabola q', which is an image of the parabola q when reflected in the point A, the latter finding can be interpreted as the property of constant distance $2a$, where $a = AB$ is the radius of the initial circle, between two points C' and C'' of the intersection of a line passing through A with either of the curves, the pretzel p and parabola q'. Being a decisive feature in the definition of a conchoid[2], it indicates the following property of the curve.

Property 1. *The pretzel curve is the conchoid of the parabola q' having the focal parameter a with respect to its focus A and with the fixed distance of magnitude $2a$.*

More precisely, it is one branch of the conchoid. The second, outer, branch is the quartic curve[3] the shape of which is close to the original parabola. Thus, the pretzel curve belongs to the focal conchoids of the conic sections; the class of conchoids briefly noted in [6]. Then the construction in Figure 2 relating point C' to point C can be called the 'conchoid construction'. More about this property and its consequences will be dealt with in Section 5.

4. The pretzel curve representations

As already mentioned, Holfeld referred to notions of the versed sine and cosine in the alternative assignment of the problem. In particular, he stated that the length of the segment BC is the fourth proportional of the versed sine, cosine, both of the radial angle $\angle MAC$, and radius of the given circle. Now we will make use of this formulation to directly derive from it the polar equations of the respective locus curves, the parabola and the pretzel curve.

To do so, we assume the usual setting of the polar coordinate system with respect to the Cartesian coordinates (see Figures 1 and 2): the pole Q is identical to A lying in the Cartesian origin, the polar axis p coincides with the positive half-axis of x and the polar angle φ is the same as the oriented angle $\angle MAC$. Using the equalities $AO = a\cos\varphi$ and $MO = a \operatorname{versin}\varphi$, we rewrite the formula (1) into $\operatorname{versin}\varphi/\cos\varphi = a/BC$, that, applying the identity $\operatorname{versin}\varphi = 1 - \cos\varphi$, directly leads to $BC = a\cos\varphi/(1-\cos\varphi)$. To get equations of either curve, the parabola q and the pretzel curve p, we consider BC as the oriented segment. This then allows us to differentiate between two possible positions of point C with respect to B. In the case where C draws the parabola we write $BC = r - a$, whereas for the pretzel curve we use $BC = a - r$. Substituting these expressions one by one into the previous formula we arrive at the polar equations of the parabola and the pretzel curve, respectively, as follows.

$$r = \frac{a}{1-\cos\varphi}\,, \qquad r = a\,\frac{1-2\cos\varphi}{1-\cos\varphi}\,. \tag{2}$$

Various ways exist of obtaining the algebraic equation of the pretzel curve, from asking the computer to do it for us, through the application of the method that was used by Holfeld, to the utilization of standard methods of analytical geometry and algebra. Principles of the automatic derivation of the equation from the geometric construction implemented in

[2] A *conchoid* of a curve c with respect to a point A is the locus of points P_1 and P_2 on a line l passing through the fixed point A (pole) so that $P_1Q = QP_2 = d$, where Q is the point of intersection of l with c and d is the fixed distance [6].

[3] Its algebraic equation is $x^4 + x^2y^2 - 2ay^3 - 2ax^2y - (a+2)^2x^2 - a(a+4)y^2 = 0$.

GeoGebra, as well as the use of the analytic representation of the task and the consequent solving of the corresponding system of polynomial equations are presented in [3]. Here, we keep the content of this paper and derive the algebraic equation from (2). Let us follow Figure 1. Assuming the hitherto used location of the task in both coordinate systems, Cartesian and polar, we denote the rectangular coordinates of point C of the curve as $C = [x, y]$ whereas its polar coordinates as $C = (r, \varphi)$. Then, substituting from the equalities $r = \sqrt{x^2 + y^2}$ and $\cos\varphi = x/\sqrt{x^2 + y^2}$ into $r = a\,(1 - 2\cos\varphi)/(1 - \cos\varphi)$, after a few steps we get the desired algebraic equation of the pretzel curve in the position shown in Figure 1

$$y^4 + x^2y^2 + 2ax^3 + 2axy^2 + 3a^2x^2 - a^2y^2 = 0. \tag{3}$$

To complete the overview of the basic methods of representation of the curve, we will briefly present its parametric equations here. As in previous cases, there are various ways to derive these equations. For example, a really effective approach is to apply the standard rational formulas $\sin\varphi = 2t/(1+t^2)$ and $\cos\varphi = (1-t^2)/(1+t^2)$ along with the polar equation (2). Another possible method to receive a rational parametrization of such a curve of genus 0 is a geometric way. Namely, intersecting the curve with a pencil of curves having already $n - 1$ points in common with the curve so that the last nth intersection point depends on the parameter of the pencil. In the case of the pretzel curve one could use a pencil $\{c(t)\}$ of conics $c(t)$. According to Bezout's theorem a conic $c(t)$ has 8 points in common with such a quartic curve. If we choose three basic points of the pencil $\{c(t)\}$ in the knots of the pretzel curve and the fourth in the 'vertex' on the symmetry axis, due to the multiplicity of knots, we have already used 7 intersecting points, so that the 8th depends rationally on the parameter t of the pencil of conics. All this we can create in GeoGebra *CAS* and finally we get the parametric equations of the pretzel curve as follows

$$x = a\,\frac{-t^4 + 4t^2 - 3}{2t^2 + 2}, \quad y = a\,\frac{t^3 - 3t}{t^2 + 1}; \quad t \in \mathbb{R}. \tag{4}$$

5. Other pretzel or knot curves

Looking at (3), we can see that the polynomial $y^4 + x^2y^2$ formed by the highest order terms is divisible by x^2+y^2. It allows us to classify the pretzel curve as the real algebraic circular curve [2]. In Section 3 the pretzel curve was identified as belonging to the class of focal conchoids of conics, of which each form depends on the kind of the conic and on the relation of the fixed distance d to the conic's latus rectum [6]. For illustration, in Figures 3 and 4 such conchoids of an ellipse and a parabola are shown respectively, produced by the conchoid construction, both with the semi-latus rectum a and the fixed distance $d = 2a$. Apparently, the pretzel curve can be considered as an affine special case of a knot curve, belonging to the larger set of knot curves which also comprises the roulettes, such as the trifolium. The question arises, as to whether this particular curve has been described elsewhere. A very similar quartic curve with three knots and touching the line at infinity can be found in [7] as the *'knot curve'* (see Figure 6). As it is represented in another coordinate frame, we transform the pretzel curve equation (3) by $x' = y$, $y' = -1 - x$ (resp., $y' = -a - x$) and, omitting the primes, receive

$$(x^2 - a^2)^2 + x^2y^2 = ay^2(2y + 3a), \tag{5}$$

where a is a positive real number (see Figure 5), while the one in [7] has the equation

$$(x^2 - a^2)^2 = ay^2(2y + 3a). \tag{6}$$

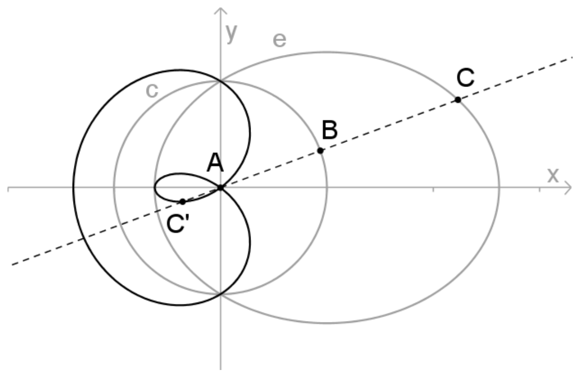

Figure 3: Focal conchoid of an ellipse

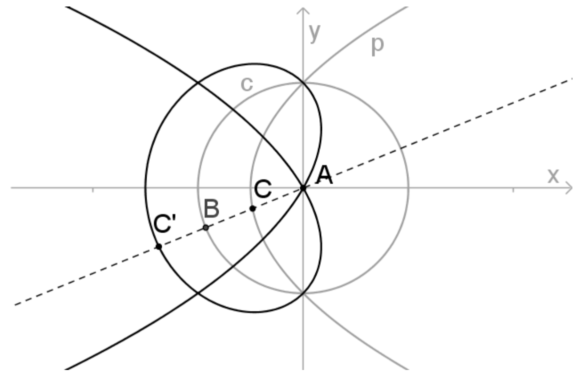

Figure 4: Focal conchoid of a parabola

Thus the pretzel curve is a linear combination of this SALMON's knot curve and the singular quartic curve. Let us note that the 'knot curve' 6 is alternatively called the 'pretzel curve' in [1], where it is treated among others together with Cassini curves and Limaçons using complex number representations.

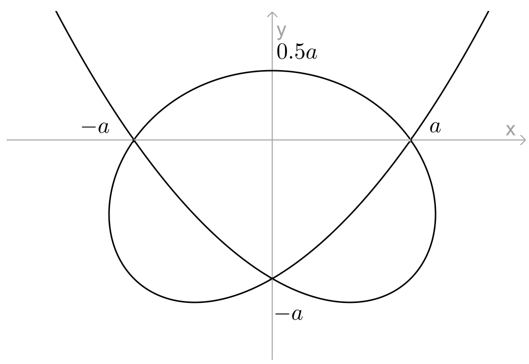

Figure 5: Pretzel curve

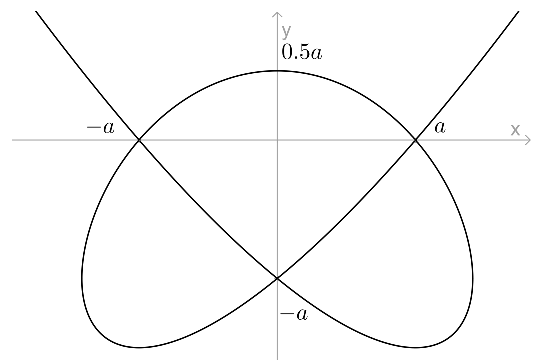

Figure 6: Knot curve

Investigating the equation (5) we reveal the following properties of the corresponding pretzel curve. Points of intersection with the coordinate axes are $\left[0,\, \frac{a}{2}\right]$ and $[0,\, -a]$ for y-axis, and $[-a,\, 0]$ and $[a,\, 0]$ for x-axis. As shown in Figure 5, the latter three points $[0,\, -a]$, $[-a,\, 0]$ and $[a, 0]$ are double points. The pretzel curve has precisely these three singular points, each of them being a crunode with one of the following pairs of distinct tangents:

$$y = \pm\sqrt{2}x \pm a\sqrt{2} \text{ at } [-a,\, 0],\quad y = \pm\sqrt{2}x \mp a\sqrt{2} \text{ at } [a,\, 0],\quad \text{and } y = \pm\frac{\sqrt{3}}{3}x - a \text{ at } [0,\, -a].$$

Property 2. *The pretzel curve is unique, it is symmetric, has three real knots, is circular and touches the line at infinity.*

Let us now focus on further selected tangents of (5). The pairs of its horizontal and vertical tangents affect its extent. The horizontal tangents are represented on the one hand by the line $y = \frac{a}{2}$, meeting the curve at $\left[0,\, \frac{a}{2}\right]$ and on the other hand by the bitangent $y = (2\sqrt{2} - 4)a$, touching the curve at two points $\left[\mp\sqrt{\sqrt{2}\,8 - 11}a,\; (2\sqrt{2} - 4)a\right]$, symmetric with respect to the y-axis. The vertical tangents are the lines $x = \mp\sqrt{2\sqrt{3} - 3}\sqrt{3}a$, symmetric with respect to the y-axis, that meet the curve at points $\left[\mp\sqrt{2\sqrt{3} - 3}\sqrt{3}a,\; (2\sqrt{3} - 4)a\right]$, respectively. There are two more bitangents to the pretzel curve, having the equations $y = \pm 2\sqrt{2}x - 4a$

and intersecting each other at point $[0, -4a]$. This recognition of bitangents can lead us to reflection on a dual curve of the curve of interest. Going back to the equation (3) of the pretzel curve and applying the reciprocation in the defining circle on it we get the following.

Property 3. *The dual curve of the pretzel curve* (3) *has degree 6 and is defined by the equation*

$$\begin{aligned} 27y^6 + 9y^4(x^2 - 10ax - 2a^2) - y^2(15x^4 + 124ax^3 - 84a^2x^2 - 48a^3x + a^4) \\ = -x(3x + 2a)(x^2 - 6ax + a^2)^2. \end{aligned} \tag{7}$$

It has four singular points $\left[(3 - 2\sqrt{2})a,\, 0\right]$, $\left[(3 + 2\sqrt{2})a,\, 0\right]$, $\left[\frac{1}{3}\,a,\ \frac{2\sqrt{2}}{3}\,a\right]$, $\left[\frac{1}{3}\,a,\ -\frac{2\sqrt{2}}{3}\,a\right]$, *the first being an isolated point and the last two belonging to the defining circle of the pretzel curve (see Figure 7), where the singular points are labeled by* S_1, S_2, S_3, *and* S_4, *respectively.*

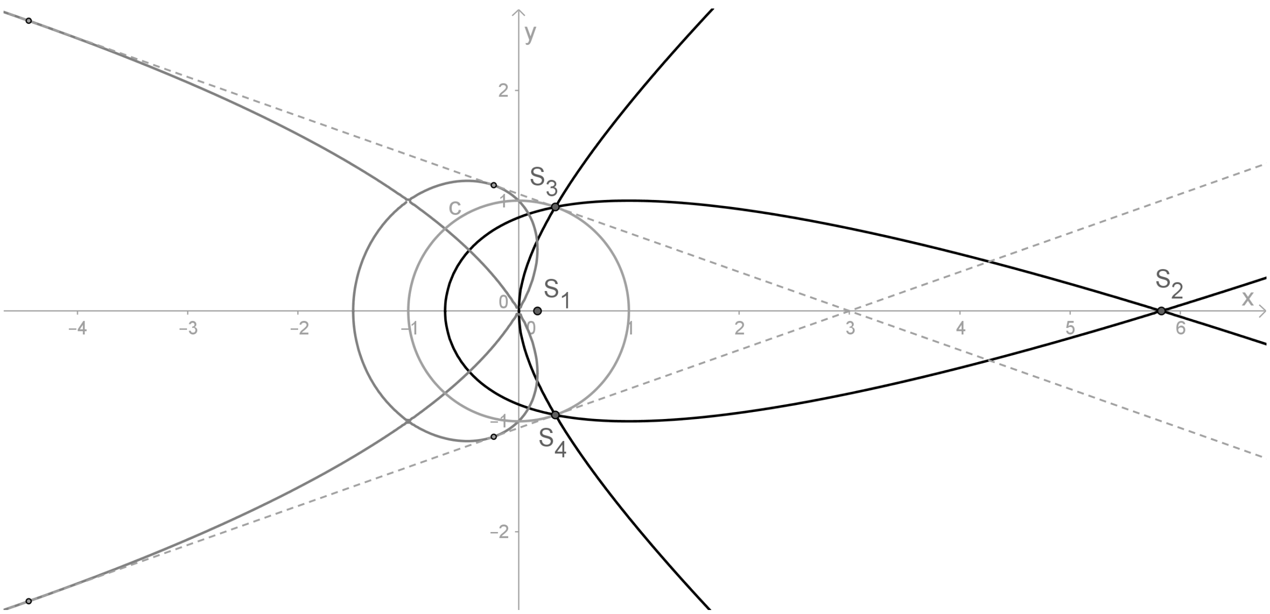

Figure 7: The pretzel curve and its dual curve; $a = 1$

6. Conclusion

The paper dealt with an algebraic circular curve of fourth degree that was named 'pretzel curve' because of its remarkable shape. Although such a curve had, to the best of the author's knowledge, not been mentioned before, it was revealed to have an apparent relation to well known classes of curves. It appeared to be a representative of the class of focal conchoids of conic sections, namely to be the focal conchoid of the parabola, being a special affine case of so-called knot curves. From the viewpoint of projective geometry, the dual curve of degree 6 of the pretzel curve was identified in the paper. As usual, new questions emerged when formulating the found properties. Some are related to the conchoidal property of the curve. It appeared to be a linear combination of specific curves. Can all of them be the result of a conchoid construction? What are the common properties of the curves of this pencil? Also of interest might be the sextactic ellipse at the proper vertex on the symmetry axis as well as the sextactic parabola at the vertex at infinity and other properties of the dual to the pretzel curve.

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