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Congruence Theorems for Quadrilaterals

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Abstract. In this article we will prove new criterions for the congruence of convex quadrilaterals. Counterexamples show that corresponding results for polygons with more than four vertices do not hold.

Key Words: Congruent-like polygons, congruent quadrilaterals, weakly congruent polygons, congruence theorems

MSC 2010: 97G40, 51M04

1. Introduction

Triangle congruence theorems are one of the basic topics in classical geometry. It is well known that the minimum number of *pieces* (sides and angles) necessary for proving that two triangles are congruent is three. On the other hand, counterexamples of pairs of triangles that are not congruent, but with three respectively congruent pieces, can be easily considered. Indeed such counterexample may have even five respectively congruent pieces, like pairs of two *consecutive Kepler Triangles* (see Figure 1) and more generally pairs of *almost congruent triangles* (see [3] and [1]).

The extension of triangle congruence theorems to polygons is more complex and certain remarks are required (see [4, Lesson 11] and [5, Chapter 8] for the definitions and the development of the basic properties related to polygons). Two convex polygons are said to be *congruent* if there is a one-to-one correspondence between their vertices such that consecutive vertices correspond to consecutive vertices, and all pairs of correspondent sides and all pairs of correspondent angles are congruent. Roughly speaking, a polygon is a closed figure formed by three or more segments joined end to end, and two polygons are congruent if we can overlap them. Formally, we can say that two polygons are congruent if, and only if, one can be transformed into the other by an isometry. In what follows, it will be useful to remind that, two polygons $\mathcal{P}$ and $\mathcal{P}'$ are congruent if every two consecutive sides of $\mathcal{P}$ are congruent to two consecutive sides of $\mathcal{P}'$, and the angles that they define are congruent.

The general study of congruence between two polygons seems too hard, and different cases such as convex, non convex or twisted polygons should be considered. Clearly the first constructive step in this direction is to study pairs of convex polygons (see [4, Definition 9.7]).

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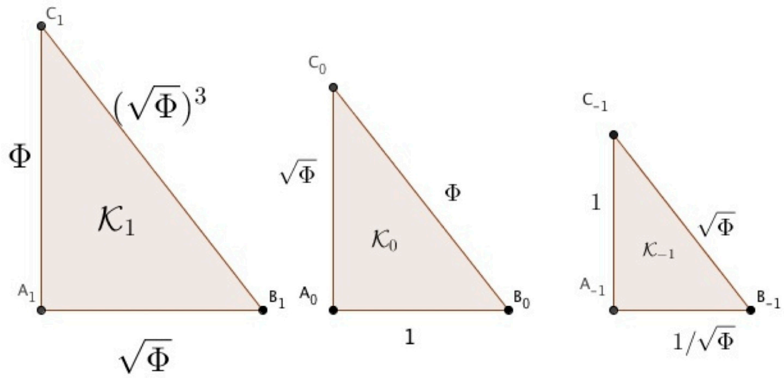

Figure 1: Examples of consecutive Kepler triangles. Φ is the Golden Mean.

Even if in this paper all polygons considered are convex, in the statement of the theorems we will highlight it.

The following definition is crucial for our investigation:

Definition 1.1. We will say that two convex polygons $\mathcal{P}$ and $\mathcal{P}'$ are *congruent-like*, if there is a bijection between the sides of $\mathcal{P}$ and $\mathcal{P}'$, and a bijection between the angles of $\mathcal{P}$ and $\mathcal{P}'$, such that each side of $\mathcal{P}$ is congruent to a corresponding side of $\mathcal{P}'$ and each angle of $\mathcal{P}$ is congruent to a corresponding angle of $\mathcal{P}'$ (note that the two bijections are 'a priori' independent).

Clearly, two congruent n-gons are congruent-like. It is easy to check that the converse holds for pairs of triangles, but it is not always true if $n > 4$ (see Figure 2).

In the following we will denote by $\mathcal{P} = (A_0, \dots, A_{n-1})$ the polygon whose consecutive vertices are $A_0, \dots, A_{n-1}$ (usually numbered counterclockwise). The length of the side A_iA_{i+1} (the index i is read mod n) will be denoted $\overline{A_iA_{i+1}}$ or a_i. The inner angle at vertex A_i as well as its radian measure will be denoted by $\hat{A}_i$, $A_{i-1}\hat{A}_iA_{i+1}$ or α_i. Moreover, we write

$$A_iA_{i+1} \to A_{i+1}A_{i+2} \to \cdots \to A_{i+n-2}A_{i+n-1} \to A_{i+n-1}A_i \to A_iA_{i+1}$$

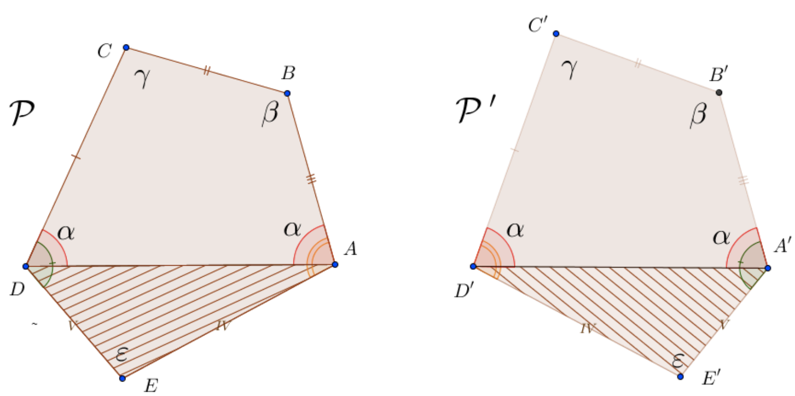

Figure 2: Starting from two congruent quadrilaterals (A, B, C, D) and (A', B', C', D') with two congruent consecutive angles of radian measure α, we may easily construct pairs of congruent-like polygons that are not congruent.

or

$$a_i \to a_{i+1} \to \cdots \to a_{i+4} \to a_{i+5} \to a_i$$

for the ordered sequence of the sides of $\mathcal{P}$, starting from A_iA_{i+1} (see Figure 3).

We will say that two congruent-like polygons $\mathcal{P}$ and $\mathcal{P}'$ are *ordered congruent-like polygons*, if whichever ordered sequence of consecutive sides of $\mathcal{P}$ is equal to an ordered sequence of sides of $\mathcal{P}'$. Figure 3 shows that two ordered congruent-like polygons need not be congruent.

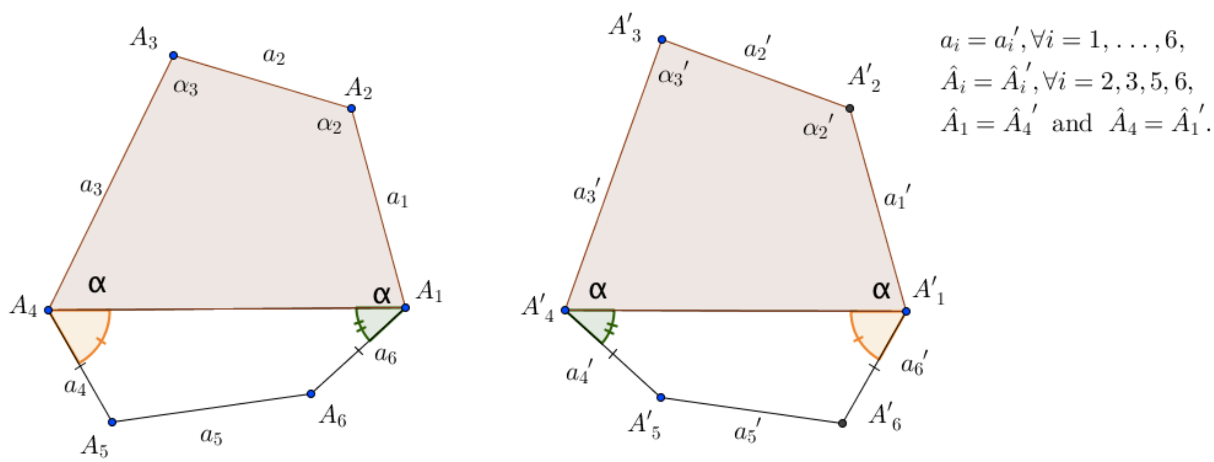

Figure 3: Starting from a quadrilateral (A, B, C, D) with two congruent consecutive angles, say α, we may easily construct pairs of ordered congruent-like n-gons that are not congruent, for every $n > 5$. Here any sequence $a_i \to a_{i+1} \to \cdots \to a_{i+4} \to a_{i+5} \to a_i$ of sides of the hexagon on the left is equal to $a'_i \to a'_{i+1} \to \cdots \to a'_{i+4} \to a'_{i+5} \to a'_i$ referred to the hexagon on the right. Note that $a_4 = a'_4 = a_6 = a'_6$.

Examples such as in Figures 2 and 3 show that the general investigation on the congruence of polygons will not be easy and the proofs will require a detailed case analysis, even in the convex cases.

For those concerning quadrilaterals congruences we highlight here that a significant number of criterions are well-known in terms of equivalence of ordered sequences of sides and angles, such as the "$SASAS$" criterion, the "$ASASA$" criterion and others. A complete list of these kinds of criterions can for instance be found in [4] and [5]. Also a result of I.E. Vance [11] showed that the minimum number of pieces necessary for the congruence of two convex quadrilaterals is 5. On the other hand, we note that if we do not require any order to the sequences of angles and sides of $\mathcal{Q}$ and $\mathcal{Q}'$, then also 7 pairs of congruent pieces of $\mathcal{Q}$ and $\mathcal{Q}'$ need not be sufficient to get the congruence of two quadrilaterals (see Figure 4).

By virtue of these considerations and the above counterexamples for general n-gons ($n > 4$), the following natural question arises: Are there two congruent-like quadrilaterals that are not congruent?

In this article, we will show that if $\mathcal{Q}$ and $\mathcal{Q}'$ are ordered congruent-like quadrilaterals, then they are congruent (Theorem 3.5). Among other results, congruence criterions for quadrilaterals (not necessarily ordered) are given (see Theorems 2.4 and 2.6). Some open questions are listed in the last section.

The paper is suitable for a large audience of readers who are referred to [8], [10], [6], and the references therein for a deeper investigation on quadrilaterals and related topics.

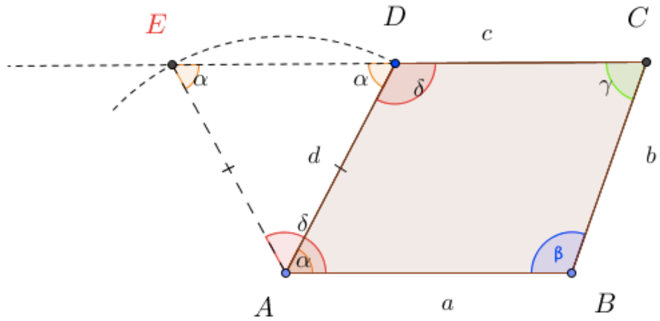

Figure 4: The trapezoids $\mathcal{Q} = (A, B, C, D)$ and $\mathcal{Q}' = (A, B, C, E)$ are not congruent though they have 7 mutually congruent pieces: four angles and three sides.

2. Criterions for congruent-like quadrilaterals

In this paper a quadrilateral $\mathcal{Q}$ will be denoted by $\mathcal{Q} = (A, B, C, D)$. The radian measures of the inner angles $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ of $\mathcal{Q}$ will be denoted by α, β, γ, δ, respectively; the lengths of the sides AB, BC, CD, and DA of $\mathcal{Q}$ will be denoted by a, b, c, d, respectively.

Clearly two congruent-like quadrilaterals will have 8 pairwise congruent pieces. In order to show that two ordered congruent-like quadrilaterals are congruent, we recall some elementary results, that allow to reduce the investigation of the possible cases.

The first one is also known as *Hinge' property* (see [7, p. 121]).

Lemma 2.1. *Let $\mathcal{T} = (A, B, C)$ and $\mathcal{T}' = (A', B', C')$ two triangles, such that $\overline{CA} = \overline{C'A'}$ and $\overline{CB} = \overline{C'B'}$. Then $\hat{C} \leq \hat{C}' \iff \overline{AB} \leq \overline{A'B'}$.*

Lemma 2.2. *Let $\mathcal{T} = (A, B, C)$ and $\mathcal{T}' = (A', B', C')$ be two triangles, such that $\overline{CA} = \overline{C'A'}$ and $\hat{C} = \hat{C}'$. If $\hat{B} \leq \pi/2$ and $\overline{CB} \leq \overline{C'B'}$, then $\overline{AB} \leq \overline{A'B'}$.*

Proof. Clearly we may suppose that $\overline{CB} < \overline{C'B'}$. Then there exists $B'' \in B'C'$ such that $\overline{C'B''} = \overline{CB}$. It follows that (A, B, C) and (A', B'', C') are congruent triangles, and in particular $\overline{A'B''} = \overline{AB}$ (see Figure 5). By hypothesis the angle $A'\hat{B}''B'$ is obtuse, thus the opposite side $A'B'$ is longer than $\overline{A'B''} = \overline{AB}$. □

Note that the above result can be read as an extension of the first criterion for triangles.

Lemma 2.3. *Let $\mathcal{Q} = (A, B, C, D)$ be a quadrilateral having three consecutive non-acute angles, say $\hat{A}$, $\hat{B}$ and $\hat{C}$. If $\mathcal{Q}$ is not a rectangle, then $a < c$ and $b < d$.*

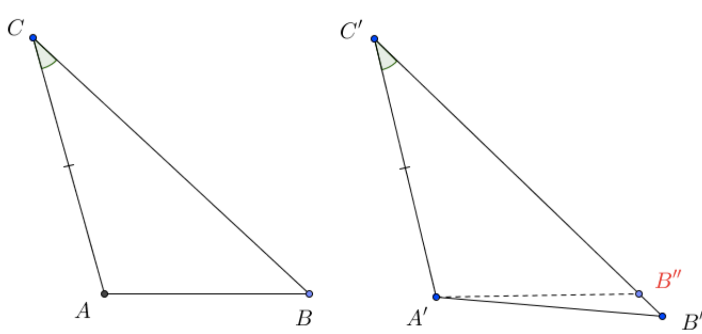

Figure 5: Inequalities in a pair of triangles with one congruent angle and adjacent side.

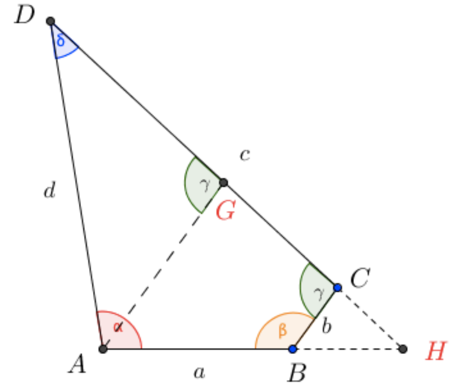

Figure 6: Non-acute angles $\hat{A}$, $\hat{B}$ and $\hat{C}$ imply $a < c$ and $b < d$.

Proof. Looking at Figure 6, it is easy to check that $\overline{BC} \leq \overline{AG} \leq \overline{AD}$.
On the other hand, if $\mathcal{Q}$ is not a rectangle, then the angles $\hat{A}$, $\hat{B}$ and $\hat{C}$ are not simultaneously right, so that at least one of the above inequalities must be strict, and $b < d$. Similarly it can be proved that $a < c$. □

We are now in the position to show our first theorem.

Theorem 2.4. *(Side-Angle-Side criterion) Let $\mathcal{Q}$ and $\mathcal{Q}'$ be congruent-like convex quadrilaterals. Then they are congruent if and only if they have two consecutive sides and the angle between them respectively congruent.*

Proof. In order to show that $\mathcal{Q} = (A, B, C, D)$ and $\mathcal{Q}' = (A', B', C', D')$ are congruent, we may suppose that they have $\overline{AB} = \overline{A'B'}$, $\overline{DA} = \overline{D'A'}$ and $\hat{A} = \hat{A}'$ (oriented counterclockwise). In particular it follows by criterions on triangles that the diagonals DA and $D'A'$ are congruent, and $\mathcal{Q}$ and $\mathcal{Q}'$ can be decomposed in a pair of congruent triangles (see Figures 7 and 8).
If $\mathcal{Q}$ and $\mathcal{Q}'$ are ordered congruent-like quadrilaterals, then even $\overline{BC} = \overline{B'C'}$ and $\overline{CD} = \overline{C'D'}$, and it easily follows that they are congruent (Figure 7).

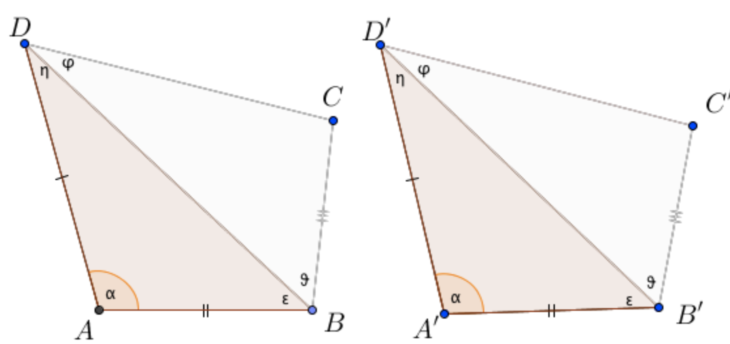

Figure 7: Here $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent.

By contradiction suppose that $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent-like quadrilaterals but not ordered. Then the sequence $\overline{DA} \to \overline{AB} \to \overline{BC} \to \overline{CD}$ must be different from $\overline{D'A'} \to \overline{A'B'} \to \overline{B'C'} \to \overline{C'D'}$ so that $\overline{BC} \neq \overline{B'C'}$, and by hypothesis we have $\overline{BC} = \overline{C'D'}$, and hence $\overline{CD} = \overline{B'C'}$ (see Figure 8).

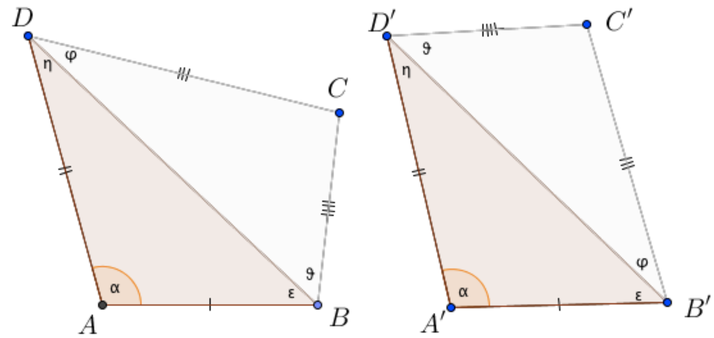

Figure 8: Here $\mathcal{Q}$ and $\mathcal{Q}'$ are not congruent-like quadrilaterals.

By hypothesis, the set of the inner angles of $\mathcal{Q}$ coincides with the set of the inner angles of $\mathcal{Q}'$. Therefore, referring to Figure 8, we have: $\eta+\varphi=\eta+\theta$ or $\eta+\varphi=\varphi+\epsilon$, then $\varphi=\theta$ or $\eta=\epsilon$. If $\varphi=\theta$, then $\overline{BC}=\overline{CD}$, $\overline{B'C'}=\overline{C'D'}$, so that in particular $\mathcal{Q}$ and $\mathcal{Q}'$ are ordered congruent-like quadrilaterals. If $\eta=\epsilon$ then $\overline{DA}=\overline{AB}$ and $\overline{D'A'}=\overline{A'B'}$, thus the sequences $\overline{DA}\to\overline{AB}\to\overline{BC}\to\overline{CD}$ and $\overline{A'B'}\to\overline{D'A'}\to\overline{D'C'}\to\overline{B'C'}$ coincide. This contradiction completes the proof. $\square$

Lemma 2.5. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be congruent-like quadrilaterals. If $\mathcal{Q}$ is a parallelogram, then $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent.*

Proof. We set $\mathcal{Q}=(A,B,C,D)$ and $\mathcal{Q}'=(A',B',C',D')$. Without loss of generality, we may suppose that $\overline{AB}=\overline{A'B'}$. By hypothesis we may assume that one of the two adjacent sides of AB is congruent to one of the adjacent sides of $A'B'$, say $\overline{DA}=\overline{D'A'}$. If $\hat{A}=\hat{A}'$, then $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent by the Side-Angle-Side criterion, applied to $\overline{DA}=\overline{D'A'}$, $\hat{A}=\hat{A}'$, $\overline{AB}=\overline{A'B'}$.
Thus we may suppose that $\hat{A}\neq\hat{A}'$. Then by hypothesis $\hat{A}'$ must be congruent to $\hat{B}=\hat{D}$. On the other hand $\overline{CD}=\overline{A'B'}$, thus $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent by the Side-Angle-Side criterion, applied to $\overline{CD}=\overline{A'B'}$, $\hat{D}=\hat{A}'$ and $\overline{DA}=\overline{D'A'}$. $\square$

Theorem 2.6. *(Angle-Side-Angle criterion) Let $\mathcal{Q}$ and $\mathcal{Q}'$ be congruent-like convex quadrilaterals. If they have a side together with the adjacent angles respectively congruent, then the quadrilaterals are congruent.*

Proof. In order to show that $\mathcal{Q}=(A,B,C,D)$ and $\mathcal{Q}'=(A',B',C',D')$ are congruent, we may suppose that they have $\overline{AB}=\overline{A'B'}$, $\hat{A}=\hat{A}'$ and $\hat{B}=\hat{B}'$.
First we note that if $\overline{B'C'}=\overline{BC}$ then we can apply the Side-Angle-Side criterion applied to $\overline{AB}=\overline{A'B'}$, $\hat{B}=\hat{B}'$ and $\overline{B'C'}=\overline{BC}$, so that $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent. By hypothesis BC must be congruent either to $C'D'$ or to $D'A'$. Therefore we may split the analysis into two main cases:

1) $\overline{C'D'}=\overline{BC}$.
In this case we have that the disposition of the sides and angles of $\mathcal{Q}'$ is prescribed. In fact, by the Side-Angle-Side criterion, we may suppose that $\overline{AD}\neq\overline{A'D'}$, and by hypothesis we have $\overline{A'D'}=\overline{DC}$, and $\overline{B'C'}=\overline{AD}$ (see Figures 9 and 10). By the Side-Angle-Side criterion, we may suppose that $\hat{D}'$ differs from $\hat{C}$. Therefore $\hat{D}'=\hat{D}$ and $\hat{C}'=\hat{C}$.

1.i) $\hat{A}+\hat{B}>\pi$.

Extending the adjacent sides of AB and $A'B'$, we determine two congruent triangles

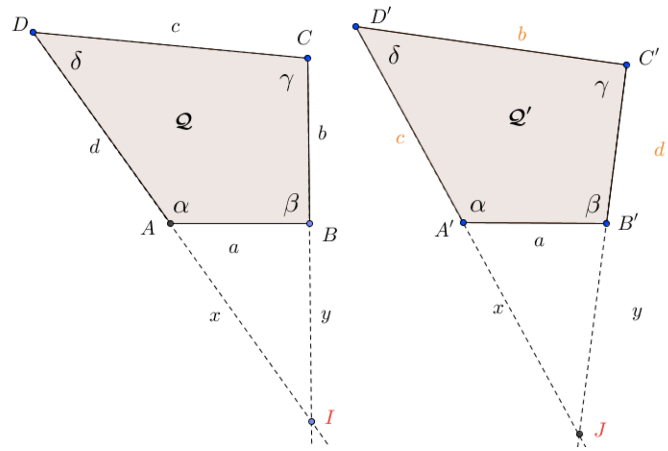

Figure 9: Case 1.i: $\overline{C'D'} = \overline{BC}$ and $\hat{A} + \hat{B} > \pi$.

(A, B, I) and (A', B', J). Moreover the triangles (I, C, D) and (J, C', D') are similar. Referring to the notation of Figure 9, the following proportion holds:

$$(y + b) : (y + d) = (x + d) : (x + c) = c : b.$$

If $b \geq c$ we have $d \geq b$ and $c \geq d$, thus $b = d = c$. And similarly, from $b \leq c$ follows that $b = d = c$. Therefore $b = c = d$, and $\mathcal{Q} = \mathcal{Q}'$ are congruent.

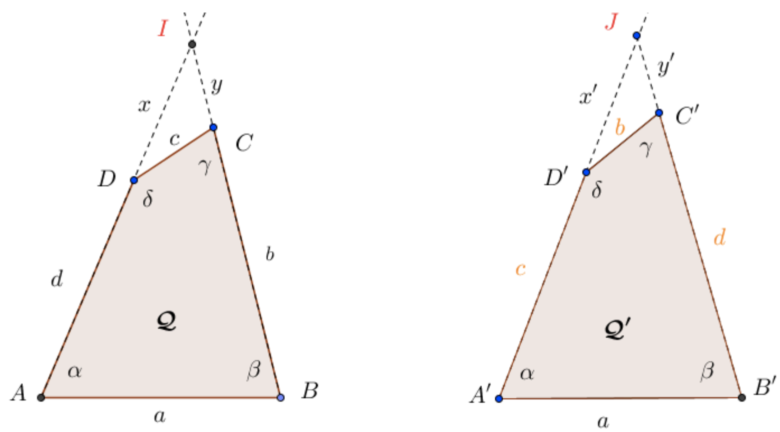

Figure 10: Case 1.ii: $\overline{C'D'} = \overline{BC}$ and $\hat{A} + \hat{B} < \pi$.

1.ii) $\hat{A} + \hat{B} < \pi$.
Extending the sides adjacent to AB and $A'B'$, we determine two congruent triangles (A, B, I) and (A', B', J) (see Figure 10). Moreover, the triangles (C, D, I) and (C', D', J) are similar, so that $x : x' = y : y' = c : b$. Thus if $b \geq c$ we have $y' \geq y$ and $x' \geq x$. On the other hand, the triangles (A, B, I) and (A', B', J) are congruent, so that $d+y' = b+y$, and $c+x' = d+x$. It follows that $d \leq b$ and $c \leq d$. This shows that $b = c = d$. Similarly if $c \geq b$, we have again $b = c = d$. Therefore $\mathcal{Q} = \mathcal{Q}'$ are congruent by case (1.i).

1.iii) $\hat{A} + \hat{B} = \pi$.
If $\hat{A} + \hat{D} < \pi$, elementary considerations show that $b < d$ and $d < c$. On the other hand, the triangles (K, C, D) and (K', C', D') are congruent, and in particular $b = c$, that is a contradiction (see Figure 11). Similarly if $\hat{A} + \hat{D} > \pi$ we get again a contradiction.

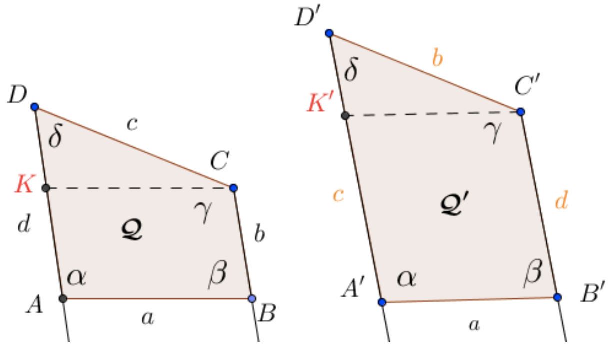

Figure 11: Case 1.iii: $\overline{C'D'} = \overline{BC}$ and $\hat{A} + \hat{B} = \pi$.

Therefore $\hat{A} + \hat{D} = \pi$, and $\mathcal{Q}$ is a parallelogram. It follows by Lemma 2.5 that $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent.

2) $\overline{D'A'} = \overline{BC}$.

It follows that either $\overline{B'C'} = \overline{DA}$ and $\overline{C'D'} = \overline{CD}$ hold or $\overline{B'C'} = \overline{CD}$ and $\overline{C'D'} = \overline{DA}$. In both cases, if $\hat{C'} = \hat{D}$ we have that $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent by the Side-Angle-Side criterion. Therefore we can assume that $\hat{C'} = \hat{C}$, and hence $\hat{D'} = \hat{D}$ (see Figures 12, 13 and 14). We split the analysis into three cases looking at the sum $\hat{A} + \hat{B}$. In each case we must take into account that either $\overline{B'C'} = \overline{DA}$ and $\overline{C'D'} = \overline{CD}$ hold or $\overline{B'C'} = \overline{CD}$ and $\overline{C'D'} = \overline{DA}$:

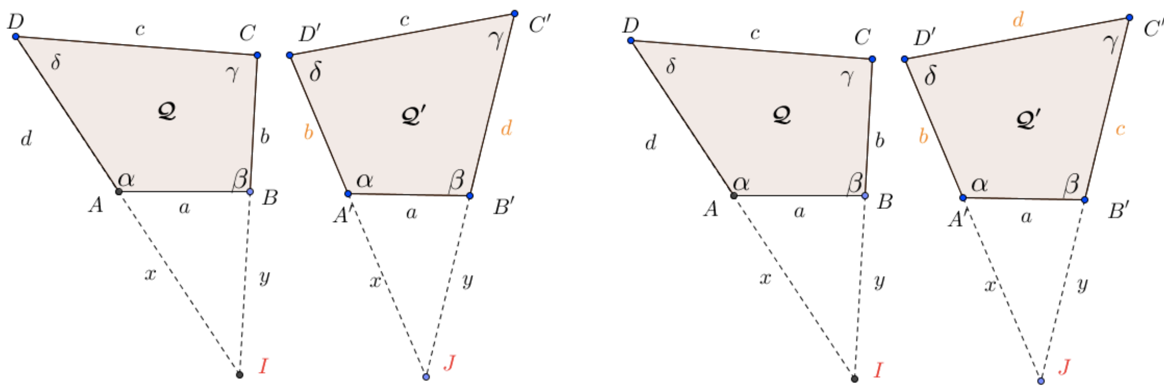

Figure 12: Case 2.i: $\overline{D'A'} = \overline{BC}$ and $\hat{A} + \hat{B} > \pi$. Here there are two possibilities.

2.i) $\hat{A} + \hat{B} > \pi$.

If $\overline{B'C'} = \overline{DA}$ and $\overline{C'D'} = \overline{CD}$ (see Figure 12, pair on the left), then the triangles (J, C', D') and (I, C, D) are congruent, so that $x + d = x + b$, that is, $d = b$ and $\mathcal{Q}$ is congruent to $\mathcal{Q}'$.

Suppose that $\overline{B'C'} = \overline{CD}$ and $\overline{C'D'} = \overline{DA}$ (see Figure 12, pair on the right). By the similarity of the triangles (J, C', D') and (I, C, D) we have: $c \leq d \iff d + x \leq b + x \iff b + y \leq c + y$ or $c \geq d \iff d + x \geq b + x \iff b + y \geq c + y$. In both cases we have $c = b = d$, and the assertion follows.

2.ii) $\hat{A} + \hat{B} < \pi$.

If $\overline{B'C'} = \overline{DA}$ and $\overline{C'D'} = \overline{CD}$ (see Figure 13 on the left), then the triangles (C, D, I) and (C', D', J) are congruent. On the other hand, (A, B, I) is congruent to (A', B', J) so that $d = b$, and $\mathcal{Q}$ is congruent to $\mathcal{Q}'$.

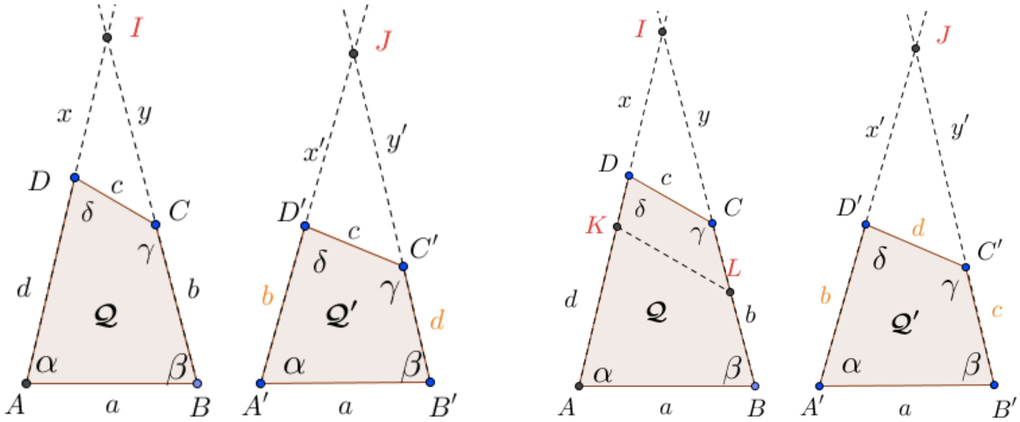

Figure 13: Case 2.ii : $\overline{D'A'} = \overline{BC}$ and $\hat{A} + \hat{B} < \pi$. Here there are two possibilities.

Suppose now that $\overline{B'C'} = \overline{CD}$ and $\overline{C'D'} = \overline{DA}$ (see Figure 13 on the right). If we set $a' = \overline{A'B'}$, $b' = \overline{B'C'}$, $c' = \overline{C'D'}$, $d' = \overline{D'A'}$, then $a = a'$, $b = d'$, $c = b'$, $d = c'$, and the assertion follows from the case 1.i .

2.iii) $\hat{A} + \hat{B} = \pi$.
Suppose that $\overline{B'C'} = \overline{DA}$ and $\overline{C'D'} = \overline{CD}$. If $\hat{A} + \hat{D} < \pi$ elementary considerations on $\mathcal{Q}$ and $\mathcal{Q}'$, respectively, show $b < d$ and $d < b$; that is a contradiction. Similarly if $\hat{A}+\hat{D} > \pi$ we get again a contradiction. Therefore $\hat{A}+\hat{D} = \pi$, and $\mathcal{Q}$ is a parallelogram. It follows by Lemma 2.5 that $\mathcal{Q}$ is congruent to $\mathcal{Q}'$.
Then we can suppose $\overline{B'C'} = \overline{CD}$ and $\overline{C'D'} = \overline{DA}$. Arguing as in the second part of 2.ii, if we set $a' = \overline{A'B'}$, $b' = \overline{B'C'}$, $c' = \overline{C'D'}$, $d' = \overline{D'A'}$, then $a = a'$, $b = d'$, $c = b'$, $d = c'$, and the assertion follows from the case 1.iii . □

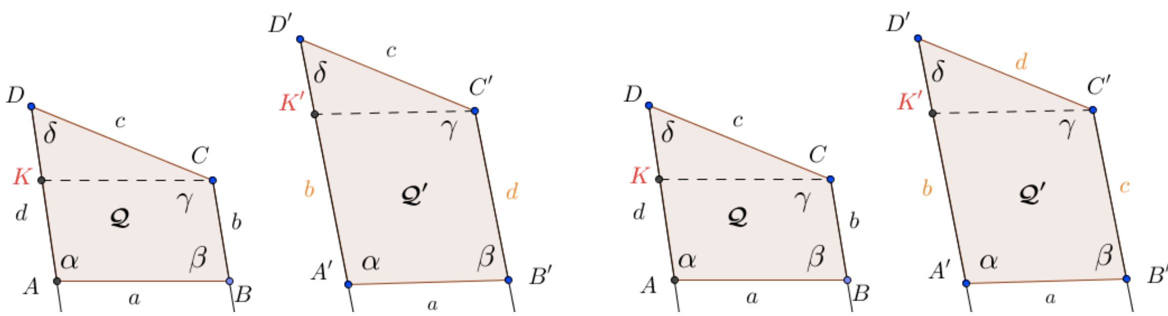

Figure 14: Case 2.iii : $\overline{D'A'} = \overline{BC}$ and $\hat{A} + \hat{B} = \pi$. Here there are two possibilities.

Remark 2.1. We highlight that the above criterions do not hold for n-gons when $n \geq 5$ as Figures 2 and 3 show.

3. Ordered congruent-like quadrilaterals

Lemma 3.1. *Let* $\mathcal{Q} = (A, B, C, D)$ *and* $\mathcal{Q}' = (A', B', C', D')$ *such that* $\overline{AB} = \overline{A'B'}$, $\overline{BC} = \overline{B'C'}$, $\overline{CD} = \overline{C'D'}$ *and* $\overline{DA} = \overline{D'A'}$. *Then* $\hat{A} > \hat{A}' \iff \hat{C} > \hat{C}' \iff \hat{B} < \hat{B}' \iff \hat{D} < \hat{D}'$.

Proof. We show the first sequence of implications. Suppose that $\hat{A} > \hat{A}'$, then by the Hinge' property $\overline{DB} > \overline{D'B'}$, and hence $\hat{C} > \hat{C}'$. Note that the sum of all internal angles of any quadrilateral is 2π, it turns out that is either $\hat{B} < \hat{B}'$ or $\hat{D} < \hat{D}'$. A new application of the Hinge' property yields: $\hat{B} < \hat{B}' \iff \overline{AC} < \overline{A'C'} \iff \hat{D} < \hat{D}'$, so that we have both the relations $\hat{B} < \hat{B}'$ and $\hat{D} < \hat{D}'$. The converse is specular. □

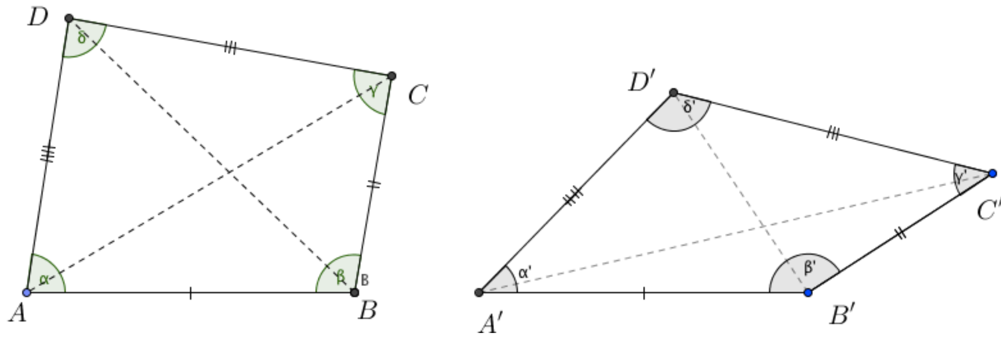

Figure 15: If one angle decreases, then the opposite one also decreases, and the other two increase.

Remark 3.1. Let $\mathcal{Q} = (A, B, C, D)$ and $\mathcal{Q}' = (A', B', C', D')$ be two ordered congruent-like quadrilaterals, then without loss of generality we may suppose that $\overline{AB} = \overline{A'B'}$, $\overline{BC} = \overline{B'C'}$, $\overline{CD} = \overline{C'D'}$, and $\overline{DA} = \overline{D'A'}$. In order to show that $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent, we may assume that $\hat{A}$ is the angle of $\mathcal{Q}$ of greatest radian measure, and that $\hat{B}$ is greater than $\hat{D}$. Therefore $\alpha + \beta \geq \pi$ and $\beta \geq \delta$.
It turns out that one of the following cases appears:

1) $\alpha \geq \beta \geq \gamma \geq \delta$;
2) $\alpha \geq \gamma \geq \beta \geq \delta$;
3) $\alpha \geq \beta \geq \delta \geq \gamma$.

According to the above cases, we will refer to $\mathcal{Q}$ as a quadrilateral of *type 1, 2 or 3*, respectively.

Lemma 3.2. *Let $\mathcal{Q} = (A, B, C, D)$ and $\mathcal{Q}' = (A', B', C', D')$ be ordered congruent-like quadrilaterals. If $\hat{A} = \hat{B}'$, $\hat{B} = \hat{C}'$, $\hat{C} = \hat{D}'$, and $\hat{D} = \hat{A}'$, then $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent.*

Proof. By hypothesis, we may assume the positions of Remark 3.1. If $\hat{A} = \hat{A}'$, the assertion follows from Theorem 2.4. So we can assume that $\hat{A} > \hat{A}' = \hat{D}$. By hypothesis and applying Lemma 3.1 we have

$$\hat{C} > \hat{C}' = \hat{B}, \quad \hat{B} < \hat{B}' = \hat{A}, \quad \hat{D} < \hat{D}' = \hat{C}.$$

Hence $\mathcal{Q}$ is of type 2, that is $\alpha \geq \gamma \geq \beta \geq \delta$, $\alpha \geq \pi/2$ and $\delta \leq \pi/2$.
Clearly we may suppose that $\mathcal{Q}$ is not a rectangle. If $\hat{B}$ is not acute, then $\hat{C}$ is even not acute so that by Lemma 2.3 applied to $\mathcal{Q}$ we have in particular that $a < c$. On the other hand, if we apply Lemma 2.3 to the quadrilateral $\mathcal{Q}'$ we have $c < a$. This is a contradiction. Then we can suppose that $\hat{B}$ is acute.
Consider now the triangles (A, B, D) and (A', B', C'), and note that $\overline{AB} = \overline{A'B'}$, $\hat{A} = \hat{B}'$ and the $A\hat{D}B$ and $A'\hat{C}'B'$ are both acute angles, as $\alpha \geq \pi/2$ (look at Figure 16, pairs of hatched triangles). An application of Lemma 2.2 yields

$$d \leq b \iff \overline{DB} \leq \overline{A'C'}. \tag{1}$$

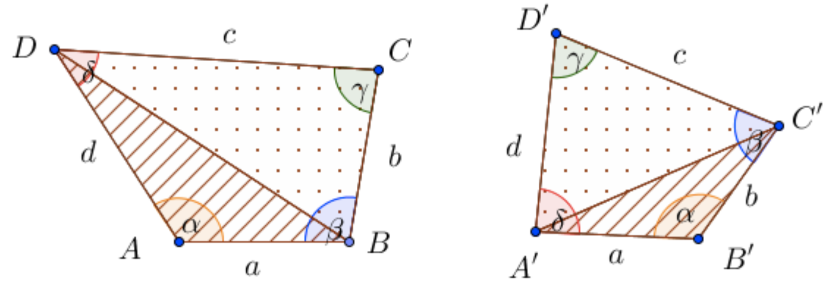

Figure 16: A special pair of ordered congruent-like quadrilaterals of type 2: they must be congruent.

Similarly, looking at the triangles (D, B, C) and (A', C', D') (look at Figure 16, pairs of dotted triangles), we note $\overline{CD} = \overline{C'D'}$, $\hat{C} = \hat{D}'$, and the angles $D'\hat{A}'C'$ and $C\hat{B}D$ are both acute ($C\hat{B}D \leq \beta$). Then by Lemma 2.2 follows that

$$d \leq b \iff \overline{A'C'} \leq \overline{DB}. \tag{2}$$

Relations (1) and (2) show that $d = b$. Therefore $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent by the Side-Angle-Side criterion for congruent-like quadrilaterals, as $\overline{DA} = \overline{C'B'}$, $\hat{A} = \hat{B}'$, $\overline{AB} = \overline{A'B'}$. □

It is useful to highlight that above result can be extended as follows:

Lemma 3.3. *Let $\mathcal{Q} = (A, B, C, D)$ and $\mathcal{Q}' = (A', B', C', D')$ be ordered congruent-like quadrilaterals. If $\hat{A} = \hat{D}'$, $\hat{B} = \hat{A}'$, $\hat{C} = \hat{B}'$, and $\hat{D} = \hat{C}'$, then $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent.*

Proof. It is enough to invert the role of $\mathcal{Q}$ and $\mathcal{Q}'$ in Lemma 3.2. □

Lemma 3.4. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be ordered congruent-like quadrilaterals. If $\hat{A} = \hat{B}'$, $\hat{B} = \hat{D}'$, $\hat{C} = \hat{A}'$, and $\hat{D} = \hat{C}'$, then $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent.*

Proof. By hypothesis, we may assume the positions of the Remark 3.1. If $\hat{A} = \hat{A}'$, the assertion follows from Theorem 2.4. So we can assume that $\hat{A} > \hat{A}' = \hat{C}$. By hypothesis and applying Lemma 3.1 we have

$$\hat{C} > \hat{C}' = \hat{D}, \quad \hat{B} < \hat{B}' = \hat{A}, \quad \hat{D} < \hat{D}' = \hat{B}. \tag{3}$$

As $\hat{D} = \hat{C}' < \hat{C}$, the case $\alpha \geq \delta \geq \beta \geq \gamma$ does not occur. Hence $\mathcal{Q}$ is either of type 1 or of type 2, that is, either $\alpha \geq \beta \geq \gamma \geq \delta$ or $\alpha \geq \gamma \geq \beta \geq \delta$.
By contradiction suppose that $\alpha + \delta \neq \pi$, so we have two cases:

1) $\alpha + \delta > \pi$.
In this case we can consider the configuration as in Figure 17.
We note that the triangles (D, A, H) and (B', C', H') are similar, and also the triangles (B, C, H) and (A', D', H') are similar, so that, referring to Figure 17, the following proportions hold:

$$d : b = x : x' = y : y' \tag{4}$$

and

$$d : b = c + x' : y + a = a + y' : c + x. \tag{5}$$

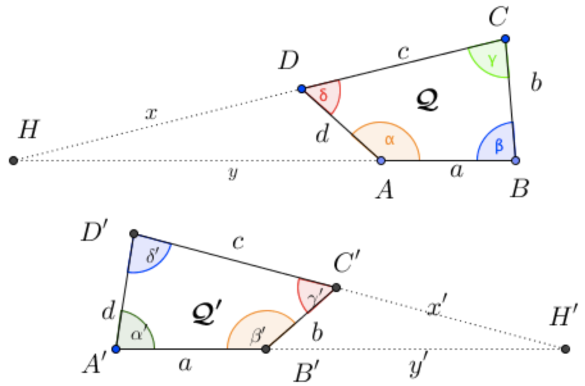

Figure 17: Case 1: $\alpha+\delta>\pi$. Here we have that $d=b$.

- Assume that $\beta \geq \gamma$, then we have $\overline{CH} \geq \overline{HB}$, that is $c+x \geq a+y$ (look at triangle (B, C, H)).
  If $d>b$, we have $c+x \geq a+y>a+y'$ by (4), so that $d<b$ by (5), a contradiction.
  If $d<b$, we have that $a+y \leq c+x<c+x'$ by (4), so that $d>b$ by (5), a contradiction.
- Assume that $\beta \leq \gamma$, then we have $\overline{CH} \leq \overline{HB}$, that is $c+x \leq a+y$.
  If $d>b$, we have that $c+x'<c+x \leq a+y$ by (4), so that $d<b$ by (5), a contradiction.
  If $d<b$, we have that $c+x \leq a+y<a+y'$ by (4), so that $d>b$ by (5), again a contradiction.

Then, if $\alpha+\delta>\pi$ then $d=b$.

2) Similarly, it can be proved that if $\alpha+\delta<\pi$ then $d=b$ (look at Figure 18).

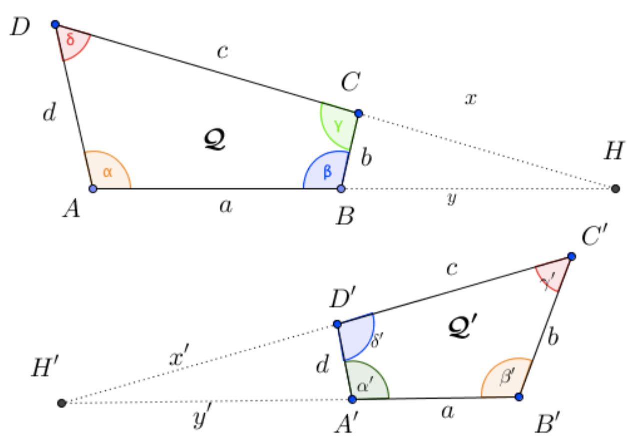

Figure 18: Case 2: $\alpha+\delta<\pi$. Here we have that $d=b$.

Therefore if $\alpha+\delta \neq \pi$ then $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent by the Angle-Side-Angle criterion (they have $d=b'$, and the respective angles $\alpha=\beta'$ and $\delta=\gamma'$).
Thus we may assume that $\alpha+\delta=\pi$. By relation (3) follows $\gamma>\delta$. In particular we have that $\alpha+\gamma=\alpha'+\beta'>\pi$, and $\mathcal{Q}$ and $\mathcal{Q}'$ are trapezoids as in Figure 19.
As $\mathcal{Q}$ is of type 1 or 2, then $\beta \geq \delta$, and hence $\alpha+\beta \geq \pi$. If $\alpha+\beta>\pi$, then $b<d$ (see figure on the left: AE is opposite to $\hat{D}$, and DA is opposite to $D\hat{E}A$ that has radian measure $\gamma>\delta$). On the other hand, if $\alpha'+\beta'>\pi$ then $d \leq b$ (see figure on the right), and this is a

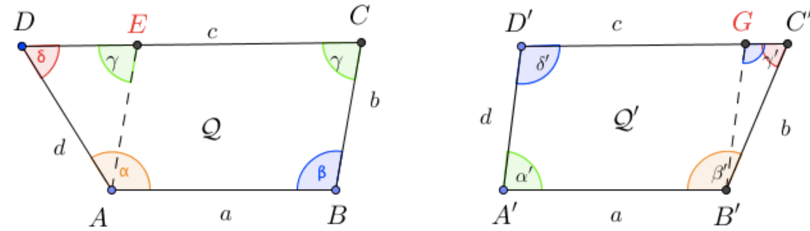

Figure 19: If $\alpha + \delta = \pi$, then $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent parallelograms.

contradiction. It follows that $\alpha + \beta = \pi$, and hence $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent parallelograms by Lemma 2.5. □

Theorem 3.5. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be ordered congruent-like convex quadrilaterals. Then $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent.*

Proof. Let $\mathcal{Q} = (A, B, C, D)$ and $\mathcal{Q}' = (A', B', C', D')$ be ordered congruent-like quadrilaterals, with the notations in agreement with Remark 3.1.
If $\hat{A} = \hat{A}'$, the assertion follows from the Side-Angle-Side criterion 2.4. So we can assume that $\hat{A} > \hat{A}'$, and by Lemma 3.1 we have $\hat{C}' < \hat{C}$, $\hat{B}' > \hat{B}$ and $\hat{D}' > \hat{D}$.
As $\hat{C}' < \hat{C}$, the case $\alpha \geq \beta \geq \delta \geq \gamma$ cannot occur. Hence $\mathcal{Q}$ is either of type 1 or 2, that is, either $\alpha \geq \beta \geq \gamma \geq \delta$ or $\alpha \geq \gamma \geq \beta \geq \delta$.

1) $\alpha \geq \beta \geq \gamma \geq \delta$.
It follows that $\hat{C}' = \hat{D}$ and $\hat{B}' = \hat{A}$. Thus $\hat{A}' \in \{\hat{B}, \hat{C}\}$, and we have two possibilities:

1.1) $\hat{C}' = \hat{D}$, $\hat{B}' = \hat{A}$, $\hat{A}' = B$, and $\hat{D}' = C$. It follows that $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent by the Angle-Side-Angle criterion 2.6 as $\hat{A} = \hat{B}'$, $\overline{AB} = \overline{A'B'}$, and $\hat{B} = \hat{A}'$.

1.2) $\hat{C}' = \hat{D}$, $\hat{B}' = \hat{A}$, $\hat{A}' = C$, and $\hat{D}' = B$. Here we may apply Lemma 3.4, so that $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent.

2) $\alpha \geq \gamma \geq \beta \geq \delta$.
It follows that $\hat{B}' \in \{\hat{A}, \hat{C}\}$, and $\hat{C}' \in \{\hat{B}, \hat{D}\}$, and we have four cases:

2.1) $\hat{B}' = \hat{A}$ and $\hat{C}' = \hat{B}$, that implies $\hat{D}' = C$and $\hat{A}' = D$. Here, by Lemma 3.2, $\mathcal{Q}$ is congruent to $\mathcal{Q}'$.

2.2) $\hat{B}' = \hat{A}$ and $\hat{C}' = \hat{D}$. This case splits in two.

  i) $\hat{A}' = C$ and $\hat{D}' = B$. Here we may apply Lemma 3.4, so that $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent.

  ii) $\hat{A}' = B$ and $\hat{D}' = C$. It follows that $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent by the Angle-Side-Angle criterion 2.6 as $\hat{A} = \hat{B}'$, $\overline{AB} = \overline{A'B'}$, and $\hat{B} = \hat{A}'$.

2.3) $\hat{B}' = \hat{C}$ and $\hat{C}' = \hat{B}$. Then $\mathcal{Q}$ and $\mathcal{Q}'$ are congruent again by the Angle-Side-Angle criterion 2.6.

2.4) $\hat{B}' = \hat{C}$ and $\hat{C}' = \hat{D}$, that implies $\hat{A}' = \hat{B}$ and $\hat{D}' = \hat{A}$. Here we may apply Lemma 3.3. Thus $\mathcal{Q}'$ is congruent to $\mathcal{Q}$.

This completes the proof. □

4. Conclusions

Citing H. Poincarè (see [2] and [9, p. 452]) we can say that

> *The definition will not be understood until you have shown not only the object defined, but the neighbouring objects from which it has to be distinguished, until you have made it possible to grasp the difference, and have added explicitly your reason for saying this or that in stating the definition.*

We think that the notion of *congruent-like polygons* may help to highlight and overcome some critical aspects of the definition of pairs of congruent polygons. In this direction another useful remark is the following:

If $\mathcal{P} = (A_0, \ldots, A_{n-1})$ and $\mathcal{P}' = (A'_0, \ldots, A'_{n-1})$ are congruent-like polygons then each side of $\mathcal{P}$ is congruent to a side of $\mathcal{P}'$, and each angle of $\mathcal{P}$ is congruent to an angle of $\mathcal{P}'$. In other words, we have the following equality referred to the sets of the measures of sides and angles of $\mathcal{P}$ and $\mathcal{P}'$:

$$\{a_0, a_1, \ldots, a_{n-1}\} = \{a'_0, a'_1, \ldots, a'_{n-1}\} \quad \text{and} \quad \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} = \{\alpha'_0, \alpha'_1, \ldots, \alpha'_{n-1}\}. \tag{6}$$

Two polygons satisfying the conditions in (6) will be called *weakly congruent.* Conversely, if $\mathcal{P}$ and $\mathcal{P}'$ are weakly congruent, and the measures of the sides and the angles of $\mathcal{P}$ are mutually different, then $\mathcal{P}$ and $\mathcal{P}'$ are congruent-like. But here we highlight that the converse is not true if the correspondences between sides and angles of $\mathcal{P}$ and $\mathcal{P}'$ are not one to one, as Figure 20 shows.

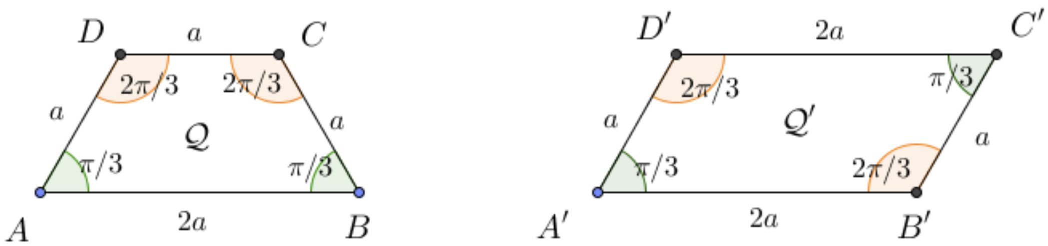

Figure 20: Pair of weakly congruent quadrilaterals that are not congruent-like. Note that erecting on sides DA and $D'A'$ an external triangle and so on, we can easily detect pairs of weakly congruent n-gons that are not congruent-like, for every $n > 4$.

Congruence theorems for quadrilaterals (and more generally for polygons) could appear to be a difficult topic for many learners. The reasons for such difficulties relate to the complexities in learning to analyze the attributes of different quadrilaterals and to distinguish between critical and non-critical aspects (see [2]). On the other hand, this kind of studies requires logical deduction to deal with many cases, together with suitable interactions between concepts and images, and so we may think that it is an important topic within mathematics education.

We conclude this article leaving some open questions. It should be a good exercise to attempt to investigate one of them:

1) Are there two convex (non-ordered) congruent-like quadrilaterals that are not congruent?
2) Are two convex ordered congruent-like pentagons necessarily congruent? In other words, does Theorem 3.5 hold for pentagons?
3) More general, one could ask: What can we state without restricting to convex polygons?

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