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On the Invariance of Brocard Angles in the Interior and Exterior Pappus Triangles of any Given Triangle

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Abstract. The paper presents groups of triangles inscribed in a given triangle ABC that have the same Brocard angle as $\triangle ABC$. Some of them are similar to $\triangle ABC$ and some are not. The central group consists of the exterior and interior Pappus triangles of $\triangle ABC$. We prove that the Miquel points of Pappus triangles produce pedal triangles that have the same Brocard angle as $\triangle ABC$. Furthermore, we prove that the locus of all points, that produce pedal triangles inside $\triangle ABC$ with the same Brocard angle, is a circle. For Miquel points of exterior Pappus triangles, we prove that all these points are located on the Brocard circle.

Key Words: Brocard angles, Brocard points, Brocard circle, Pappus triangles, Miquel points, symmedian point

MSC 2010: 51M04, 51M25

1. Brocard points and Brocard angle

It is known that every triangle ABC has two Brocard points M_1 and M_2 (Henry Brocard, French mathematician, 1845–1922). These points are isogonal conjugates (Figure 1), i.e., the lines AM_2, BM_2 and CM_2 are obtained by reflecting the lines AM_1, BM_1 and CM_1 in the bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively. The Brocard points satisfy

$$\angle M_1BA = \angle M_1AC = \angle M_1CB = \omega \quad \text{and} \quad \angle M_2AB = \angle M_2BC = \angle M_2CA = \omega\,.$$

The points M_1 and M_2 are also called the *first Brocard point* and the *second Brocard* point, respectively. Angle ω is called the *Brocard angle* of the triangle ABC [2, pp. 99–109] or [7]. It can be found by the formulas

$$\cot\omega = \frac{AB^2 + BC^2 + AC^2}{4\Delta}\,, \tag{1}$$

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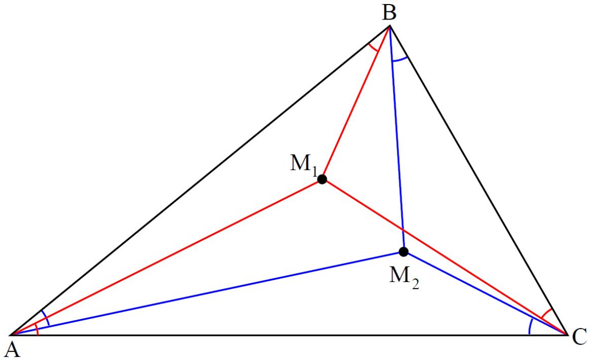

Figure 1: Two Brocard points M_1 and M_2

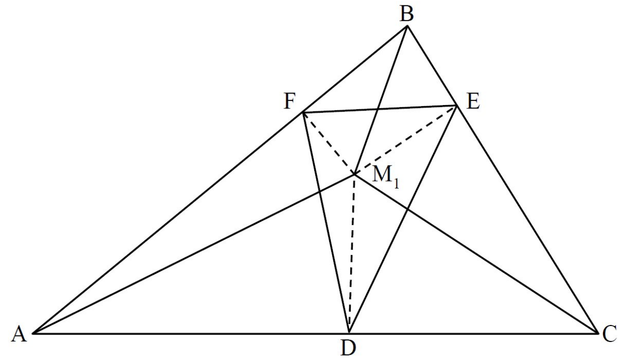

Figure 2: $\triangle DEF$ is the pedal triangle of M_1

where Δ denotes the area of the triangle, and

$$\cot\omega = \cot A + \cot B + \cot C, \tag{2}$$

$$\cot\omega = \frac{1 + \cos A \cdot \cos B \cdot \cos C}{\sin A \cdot \sin B \cdot \sin C}, \tag{3}$$

and others.

It is known that $\omega \le 30°$, where equality holds only for equilateral triangles [2, p. 103]. Note, that formula (2) implies that two similar triangles have the same Brocard angle.

2. Properties of Brocard points

There are many interesting properties related to the Brocard points and the Brocard angle. We present only those related to this paper.

1. The pedal triangles of Brocard points are congruent to each other, and similar to the triangle ABC (Figure 2). Moreover $\frac{FD}{AB} = \frac{FE}{BC} = \frac{ED}{AC} = \sin\omega$. Then $\frac{\Delta_1}{\Delta} = \sin^2\omega$, where Δ_1 denotes the area of pedal triangle FDE (Reminder: for the pedal triangle DEF of the point M_1, the segments DM_1, EM_1 and FM_1 are perpendicular to the sides of $\triangle ABC$).
2. The distance between the two Brocard points is equal to $M_1M_2 = 2R\sin\omega\sqrt{1-4\sin^2\omega}$, where R is the circumradius of triangle ABC. The distance d between the incenter O of the triangle ABC and each of the Brocard points is $d = R\sqrt{1-4\sin^2\omega}$ [9]. From these two formulas follows that in an isosceles triangle M_1OM_2 $\angle M_1OM_2 = 2\omega$. Therefore, for two triangles ABC and $A'B'C'$ that have the same Brocard angle, the triangles M_1OM_2 and $M_1'O'M_2'$ are similar with similarity ratio R/R'.
3. There is an interesting property related to an equilateral triangle. Let O be the circumcenter of the equilateral triangle ABC and $\mathcal{O}$ be any circle inside the triangle which has radius d and center O. Then all points M that lie on $\mathcal{O}$ have pedal triangles with the same area and the same Brocard angle (Figure 3).

Really, $\frac{\Delta_1}{\Delta} = \frac{R^2-d^2}{4R^2}$, where Δ_1 is the area of pedal triangle DEF of point M [3, pp. 139–140]. So we obtain

$$\Delta_1 = \frac{R^2-d^2}{4R^2} \cdot \frac{3\sqrt{3}\,R^2}{4} = \frac{3\sqrt{3}\,(R^2-d^2)}{16},$$

i.e., for all points M on $\mathcal{O}$ the pedal triangles have the same area.

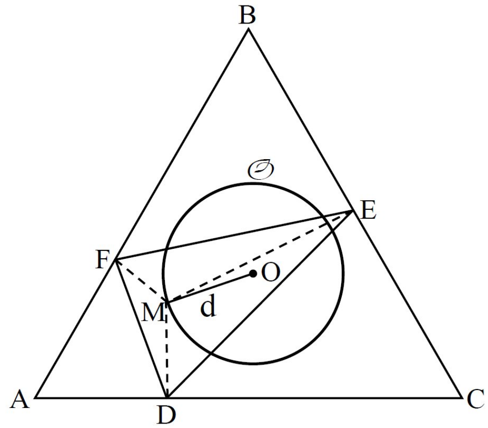

Figure 3: $\triangle DEF$ is the pedal triangle of point M lying on the circle $\mathcal{O}$ with the center O, the circumcenter of $\triangle ABC$

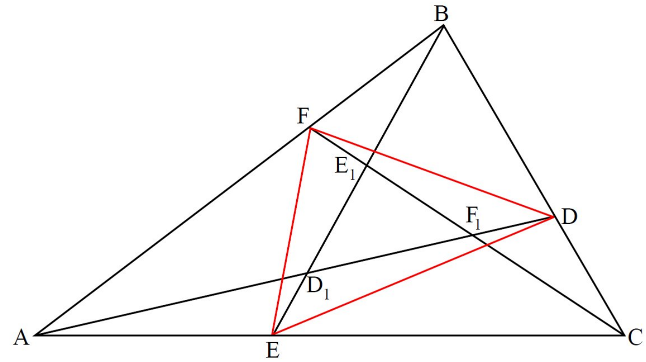

Figure 4: Exterior and interior Pappus triangles

It is known that for any triangle the following equality holds [1, p. 70]:

$$AM^2 + BM^2 + CM^2 = AO^2 + BO^2 + CO^2 + 3d^2.$$

Therefore, for the equilateral triangle we have $AM^2 + BM^2 + CM^2 = 3R^2 + 3d^2$. On the other hand, according to the sines' law, follows

$$AM \sin\angle A = FD, \quad BM = FE \sin\angle B, \quad CM = DE \sin\angle C,$$

and so

$$\sin^2 60^\circ \left(AM^2 + BM^2 + CM^2\right) = EF^2 + ED^2 + FD^2 = \frac{3}{4}(3R^2 + 3d^2)$$

and

$$\frac{EF^2 + ED^2 + FD^2}{4\Delta_1} = \frac{9(R^2 + d^2)}{16} \cdot \frac{16}{3\sqrt{3}\,(R^2 - d^2)} = \frac{\sqrt{3}\,(R^2 + d^2)}{R^2 - d^2}.$$

From this formula and from (1) we obtain the following for the pedal triangle DEF of any point M on $\mathcal{O}$:

$$\cot\omega = \frac{\sqrt{3}\,(R^2 + d^2)}{R^2 - d^2},$$

i.e., all these pedal triangles have the same Brocard angle.

3. Exterior and interior Pappus triangles

One of the Pappus theorems (Pappus of Alexandria, one of the great Greek mathematicians of antiquity, about 290 – 350 AD) discusses a triangle ABC where points D, E, F on the sides BC, AC, AB divide the sides in the same ratio [6, p. 53, problems 448–450].

For any given triangle ABC, let the points D, E, F on the sides BC, AC and AB (see Figure 4) satisfy the ratio

$$\frac{BF}{FA} = \frac{AE}{EC} = \frac{CD}{DB} = k, \quad 0 \le k < \infty.$$

We denote the points of intersection of BE and AD, of AD and CF and of BE and CF by D_1, F_1 and E_1 respectively. Let us call triangle DEF the *exterior k-related Pappus triangle* of $\triangle ABC$. Similarly, let us call triangle $D_1E_1F_1$ the *interior k-related Pappus triangle.*

It turns out that Pappus triangles have a number of interesting qualities in relation to the Brocard angle. We will examine some of them in this article. First, we should mention that any exterior or interior Pappus triangle of $\triangle ABC$ has the same Brocard angle as the triangle ABC. This fact appears as Theorem 476 in the book [3, p. 284], for exterior Pappus triangles without detailed proof. For interior Pappus triangles this fact is shown in [3, p. 284] as an exercise. These two qualities of Pappus triangles have a natural extension — they are also true for the external division of the sides of triangle ABC.

We will present detailed proofs of the statements formulated above.

Theorem 1. *For a given triangle ABC, all its exterior and interior k-related Pappus triangles have the same Brocard angle as $\triangle ABC$.*

We emphasize that the Brocard angle of the exterior and interior k-related Pappus triangles does not depend on k, i.e., it does not depend on the division ratio of the sides of $\triangle ABC$.

Proof a) for *exterior k-related Pappus triangles:*

Lemma 1. $\dfrac{\Delta_1}{\Delta} = \dfrac{k^2-k+1}{(k+1)^2}$, *where Δ_1 is the area of the triangle FDE.*

Proof. According to Routh's theorem [4], for a triangle ABC with points D, E, F dividing the sides in the ratios $CD/DB = \alpha$, $AE/EC = \beta$, $BF/FA = \gamma$, we have

$$\frac{\Delta_1}{\Delta} = \frac{1+\alpha\beta\gamma}{(1+\alpha)(1+\beta)(1+\gamma)}.$$

Since for the k-related Pappus triangle FDE holds $k = \alpha = \beta = \gamma$, we obtain

$$\frac{\Delta_1}{\Delta} = \frac{1+k^3}{(1+k)^3} = \frac{k^2-k+1}{(k+1)^2},$$

as stated. □

Lemma 2. $\dfrac{EF^2+FD^2+DE^2}{a^2+b^2+c^2} = \dfrac{k^2-k+1}{(k+1)^2}$, *where $BC = a$, $AC = b$, $AB = c$.*

Proof. From the law of cosines for the triangle FAE, one can easily obtain the formula

$$EF^2 = \frac{c^2(1-k)+b^2(k^2-k)+a^2k}{(k+1)^2}.$$

Similarly, one can obtain the formulas for FD^2 and DE^2. These formulas prove the statement of the lemma. □

Corollary 1. $\dfrac{\Delta_1}{\Delta} = \dfrac{EF^2+FD^2+DE^2}{a^2+b^2+c^2}.$

This formula is derived from the Lemmas 1 and 2.

Corollary 2. $\dfrac{EF^2+FD^2+DE^2}{4\Delta_1} = \cot\omega.$

This statement is derived from (1) and Corollary 1.

So for each k, where $0 \le k < \infty$, the triangles ABC and FDE have the same Brocard angle; thus Theorem 1 is proved for exterior k-related Pappus triangles.

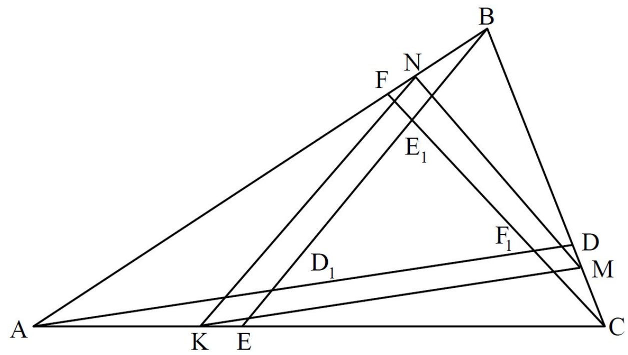

Figure 5: $\triangle KMN$ is similar to the Pappus triangle $D_1F_1E_1$

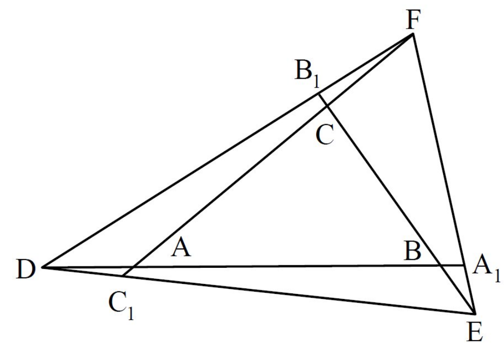

Figure 6: The case of the external division of sides of $\triangle ABC$

Proof of Theorem 1 b) for *interior k-related Pappus triangles:*
For this case, we prove the following lemma.

Lemma 3. *There exist points K, M, N on the sides AC, BC, AB, respectively, for which $NK \parallel BE$, $KM \parallel AD$ and $MN \parallel FC$ (Figure 5).*

Proof. The points K, M and N on the sides AC, CB and BA, respectively, satisfy

$$\frac{AK}{KC} = \frac{CM}{MB} = \frac{BN}{NA} = \frac{k}{k+1}\,.$$

Then $\dfrac{AC}{AK} = \dfrac{2k+1}{k}$. Since $\dfrac{AE}{EC} = k$, $\dfrac{AC}{AE} = \dfrac{k+1}{k}$ and so $\dfrac{AE}{AK} = \dfrac{2k+1}{k+1} = \dfrac{AB}{NA}$, i.e., $KN \parallel EB$. Similarly, one can obtain that $KM \parallel AD$ and $MN \parallel CF$. □

Corollary 3. *From Lemma 3 follows that $\triangle KMN$ is similar to the interior k-related Pappus $\triangle D_1F_1E_1$. Therefore these triangles have the same Brocard angle, which also is the Brocard angle of $\triangle ABC$.*

Corollary 4. *Since*

$$\frac{KN}{BE} = \frac{KM}{AD} = \frac{MN}{CF} = \frac{k+1}{2k+1}\,,$$

the three cevians BE, AD and CF form a triangle that is similar to $\triangle NKM$ and so it has the same Brocard angle as the original $\triangle ABC$. In particular, the triangle formed by the three medians of $\triangle ABC$, has the same Brocard angle as $\triangle ABC$.

This concludes the proof of Theorem 1. □

Theorem 2. *Let D, E and F be any points on extension of the sides of $\triangle ABC$ (Figure 6) such that*

$$\frac{AD}{BD} = \frac{EB}{EC} = \frac{CF}{FA} = k\,,$$

where $k < 1$. Then $\triangle DEF$ has the same Brocard angle as $\triangle ABC$.

Proof. Let denote the points of intersection of straight lines DB, FC, EA with straight lines FE, ED and FD by A_1, B_1, C_1, respectively. From $\dfrac{AD}{AD+AB} = k$ follows $AD = \dfrac{AB\cdot k}{1-k}$ and $BD = \dfrac{AB}{1-k}$. Similarly $BE = \dfrac{BC\cdot k}{1-k}$ and $EC = \dfrac{BC}{1-k}$.

By using Menelaus' theorem for $\triangle EDB$ that is intersected by a straight line passing through the points C, A and C_1, we obtain

$$\frac{EC}{BC}\cdot\frac{AB}{AD}\cdot\frac{DC_1}{C_1E}=1\,.$$

Then

$$\frac{BC/(1-k)}{BC}\cdot\frac{AB}{AB\cdot k/(1-k)}\cdot\frac{DC_1}{C_1E}=1\,,\quad \text{i.e.,}\quad \frac{DC_1}{C_1E}=k\,.$$

Similarly we obtain that $\frac{FA_1}{A_1E}=k$ and $\frac{EB_1}{B_1D}=k$. Therefore, $\triangle ABC$ is the interior k-related Pappus triangle of $\triangle DEF$, and consequently both have the same Brocard angle. □

4. Other triangles inscribed in $\triangle ABC$ with the same Brocard angle

An exterior Pappus triangle is not the only triangle inscribed in $\triangle ABC$ that has the same Brocard angle. There are many additional examples for such triangles, three of which will be shown below:

1. An example of a triangle that is not a Pappus triangle but similar to some exterior Pappus triangle.
2. An example of a triangle that is not a Pappus triangle itself and not similar to $\triangle ABC$, but has the same Brocard angle.
3. An example of a triangle that is not a Pappus triangle but is similar to $\triangle ABC$ (dividing $\triangle ABC$ into four triangles that are similar to $\triangle ABC$).

5. Pappus triangles and pedal triangles of a given triangle ABC

Let $\triangle LPS$ be a pedal triangle of any point M inside $\triangle ABC$. The question arises: Can the pedal triangle be a Pappus triangle? The following lemma gives the answer.

Lemma 4. *For a given triangle ABC, its only Pappus pedal triangle is the pedal triangle of the circumcenter of $\triangle ABC$.*

Proof. It is known [5, pp. 85–86] that $AS^2+CP^2+BL^2=SC^2+PB^2+LA^2$ (Figure 7). Since $SC=AC-AS$, $PB=BC-CP$, $AL=AB-BL$, one can easily obtain that

$$AC\cdot AS+BC\cdot CP+AB\cdot BL=\frac{1}{2}\,(AB^2+AC^2+BC^2).$$

If the pedal triangle LPS is also a k-related Pappus triangle of $\triangle ABC$ then

$$\frac{AS}{AC}=\frac{CP}{BC}=\frac{BL}{AB}=\frac{k}{k+1}\,,$$

and so

$$\frac{k}{k+1}(AB^2+AC^2+BC^2)=\frac{1}{2}\,(AB^2+AC^2+BC^2),\quad \text{hence}\quad k=1,$$

i.e., L, P and S are the midpoints of the sides of $\triangle ABC$ and M is its circumcenter. □

In the coming sections we show two examples of pedal triangles inscribed in $\triangle ABC$ that are not Pappus triangles, but have the same Brocard angle as $\triangle ABC$.

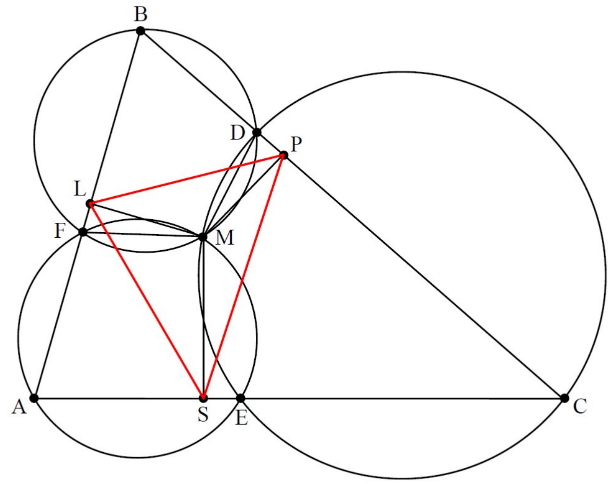

Figure 7: M is the Miquel point of $\triangle ABC$

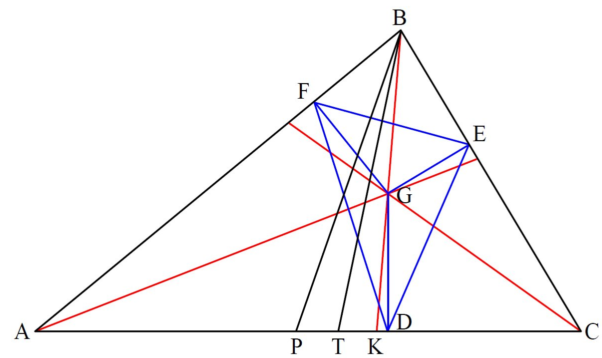

Figure 8: G is the symmedian point of $\triangle ABC$

6. Brocard angle and Miquel point

Let D, E, F be three points on the sides of $\triangle ABC$ (Figure 7). Then, according to Miquel's theorem, the circumcircles of the triangles AFE, BFD and CDE pass through the common point M called the *Miquel point* [3, pp. 131–133].

Lemma 5. *Let M be the Miquel point of the triangle ABC and LPS be the pedal triangle of M. Then $\triangle LPS$ is similar to $\triangle FDE$.*

Proof. $\angle LFM + \angle BDM = 180°$ and also $\angle PDM + \angle BDM = 180°$. Then $\angle LFM = \angle PDM = \alpha$, $\angle LMF = \angle PMD$ and $\angle LMP = \angle FMD$. Since $\dfrac{LM}{FM} = \sin\alpha = \dfrac{PM}{DM}$, the triangle LMP is similar to $\triangle FMD$ and $\dfrac{LP}{FD} = \sin\alpha$. Similarly, one can easily check that $\dfrac{LS}{FE} = \sin\alpha$ and $\dfrac{PS}{DE} = \sin\alpha$, i.e., the triangles DEF and PSL are similar. □

Corollary 5. *If $\triangle DEF$ is an exterior Pappus triangle of $\triangle ABC$, then the pedal triangle PSL of the Miquel point M has the same Brocard angle as $\triangle ABC$.*

7. Brocard angle and symmedian point

For a given triangle ABC, the symmedian from vertex B is the cevian BK obtained from median BP by reflection in the angle bisector BT (Figure 8). The three symmedians meet at one point G called the *symmedian point* (or Lemoine point or Grebe point) [2, pp. 57–58].

Lemma 6. *The pedal triangle FED of the symmedian point G has the same Brocard angle as $\triangle ABC$.*

Proof. We prove that $\triangle FED$ is similar to the triangle formed by medians m_a, m_b and m_c of $\triangle ABC$.

It is known that

$$\frac{GE}{a} = \frac{GD}{b} = \frac{GF}{c} = 0.5\tan\omega,$$

where ω is the Brocard angle of $\triangle ABC$ [2, p. 109]. Then

$$FD^2 = (0.5\tan\omega)^2(b^2+c^2+2bc\cos\angle A) = (0.5\tan\omega)^2(2b^2+2c^2-a^2).$$

Since $m_a^2 = 0.25\,(2b^2+2c^2-a^2)$ [3, p. 68], $FD^2 = m_a^2\tan^2\omega$, i.e., $FD = m_a\tan\omega$. Similarly we obtain that $FE = m_b\tan\omega$ and $ED = m_c\tan\omega$. Therefore $\triangle FED$ is similar to the triangle formed by the medians m_a, m_b and m_c, and both have the same Brocard angle that is equal to the Brocard angle of $\triangle ABC$. □

8. Brocard angle and dividing a triangle into four similar triangles

Lemma 7. *Let BD be the symmedian of $\triangle ABC$, $DE \parallel BC$ and $DF \parallel AB$ (Figure 9). Then each one of the triangles AED, DFC, FBE and EDF is similar to $\triangle ABC$ and so they have the same Brocard angle.*

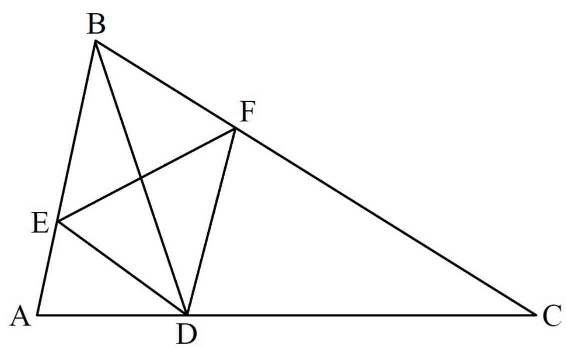

Figure 9: Dividing $\triangle ABC$ into four similar triangles with the help of the symmedian BD

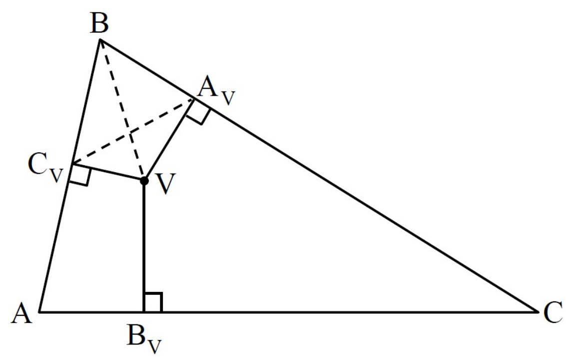

Figure 10: The quadrilateral $A_V V\, C_V B$ is inscribed in the circle with diameter BV

Proof. Obviously, the triangles AED and DFC are similar to $\triangle ABC$. Then $\dfrac{BC}{BF} = \dfrac{AB}{AE}$. Since for the symmedian BD holds that $\dfrac{AD}{DC} = \dfrac{AB^2}{BC^2}$ [2, p. 58], we obtain

$$\frac{AE^2}{BF^2} = \frac{AD}{DC} = \frac{AE}{DF}, \quad \text{hence} \quad \frac{AE}{ED} = \frac{ED}{DF}.$$

From this follows that the triangles AED and EDF are similar, and then all the triangles AED, DFC, FBE, and EDF are similar to $\triangle ABC$. □

9. Locus of Miquel points of exterior Pappus triangles

We have already proved that if M is the Miquel point created from the exterior k-related Pappus triangle DEF, then the pedal triangle of M is similar to the Pappus triangle DEF; so it has the same Brocard angle. When k changes from 0 to ∞, all Miquel points appropriated to Pappus triangles are created. What is the locus of these points? To answer this question we will prove first the following theorem.

Theorem 3. *Let P be some point in $\triangle ABC$ and let $A_PB_PC_P$ be the pedal triangle of P. Then the locus of all points V, whose pedal triangles have the same Brocard angle as $\triangle A_PB_PC_P$, is a circle.*

Proof. We denote the Brocard angle of $\triangle A_P B_P C_P$ by ω. According to Eq. (1),

$$\cot\omega = \frac{{A_P B_P}^2 + {B_P C_P}^2 + {A_P C_P}^2}{4\Delta},$$

where Δ is the area of $\triangle A_P B_P C_P$. We consider $\triangle ABC$ in some Cartesian coordinate system and denote the respective coordinates of the vertices A, B and C by (x_A, y_A), (x_B, y_B) and (x_C, y_C). Let point $V = (x, y)$ belong to the locus (Figure 10). Since the quadrilateral $A_V V C_V B$ is inscribed in the circle with diameter BV, we have

$$A_V C_V^2 = BV^2 \sin^2\angle B = \left[(x - x_B)^2 + (y - y_B)^2\right]\sin^2\angle B.$$

Similarly, we obtain

$$B_V C_V^2 = AV^2 \sin^2\angle A = \left[(x - x_A)^2 + (y - y_A)^2\right]\sin^2\angle A$$

and

$$B_V A_V^2 = CV^2 \sin^2\angle C = \left[(x - x_C)^2 + (y - y_C)^2\right]\sin^2\angle C.$$

So we can conclude that

$$A_V B_V^2 + B_V C_V^2 + A_V C_V^2 = Mx^2 + My^2 + Nx + Py + Q,$$

where M, N, P, and Q are constants for $\triangle ABC$.
The area Δ of the pedal triangle $A_V B_V C_V$ of the point V can be calculated by the formula $\Delta = \frac{R^2 - d^2}{4R^2}\cdot\Delta_{ABC}$, where Δ_{ABC} is the area of triangle ABC, R is its circumradius, and d is the distance between V and the circumcenter O of the triangle ABC. Then

$$4\Delta = 1 - \frac{\Delta_{ABC}}{R^2}\left[(x_O - x)^2 + (y_O - y)^2\right] = Fx^2 + Fy^2 + Ex + Hy + J,$$

where F, E, H, and J are constants. So

$$\cot\omega = \frac{Mx^2 + My^2 + Nx + Py + Q}{Fx^2 + Fy^2 + Ex + Hy + J}.$$

Since $\cot\omega$ is constant, one can easily conclude, that the locus of the point $V = (x, y)$ is a circle. $\square$

Remark. It is easy to see that property 3 of the Brocard points mentioned above and related to an equilateral triangle, is a corollary of Theorem 3.

If O is the circumcenter of $\triangle ABC$ and G is the symmedian point, then the circle with diameter OG is called the *Brocard circle*. It is known that the Brocard circle passes through the two Brocard points of $\triangle ABC$, which are symmetric with respect to OG. The diameter OG can be calculated by the formula

$$OG = \frac{R\sqrt{1 - 4\sin^2\omega}}{\cos^2\omega},$$

where R is the circumradius and ω is the Brocard angle of $\triangle ABC$ [8].

Theorem 4. *For any given triangle ABC, all Miquel points of exterior Pappus triangles of $\triangle ABC$ are located on the Brocard circle of $\triangle ABC$.*

Proof. According to Theorem 3, all Miquel points are on a circle because they all have the same Brocard angle as $\triangle ABC$. In addition, the pedal triangles of the circumcenter O, of the symmedian point G and of the two Brocard points of $\triangle ABC$ have the same Brocard angle as $\triangle ABC$. Then from Theorem 3 we conclude that all Miquel points, O, G, and the two Brocard points are located on the same circle, which is the Brocard circle of $\triangle ABC$. □

Corollary. *The triangle inscribed in $\triangle ABC$, that has the same Brocard angle as $\triangle ABC$ and has minimal area, is a pedal triangle of the symmedian point G.*

Indeed, on the Brocard circle of $\triangle ABC$, point G is most distant from O. Since

$$\Delta = \frac{R^2 - d^2}{4R^2} \cdot \Delta_{ABC}\,,$$

the minimal area Δ of the pedal triangles of the points on the Brocard circle is reached for the maximal value of d, i.e., for the pedal triangle of point G. In this case

$$d^2 = OG^2 = \frac{R^2(1 - 4\sin^2\omega)}{\cos^4\omega} \quad \text{and so} \quad \Delta_{\min} = \frac{3}{4}\tan^2\omega \cdot \Delta_{ABC}.$$

References

[1] N. Altshiller-Court: *College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle*. 2nd ed., rev. enl., Barnes and Noble, New York 1952.

[2] R. Honsberger: *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*. Math. Assoc. Amer., Washington DC 1995.

[3] R.A. Johnson: *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*. Houghton Mifflin, Boston/MA 1929.

[4] V. Oxman, A. Sigler, M. Stupel: *Surprising Relations between the Areas of Triangles in the Configuration of Routh's Theorem*. J. Geometry Graphics **18**/2, 197–201 (2014).

[5] A.S. Posamentier, C.T. Salkind: *Challenging Problems in Geometry*. 2nd revised ed., Dover Publishing Co., New York 1996.

[6] G. Titeica: *Culegere de Probleme de Geometrie* [in Romanian]. Editura Technica, Bucuresti 1965.

[7] E.W. Weisstein: *Brocard Angle*. From MathWorld – A Wolfram Web Resource, `http://mathworld.wolfram.com/BrocardAngle.html`.

[8] E.W. Weisstein: *Brocard Circle*. From MathWorld – A Wolfram Web Resource, `http://mathworld.wolfram.com/BrocardCircle.html`.

[9] E.W. Weisstein: *Brocard Points*. From MathWorld – A Wolfram Web Resource, `http://mathworld.wolfram.com/BrocardPoints.html`.

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