Journal for Geometry and Graphics
Volume 21 (2017), No. 1, 79–88.

Some More Surprising Properties of the "King" of Triangles

Ruti Segal[1], **Avi (Berman) Sigler**[2], **Moshe Stupel**[3]

[1] *Shaanan Religious College of Education (and Oranim College of Education)*
P.O.B. 906, Haifa, Israel
email: `rutisegal@gmail.com`

[2] *Shaanan Religious College of Education, Haifa, Israel*
email: `avibsigler@gmail.com`

[3] *Shaanan Religious College of Education (and Gordon College College of Education)*
email: `stupel@bezeqint.net`

Abstract. The article describes eleven "new" unique geometric properties of an equilateral triangle, which are apparently not recognized to the most how engaged Euclidean Geometry. Mathematical proofs were given to all of the properties, using various mathematical tools. Other research directions were proposed that allow for additional properties discovery.

Key Words: equilateral triangle, geometry of triangles, geometric properties discovery

MSC 2010: 51M04, 51M25

1. Introduction

The equilateral triangle is the most elaborate triangle among other triangles, since all three angles are of 60° and the lengths of the all sides are equal. The equilateral triangle appears to be quite simple when compared to other types of triangles: scalene triangles, isosceles triangles, right-angled triangles, acute-angled triangles and obtuse-angled triangles. It turns out this is not the case. Basic properties of an equilateral triangle are well known, such as:

- All the angle bisectors, the medians, the altitudes and the perpendicular bisectors of the triangle intersect at a single point.
- This point of intersection is both the center of the circle circumscribing the triangle and the center of the circle inscribed in the triangle. It is also the center of gravity of the triangle, and the Fermat point of the triangle.

ISSN 1433-8157/$ 2.50

- The three medians of the triangle divide it into six congruent triangles (this property also holds for the other lines: angle bisectors, altitudes etc.). It is important to note that in an arbitrary triangle we would obtain six triangles of equal area, which in general are not congruent.

These properties appear in the textbooks of Euclidean geometry. New unique properties are surprising and appear in few resources [1, 2, 3, 5]. Therefore we believe that our "new" properties that appear in the article are original at least to us.

The properties were discovered by presenting ideas and testing their correctness by combining different mathematical tools and using known theorems.

The more one considers the equilateral triangle and thinks multi-directionally, the more one can find additional properties and relations. The process of investigation and discovery contributes to know hidden properties in the equilateral triangle and to see the beauty of mathematics.

2. Eleven Properties

Property 1. *Given is an equilateral triangle $\triangle ABC$. If D is a point on BC (see Figure 1) then there holds*

$$AD > BD, DC \quad \text{and} \quad AD < AB.$$

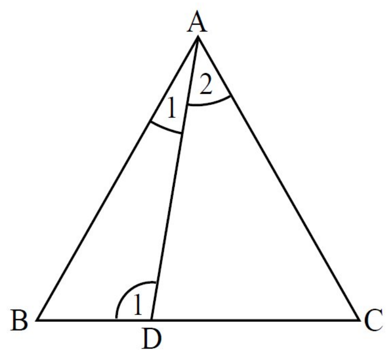

Figure 1: Inequality between triangle sides

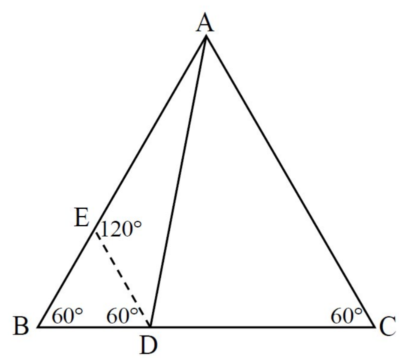

Figure 2: Building a triangle with 120° angle

Proof. $\angle ADB > \angle C$, therefore $\angle ADB > \angle B = 60°$. Hence $AB > AD$, since in the triangle $\triangle ABD$ the larger side lies opposite the larger angle.
Each of the angles, $\angle BAD$ and $\angle DAC$, is smaller than $60°$. Therefore in the triangle $\triangle ABC$ we have $BD < AD$ since $\angle B > \angle BAD$ and in the triangle $\triangle ADC$ we have $DC < AD$ since $\angle C > \angle DAC$. □

Property 2. *Given an equilateral triangle $\triangle ABC$ and some point D on the side BC (see Figure 2), then from the segments AD, BD and DC one can build a triangle, one of whose angles is equal to* $120°$.

Proof. We draw a line DE parallel to AC with $E \in AB$. It is clear that the triangle $\triangle BED$ is equilateral. Therefore $BD = DE$. The trapezoid $DEAC$ is an isosceles trapezoid, and therefore $DC = EA$. Hence it follows that the sides of the triangle $\triangle AED$ are equal to the segments AD, BD and CD. In the triangle $\triangle AED$ we have $\angle AED = 120°$ (supplementary angle to an angle of $60°$). Thus the property has been proven. □

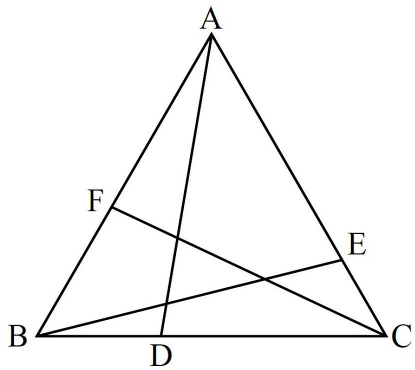

Figure 3: Triangle inequality of segment lengths

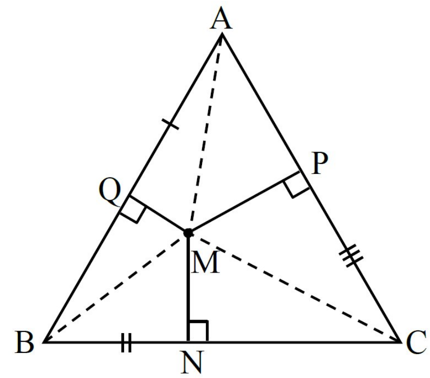

Figure 4: $AQ + BN + CP = \frac{3}{2}\,a$

Property 3. *Given is an equilateral triangle $\triangle ABC$. The points D, E and F are located somewhere on the sides BC, AC and AB, respectively (see Figure 3).*
Then the lengths of the segments CF, BE and AD satisfy the triangle inequality, and the triangle that can be constructed from them is acute-angled.

Proof. It is enough to show, without loss of generality, that $AD^2 + CF^2 > BE^2$, since if this holds, then clearly $AD + CF > BE$.
We denote by a the side length of the triangle $\triangle ABC$ and by h the altitude ($h = \frac{a}{2}\sqrt{3}$). From Property 1 we have:

$$a^2 > AD^2 \geq h^2 = \frac{3a^2}{4}\,, \qquad a^2 > CF^2 \geq h^2 = \frac{3a^2}{4}\,, \qquad a^2 > BE^2 \geq h^2 = \frac{3a^2}{4}\,.$$

We plug the sum of the first two inequalities into the third inequality:

$$AD^2 + CF^2 \geq 2h^2 = \frac{3}{2}\,a^2 > BE^2.$$

In the same manner we obtain:

$$AD^2 + BE^2 > CF^2 \quad \text{and} \quad CF^2 + BE^2 > AD^2.$$

The proof that the triangle whose sides are the segments BE, CF and AD is acute-angled is obtained by applying the Law of Cosines:

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}\,.$$

When $b^2 + c^2 > a^2$, we have $\cos\alpha > 0$ and therefore $\alpha < 90^\circ$, which also applies to the other angles of the triangle. □

Property 4. *Given is an equilateral triangle $\triangle ABC$. Let M be some point inside the triangle, from which altitudes are drawn to the sides of the triangle (see Figure 4). We denote by N, P and Q the points of intersection of the altitudes with the sides of the triangle. Then there holds*

$$AQ + BN + CP = \frac{3}{2}\,a\,,$$

in other words, independent of the location of the point M, the sum of these segments equals half the perimeter of the triangle.

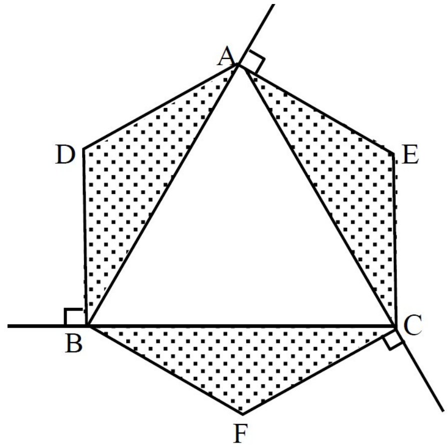

Figure 5: Proof of Property 4

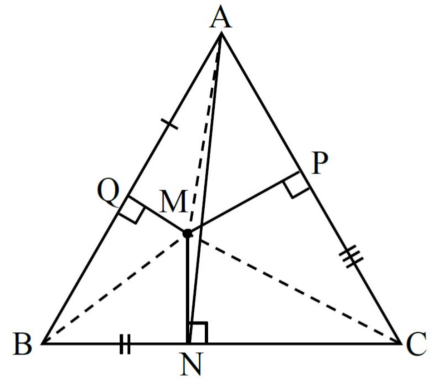

Figure 6: $2(MN+MP+QM) \leq AM+BM+CM$

Proof. From the Pythagorean Theorem in the triangles $\triangle AQM$ and $\triangle BQM$ we have

$$QM^2 = AM^2 - AQ^2 = BM^2 - BQ^2 \implies AM^2 - BM^2 = AQ^2 - BQ^2.$$

In the same manner, for the other two pairs of triangles, we obtain

$$BM^2 - MC^2 = BN^2 - NC^2 \quad \text{and} \quad MC^2 - AM^2 = PC^2 - AP^2.$$

Summing up the three equations, we obtain

$$AQ^2 + BN^2 + PC^2 = BQ^2 + NC^2 + AP^2,$$

and by expressing the segments BQ, NC and AP using the side a of the triangle, we obtain

$$3a^2 - 2a(AQ + BN + PC) = 0,$$

or, $AQ + BN + PC = \frac{3}{2}\,a$, which is half the perimeter of the triangle. □

Note: This property is also maintained for points M in the regions AEC, CFB, and BDA, outside the triangle, because then the verticals still reach all three sides of the triangle (see Figure 5).

Property 5. *Given is an equilateral triangle $\triangle ABC$. Let M be some point inside the triangle, from which altitudes are drawn to the sides of the triangle (see Figure 6). We denote by N, P and Q the points of intersection of the altitudes with the sides of the triangle. Then*

$$2(MN + MP + MQ) \leq AM + BM + CM,$$

and the equality holds when M is the orthocenter of the triangle.

Proof. It is clear that $AM + MN \geq AN \geq h$ (h is the altitude of the triangle). In the same manner, one obtains the relations $CM + MQ \geq h$ and $BM + MP \geq H$. By summing up these three relations, one obtains

$$(AM + BM + CM) + \underbrace{(MN + MQ + MP}_{h} \geq 3h\,.$$

From the known property that the sum of the distances of any point in an equilateral triangle from the sides of the triangle is a constant value that equals the altitude of the triangle, i.e., $MN+MQ+MP=h$, we obtain that $AM+BM+CM \geq 2h$, and therefore $AM+BM+CM \geq 2(MN+MP+MQ)$. □

Note: There exists a theorem, which generalize this property to any arbitrary triangle (Erdös, Mordell and Barrow, 1935 [2]). It must be emphasized that the proof of this theorem for an arbitrary triangle is very difficult.

Property 6. *Given the same data as in Property 5, there holds:*
The values of $CM \cdot MQ$, $BM \cdot MP$ and $AM \cdot MN$ satisfy the triangle inequality.

Proof. At the first stage we prove the following trigonometric claim:

If the angles α, β and γ satisfy $\alpha, \beta, \gamma < 180°$ and also $\alpha+\beta+\gamma = 360°$, then the values of $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ satisfy the triangle inequality.

Proof of the trigonometric claim: If γ_1, β_1 and α_1 are the angles of a triangle whose sides are a, b and c, respectively, then from the Law of Sines we have

$$a = 2R\sin\alpha_1, \quad b = 2R\sin\beta_1, \quad c = 2R\sin\gamma_1.$$

Since the lengths a, b and c of the sides satisfy the triangle inequality, then $\sin\gamma_1$, $\sin\beta_1$ and $\sin\alpha_1$ also satisfy the triangle inequality. Denoting $\gamma = 180° - \gamma_1$, $\beta = 180° - \beta_1$ and $\alpha = 180° - \alpha_1$, there holds

$$\sin\gamma = \sin\gamma_1, \quad \sin\beta = \sin\beta_1, \quad \sin\alpha = \sin\alpha_1 \quad \text{and also} \quad \alpha+\beta+\gamma = 360°.$$

Hence, it is also clear that $\sin\alpha$, $\sin\beta$ and $\sin\gamma$ satisfy the triangle inequality.

Proof of Proposition 6: We multiply each of the products $CM \cdot MQ$, $BM \cdot MP$ and $AM \cdot MN$ with $\frac{a}{2}$ and obtain

$$\begin{aligned}
AM \cdot MN \cdot \frac{a}{2} &= AM \cdot \frac{MN \cdot a}{2} = AM \cdot S_{\triangle BMC} = \frac{AM \cdot BM \cdot CM}{2} \cdot \sin\angle BMC, \\
BM \cdot MP \cdot \frac{a}{2} &= BM \cdot S_{\triangle AMC} = \frac{AM \cdot BM \cdot CM}{2} \cdot \sin\angle AMC, \\
CM \cdot MQ \cdot \frac{a}{2} &= CM \cdot S_{\triangle AMB} = \frac{AM \cdot BM \cdot CM}{2} \cdot \sin\angle AMB.
\end{aligned}$$

In each of the last three products there is a common factor multiplied by the sine of an angle. Since it was proven that $\sin\angle AMB$, $\sin\angle AMC$ and $\sin\angle BMC$ satisfy the triangle inequality, it is also clear that the same holds for the products $CM \cdot MQ$, $BM \cdot MP$ and $AM \cdot MN$. □

Property 7. *Given the same data as in Property 5, there holds $\frac{AM}{QP} = \frac{BM}{QN} = \frac{CM}{PN}$.*
In other words, the triangle $\triangle QNP$ is similar to the triangle that can be constructed from segments with the lengths AM, BM and CM.

Proof. The quadrilateral $AQMP$ can be inscribed in a circle in which AM is a diameter. From the Law of Sines in the triangle , we obtain $QP = AM \cdot \sin 60°$. We continue in the same manner for the quadrilaterals $BQMN$ and $CPMN$ and get $QN = BM \cdot \sin 60°$ and $PN = CM \cdot \sin 60°$. From the three relations follows

$$\frac{AM}{QP} = \frac{BM}{QN} = \frac{CM}{PN} = \frac{1}{\sin 60°}.$$ □

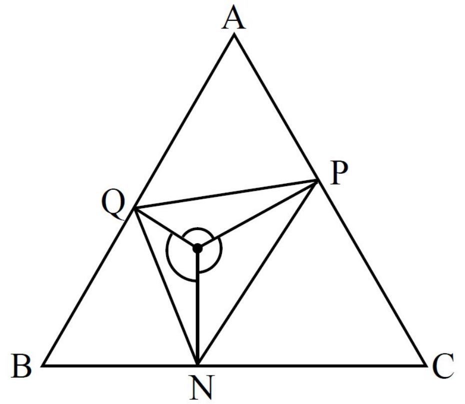

Figure 7: Inequality of areas

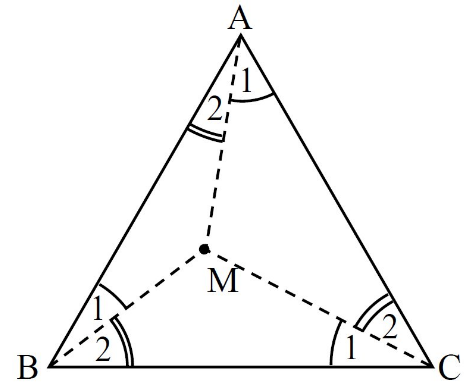

Figure 8: One angle larger or smaller than 30°

Property 8. *Given the same data as in Property 5, there holds* $S_{\triangle QPN} \leq \frac{1}{4} S_{\triangle ABC}$.

Proof. At the first stage we shall prove the following inequality:

$$(x+y+z)^2 \geq 3xy + 3yz + 3xz \quad \text{for all} \quad x, y, z \in \mathbb{R}. \tag{1}$$

For this purpose we first prove that the following inequality always holds:

$$x^2 + y^2 + z^2 \geq xy + yz + xz. \tag{2}$$

The proof relies on summing up the three known inequalities

$$x^2 + y^2 \geq 2xy, \qquad y^2 + z^2 \geq 2yz, \qquad x^2 + z^2 \geq 2xz$$

(because $(x-y)^2 \geq 0$).

Proof of the inequality (1): We expand $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$. Substituting the inequality (2), we obtain $(x+y+z)^2 \geq 3xy + 3yz + 3xz$.

We denote the distances from the point M to the sides by $MN = x$, $MP = y$ and $QM = z$ (see Figure 7). We already mentioned the property that the altitude of the triangle equals $h = x + y + z$, and hence

$$\begin{aligned} h^2 &= (x+y+z)^2 \geq 3(xy+yz+xz) = 3\left(\frac{2S_{\triangle QMN}}{\sin 120^\circ} + \frac{2S_{\triangle MPN}}{\sin 120^\circ} + \frac{2S_{\triangle QMP}}{\sin 120^\circ}\right) \\ &= \frac{6S_{\triangle QNP}}{\sin 120^\circ} = \frac{12}{\sqrt{3}} S_{\triangle QNP} \qquad \text{or} \qquad S_{\triangle QNP} \leq \frac{h^2\sqrt{3}}{2}. \end{aligned}$$

It is known that the expression for the area of an equilateral triangle in terms of the altitude h is $S_{\triangle ABC} = \frac{h^2}{\sqrt{3}}$, hence $h^2 = \sqrt{3}\, S_{\triangle ABC}$, and therefore, by substituting in the obtained inequality, we have $S_{\triangle QNP} \leq \frac{1}{4} S_{\triangle ABC}$. The maximum of the area of the triangle is obtained when $x = y = z$, in other words, the point M is the center of the equilateral triangle. □

Property 9. *The vertices of an equilateral triangle cannot be the grid points of a Cartesian system of coordinates. (Grid points are points, both of whose coordinates are integers).*

Proof. An indirect proof shall be given. If the side length of the triangle is a then its area is $\frac{a^2\sqrt{3}}{4}$. If the points A and B are grid points, then it is clear that a^2 is an integer, based on the Pythagorean Theorem. However, if C is also a grid point, then the area of the triangle is an integer or half of an integer, due to the formula for calculating the area of a triangle in a system of coordinates as

$$S = \pm\frac{1}{2}\begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}.$$

Hence, the expression for the area is rational. However, a contradiction is obtained because $\sqrt{3}$ is irrational. □

Property 10. *Given is an equilateral triangle $\triangle ABC$, and an internal point M in the triangle which is not the point of intersection of the angle bisectors of the triangle. We denote the angles formed at the three vertices as shown in Figure 8. Then the following property holds: At least one of the angles $\angle A_1$, $\angle B_1$ and $\angle C_1$ is smaller than $30°$, and, at least one of the angles $\angle A_1$, $\angle B_1$ and $\angle C_1$ is larger than $30°$.*

Proof. To prove the property, we first prove the following claim, which is the trigonometric representations of Ceva's theorem:

$$\sin\angle A_1 \cdot \sin\angle B_1 \cdot \sin\angle C_1 = \sin\angle A_2 \cdot \sin\angle B_2 \cdot \sin\angle C_2 . \tag{3}$$

By using the Law of Sines in each of the triangles formed by connecting the point M with the vertices of the triangle, we obtain

$$\begin{aligned} &\text{in triangle } \triangle AMB: && \frac{AM}{MB} = \frac{\sin\angle B_1}{\sin\angle A_2}, \\ &\text{in triangle } \triangle BMC: && \frac{BM}{MC} = \frac{\sin\angle C_1}{\sin\angle B_2}, \\ &\text{in triangle } \triangle CMA: && \frac{CM}{MA} = \frac{\sin\angle A_1}{\sin\angle C_2}. \end{aligned}$$

By multiplying these three relations, we obtain

$$\frac{\sin\angle A_1 \cdot \sin\angle B_1 \cdot \sin\angle C_1}{\sin\angle A_1 \cdot \sin\angle B_1 \cdot \sin\angle C_1} = 1 .$$

Thus the claim (3) is proved.

Therefore, if each of the angles $\angle A_1$, $\angle B_1$ and $\angle C_1$ is larger than $30°$, then each of the angles $\angle A_2$, $\angle B_2$ and $\angle C_2$ will be smaller than $30°$, and the relation (3) does not hold. Conversely, if all of the angles $\angle A_1$, $\angle B_1$ and $\angle C_1$ are smaller than $30°$, then each of the angles $\angle A_2$, $\angle B_2$ and $\angle C_2$ will be larger than $30°$, and again, relation (3) does not hold. □

Property 11. *Given is an equilateral triangle $\triangle ABC$, and some points D, E and F on the sides BC, AC and AB, respectively. The points and the segments that connect them divide the triangle into four triangles (see Figure 9). We use the notations π for the product of the side lengths of the triangle, P for the sum of the side lengths, S for the area, and Σ^2 for the sum of the squares of the side lengths of the triangle. Then the following inequalities hold:*

$$\begin{aligned} &\text{(a)} && P_{\triangle DEF} \geq \min\{P_{\triangle AFE},\ P_{\triangle BDF},\ P_{\triangle CED}\}, \\ &\text{(b)} && \pi_{\triangle DEF} \geq \min\{\pi_{\triangle AFE},\ \pi_{\triangle BDF},\ \pi_{\triangle CED}\}, \\ &\text{(c)} && \Sigma^2_{\triangle DEF} \geq \min\left\{\Sigma^2_{\triangle AFE},\ \Sigma^2_{\triangle BDF},\ \Sigma^2_{\triangle CED}\right\}, \\ &\text{(d)} && S_{\triangle DEF} \geq \min\{S_{\triangle AFE},\ S_{\triangle BDF},\ S_{\triangle CED}\}. \end{aligned}$$

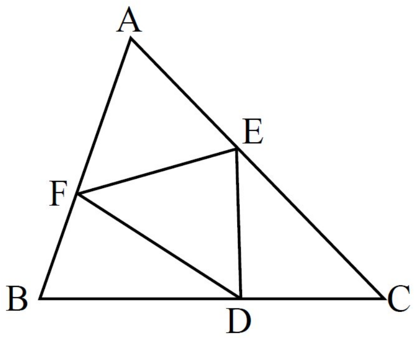

Figure 9: Inequalities between areas, perimeters, multiples and squared triangle sides

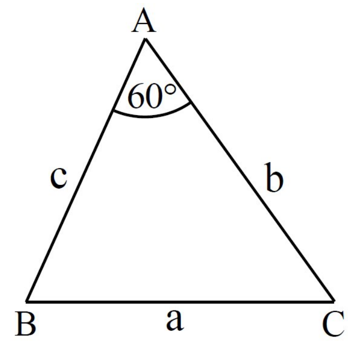

Figure 10: Inequalities between areas, perimeters, multiples and squared triangle sides

This implies, for example, that when an equilateral triangle is divided into four triangles, as shown in the figure, the probability that the middle triangle shall have the smallest perimeter is zero.

As a preparation to the proof of Proposition 11 we provide the following

Lemma. *In every triangle $\triangle ABC$ with $\angle A = 60°$ (see Figure 10) the following relations hold:*

$$2a^2 \geq b^2 + c^2, \tag{4}$$
$$a^2 > bc, \tag{5}$$
$$2a \geq b + c. \tag{6}$$

Proof. From the Law of Cosines follows

$$a^2 = b^2 + c^2 - bc, \quad \text{hence} \quad b^2 + c^2 = a^2 + bc \leq a^2 + \frac{b^2 + c^2}{2}$$

(due to the inequality between the geometric average and average of the squares). By transposing terms we obtain the relation (4).
From the inequality between averages, there holds $b^2 + c^2 \geq 2bc$, and therefore from (4) we obtain (5), $a^2 \geq bc$.
From (4) follows $4a^2 \geq 2b^2 + 2c^2 \geq (b + c)^2$, and by taking the square roots we obtain (6). □

Proof of Property 11, (a): From (6) one obtains

after adding EF to both sides: $2EF \geq AF + AE$,
after adding DF to both sides: $2DF \geq BF + BD$,
after adding DE to both sides: $2ED \geq DC + EC$.

The sum of the three relations yields

$$EF + DF + DE = P_{\triangle DEF} \geq \frac{P_{\triangle AEF} + P_{\triangle BDF} + P_{\triangle DCE}}{3}.$$

Thus property (a) has been proved.

Property (b): From (5) in the lemma, one obtains

after multiplying both sides with EF: $AF \cdot AE \leq FE^2$,
after multiplying both sides with FD: $BF \cdot BD \leq FD^2$,
after multiplying both sides with ED: $DC \cdot EC \leq ED^2$.

The product of the three inequalities yields

$$\pi^3_{\triangle DEF} \geq \pi_{\triangle AFE} \cdot \pi_{\triangle CED} \cdot \pi_{\triangle BDF} \, .$$

Thus property (b) has been proved.

Property (c): From relation (4) one obtains

$$2FE^2 \geq AF^2 + AE^2, \qquad 2FD^2 \geq BF^2 + BD^2, \qquad 2ED^2 \geq CD^2 + CE^2.$$

The sum of the three inequalities gives

$$2(FE^2 + FD^2 + ED^2) \geq AF^2 + AE^2 + BF^2 + BD^2 + CD^2 + CE^2.$$

We add $\Sigma^2_{\triangle DEF} = FE^2 + FD^2 + ED^2$ to both sides of this inequality and obtain

$$\Sigma^2_{\triangle DEF} \geq \frac{\Sigma^2_{\triangle AEF} + \Sigma^2_{\triangle BDF} + \Sigma^2_{\triangle CED}}{3}.$$

Thus we have proved property (c).

Property (d): In every triangle the largest angle is larger than or equal to $60°$. Let us assume that the angle $\angle FDE$ is the largest in $\triangle DEF$. We distinguish two cases:

Case I: $60° \leq \angle FDE \leq 120°$,

Case II: $\angle FDE > 120°$.

Proof for Case I: As product of the three inequalities $\sin \angle FDE \geq \sin 60°$, $FD^2 \geq BF \cdot BD$ and $ED^2 \geq DC \cdot DE$ we obtain $S^2_{\triangle DEF} \geq S_{\triangle BDF} \cdot S_{\triangle CED}$. Therefore $S_{\triangle DEF} \geq \min\{S_{\triangle BDF},\ S_{\triangle CED}\}$ and further $\geq \min\{S_{\triangle AFE},\ S_{\triangle BDF},\ S_{\triangle CED}\}$.

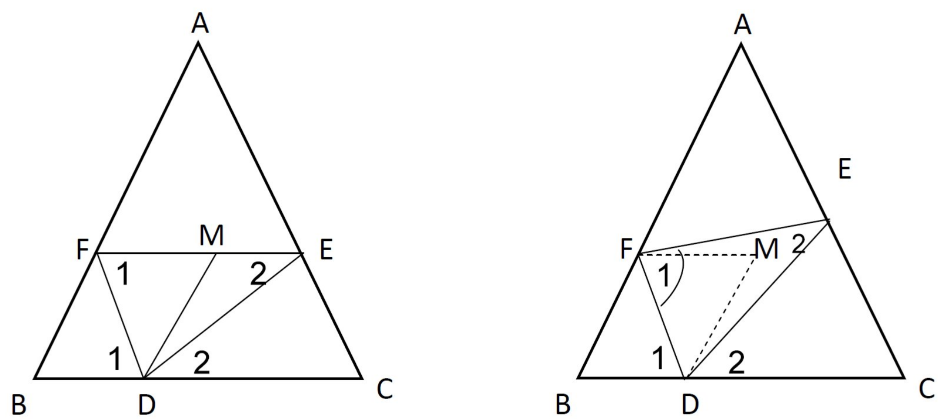

Figure 11: Proof of Case II, left: subcase a), right: subcase b)

Proof for Case II: Let point M extend the triangle FBD to the parallelogram $FBDM$. Then $\angle BDM = 120° < \angle BDE$. On the other hand, using the notation given in Figure 11, left, we have $\angle D_1 + \angle D_2 = \angle F_1 + \angle E_2$.

a) In the case $\angle D_1 = \angle F_1$ the side EF passes through M and is parallel to BC. Hence $S_{\triangle DEF} > S_{\triangle DMF} = S_{\triangle BDF}$.

b) Without loss of generality, the case $\angle D_1 \neq \angle F_1$ can be reduced to $\angle D_1 < \angle F_1$. Then M lies in the interior of $\triangle DEF$. Therefore again $S_{\triangle DEF} > S_{\triangle DMF} = S_{\triangle BDF}$, where the latter is $\geq \min\{S_{\triangle AFE},\ S_{\triangle BDF},\ S_{\triangle CED}\}$. $\square$

Remark. Property (d) is valid for all triangles.

3. Conclusions

In this study, we provided 11 essential properties that must be valid in all equilateral triangles. It would be interesting to ask which properties are sufficient to characterize equilateral triangle among all triangles. We found that Properties 2, 4, 6, and 7 are sufficient to characterize equilateral triangle.

Directions for further investigation of special properties in an equilateral and arbitrary triangle:

(a) Direction of investigation — adding data (“what if in addition?”)

(b) Direction of investigation — replacing data (“what if instead?”)

(c) Direction of investigation — generalization (“what if not?”)

(d) Direction of investigation — Is the property found in the equilateral triangle also valid in an arbitrary triangle?

These questions are supported by Brown and Walter [1], who proposed the WIN method and opened up courses for prospective teacher’s based upon this method. Polya [6] and Brown & Walter [1] described in a similar way enquiry skills: changing data, reducing data, adding data, analogy, looking for invariants, and checking extreme cases. The WIN method includes all the skills mentioned by Polya as well as the essential elements of diagnostics and creating an attributes list.

References

[1] S.I. Brown, M.I. Walter: *Problem Posing in Mathematics Education.* In S.I. Brown, M.I. Walter (eds.): *Problem Posing Reflection and Application*, Lawrence Erlbaum Associates, Hillsdale, New Jersey, 1993, pp. 16–27.

[2] P. Erdös, L.J. Mordell, D.F. Barrow: *Problem 3740.* Amer. Math. Monthly **42**, p. 396 (1935); solutions, ibid., **44**, 252–254 (1937).

[3] R. Honsberger: *Equilateral Triangles.* In *Mathematical Gems I*, Math. Assoc. of America, Washington DC 1973, Chapter 3.

[4] N. Khanderia, A. Katz, K.N. Thanh: *Properties of Equilateral Triangles.* `https://brilliant.org/wiki/properties-of-equilateral-triangles/`, 2016.

[5] J. Sándor: *On the geometry of equilateral triangles.* Forum Geometricorum **5**, 107–117 (2005).

[6] G. Polya: *Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving.* Vol. 1, John Wiley & Sons, New York 1962.

[7] De Prithwijit: *Curious properties of the circumcircle and incircle of an equilateral triangle.* Mathematical Spectrum **41**/(1), 32–35 (2008).

Received May 31, 2016; final form December 13, 2016