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Rotor Coordinates and Vector Trigonometry

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Abstract. Rational trigonometry is a purely algebraic approach to trigonometry which uses quadrance and spread instead of distance and angle for metrical measurements. In this paper we introduce a variant called vector trigonometry, which is useful for planar applied engineering problems where vector quantities are involved. We derive basic trigonometric laws involving rotor coordinates of length and half-slope.

Key Words: rational trigonometry, vector trigonometry, rotor coordinates, half-slope

MSC 2010: 51N20, 70A05, 97G60

1. Introduction

Rational trigonometry, introduced in 2005 in [8], see also [9], is a purely algebraic approach to trigonometry which uses *quadrance* and *spread* instead of *distance* and *angle* for metrical measurements. This approach has now led to a range of new developments in planar Euclidean geometry [5], [6], chromogeometry [10], [11], and non-Euclidean geometries [12], [13], [14], [15], [1], and [2].

In this paper we introduce a variant of *rational trigonometry* called *vector trigonometry,* which is useful for planar applied engineering problems where vector quantities are involved. This is an applications oriented framework which replaces the usual polar coordinates r and θ of a vector $\mathbf{v}$ with *rotor coordinates* r and h, and we write $\mathbf{v} = |\,r, h\rangle$. The quantity r is the usual length, so this trigonometry does have an approximate aspect, in that approximate square roots will be needed. The *half-slope* h may be defined in terms of the rational parametrization of the unit circle c_U with equation $x^2 + y^2 = 1$, given by

$$\mathbf{e}(h) \equiv \left(\frac{1-h^2}{1+h^2}, \frac{2h}{1+h^2} \right) \equiv (C(h), S(h)) \,. \tag{1}$$

We say that h is the half-slope of the vector $\mathbf{v} = \mathbf{e}(h)$, or any positive multiple of $\mathbf{v}$. It turns out that $h = \tan(\theta/2)$; but the actual definition is independent of both angles and circular functions. Figure 1 shows a vector $\mathbf{v}$ and its half-slope h, viewed geometrically.

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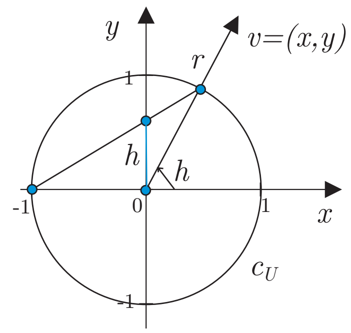

Figure 1: Rotor coordinates $|r,h\rangle$ for $\mathbf{v}=(x,y)$

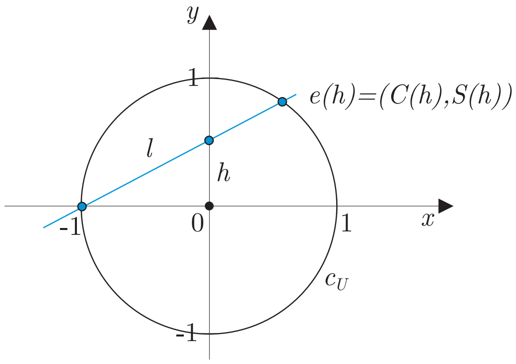

Figure 2: Rational parametrization of the unit circle

Aspects of this theory will be familiar to readers, coming from stereographic projection and the projective rational parametrization of a conic, and connecting to the Cayley transform of linear algebra, and also to well-known trigonometric formulas. The half-slope as the quantity $\tan(\theta/2)$ is implicit in the Weierstrass substitution when integrating circular functions, and has also historically played an important role in kinematics. In fact the *kinematic mapping* developed by W. Blaschke and J. Grünwald (1911) maps the point (x,y,z) of Euclidean 3-space onto the rotation in the plane $z=0$ with center (x,y) and angle θ such that $z=\cot(\theta/2)=1/h$ (see [4, p. 399ff], [3]). The present paper attempts to frame the corresponding trigonometry, avoiding transcendental functions, and hence allowing a wider range of practical examples and computations with vectors, even by hand.

1.1. A quick review of RT

We now give a brief overview of the principal notions of Euclidean rational trigonometry ([8], [9]) and then introduce the corresponding ideas for vector trigonometry. Given a vector $\mathbf{v}\equiv(x,y)$, its **quadrance** is the number $Q(\mathbf{v})\equiv x^2+y^2$. Length is then regarded as a secondary concept, namely the square root of the quadrance. Given two vectors $\mathbf{v}_1\equiv(x_1,y_1)$ and $\mathbf{v}_2\equiv(x_2,y_2)$, the **spread** between them is the number

$$s(\mathbf{v}_1,\mathbf{v}_2)\equiv\frac{(x_1y_2-x_2y_1)^2}{(x_1^2+y_1^2)(x_2^2+y_2^2)}. \tag{2}$$

The spread $s(\mathbf{v}_1,\mathbf{v}_2)$ is the square of the usual sine of an angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, but pleasantly requires no prior definitions of angular measure or circular functions. The purely algebraic aspect means that quadrance and spread are valid concepts over a general field F, and in fact they can be framed in more general geometries built from other bilinear forms.

If lines l_1 and l_2 have direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then we define $s(l_1,l_2)\equiv s(\mathbf{v}_1,\mathbf{v}_2)$. Note that this quantity is independent of order or orientation, and of re-scaling; the spread is really defined between lines, not rays. There are some closely related secondary concepts. The **cross** between the two lines is

$$c(l_1,l_2)\equiv\frac{(x_1x_2+y_1y_2)^2}{(x_1^2+y_1^2)(x_2^2+y_2^2)}=1-s(l_1,l_2)$$

while the **twist** is

$$t(l_1,l_2)\equiv\frac{s(l_1,l_2)}{c(l_1,l_2)}=\frac{(x_1y_2-x_2y_1)^2}{(x_1x_2+y_1y_2)^2}.$$

Since the twist is always a square, we define also the **turn**

$$u(l_1, l_2) \equiv \frac{x_1 y_2 - x_2 y_1}{x_1 x_2 + y_1 y_2}$$

which is an *oriented* quantity; in fact $u(l_2, l_1) = -u(l_1, l_2)$. These concepts were introduced and applied in [8].

In this paper we introduce a *directed* version of these ideas, allowing us to define a trigonometry on the (oriented) vectors themselves. This is well-suited for practical engineering and scientific applications in which direction plays a role. We will develop some basic trigonometry with this new technology, and see that the associated algebra itself is quite symmetrical and pleasant.

2. The unit circle and the Cayley transform

Polar coordinates arise from the transcendental parametrization of the unit circle c_U with equation $x^2 + y^2 = 1$ given by $\varphi(\theta) \equiv (\cos\theta, \sin\theta)$. In practice this generates vectors which are only *approximately* of unit length. There is a much older, and more exact, *rational parametrization:*

$$\mathbf{e}(h) \equiv (C(h), S(h)) \tag{3}$$

where

$$C(h) \equiv \frac{1-h^2}{1+h^2} \quad \text{and} \quad S(h) \equiv \frac{2h}{1+h^2} \tag{4}$$

are the **capital** C and **capital** S **functions** respectively.

This implicitly goes back to Euclid's construction of Pythagorean triples. Geometrically $\mathbf{e}(h)$ is the point where the line l through $[-1, 0]$ and $[0, h]$ meets c_U. If h is rational, then l will have rational coordinates, and since one of its meets with c_U is rational, the other will be also. The converse also holds; any rational point on c_U is of the form $\mathbf{e}(h)$, provided we also allow h to take on the *extended value* ∞, so that $e(\infty) = (-1, 0)$. Other common examples are $e(0) = (1, 0)$, $e(1) = (0, 1)$ and $e(-1) = (0, -1)$.

The rational parametrization has a modern formulation in terms of linear algebra. If X is a skew-symmetric matrix for which $I + X$ is invertible, then the **Cayley transform** of X may be defined to be the orthogonal matrix

$$c(X) \equiv \frac{I - X}{I + X}.$$

In the 2×2 case, if

$$X = \begin{pmatrix} 0 & -h \\ h & 0 \end{pmatrix} \quad \text{then} \quad c(X) = \begin{pmatrix} C(h) & S(h) \\ -S(h) & C(h) \end{pmatrix} \equiv \sigma_h. \tag{5}$$

If we also define

$$\sigma_\infty \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

which is consistent (in a limiting sense) with $h = \infty$, then the orthogonal matrices σ_h for h an extended rational number (that is, including the value ∞) bijectively represent rational rotations. This gives us an *algebraic* alternative to the usual complex exponential map between the line and the group of rotations.

3. The rational functions C, S, T and M

The capital C and S functions $C(h)$ and $S(h)$ given by (4) may be combined to define the **capital T function**

$$T(h) \equiv \frac{S(h)}{C(h)} = \frac{2h}{1-h^2}.$$

These three functions have graphs, over the rational numbers, as shown in Figure 3.

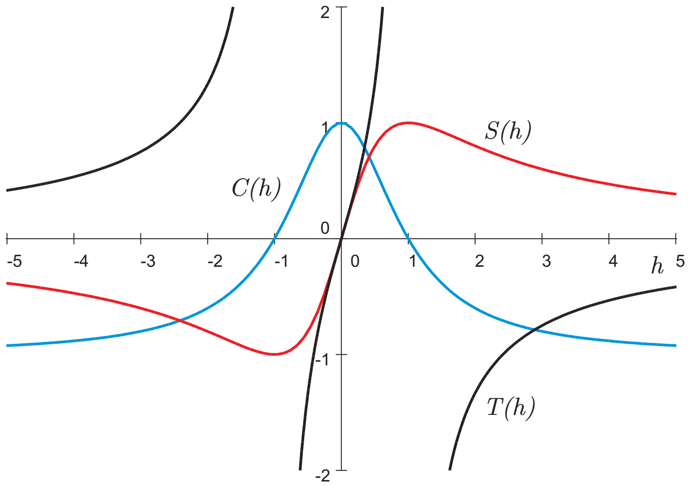

Figure 3: Graphs of $C(h)$, $S(h)$ and $T(h)$

They satisfy analogs of well-known properties of the transcendental circular functions $\cos\theta$, $\sin\theta$ and $\tan\theta$. The most obvious such relations are

$$C(h)^2 + S(h)^2 = 1 \tag{7}$$

together with the symmetry conditions

$$C(-h) = C(h), \quad S(-h) = -S(h) \quad \text{and} \quad T(-h) = -T(h).$$

Another important function is the **capital M function**

$$M(h) \equiv \frac{2}{1+h^2} = 1 + C(h) = \frac{S(h)}{h}$$

whose main significance is that it describes the rotationally invariant measure on the circle. It is also involved in the following formulas.

Theorem 1 (C and S derivative)**.** *The derivatives of C and S are*

$$\frac{dC}{dh}(h) = -S(h)M(h) \quad \textit{and} \quad \frac{dS}{dh}(h) = C(h)M(h).$$

Proof. This is a first-year calculus computation. □

Theorem 2 (C and S second order derivative)**.** *Both $C(h)$ and $S(h)$ satisfy the second order differential equation*

$$\frac{1}{M(h)}\frac{d}{dh}\left(\frac{1}{M(h)}\frac{df}{dh}\right) + f = 0.$$

Proof. This follows by combining both formulas of the previous theorem. □

4. Rotor coordinates

If h is a rational number, then $\mathbf{e}(h) \equiv (C(h), S(h))$ is a rational vector of unit length. For any rational number $r > 0$, the vector $\mathbf{v} = r\mathbf{e}(h)$ is then also a rational vector, and the usual Cartesian coordinates for $\mathbf{v}$ are

$$x = r\left(\frac{1-h^2}{1+h^2}\right) \quad \text{and} \quad y = r\left(\frac{2h}{1+h^2}\right). \tag{8}$$

The number

$$r = r(\mathbf{v}) \equiv \sqrt{x^2+y^2} \tag{9}$$

is the **length** of $\mathbf{v}$, and

$$h = h(\mathbf{v})$$

is the **half-slope** of $\mathbf{v}$. In the special case of $\mathbf{w} \equiv (0, -1)$, we define $h(r\mathbf{w}) = \infty$ for any $r > 0$.

The quantities r and h determine $\mathbf{v}$, and will be called **rotor coordinates** for $\mathbf{v}$, written

$$\mathbf{v} = |\, r, h\rangle\,.$$

The above formulas extend to more general vectors $\mathbf{v} = (x, y)$, but in this case r will typically exist in a *quadratic extension* of the field containing x and y, which also contains h because of the following important result. We give two proofs.

Theorem 3 (Half-slope formula)**.** *If* $\mathbf{v} \equiv (x, y)$ *has length* $r \equiv \sqrt{x^2+y^2}$ *and* $y \neq 0$*, then*

$$h(\mathbf{v}) = \frac{r-x}{y}\,. \tag{10}$$

Proof. To find $h \equiv h(\mathbf{v})$, normalize to obtain the unit vector

$$\frac{\mathbf{v}}{r} = \left(\frac{x}{r}, \frac{y}{r}\right)$$

which is collinear with the vectors $(-1, 0)$ and $(0, h)$ as in Figure 1. It follows from similar triangles that

$$\frac{h}{1} = \frac{y/r}{1+x/r} = \frac{y}{r+x} = \frac{r-x}{y},$$

the last equality since $y^2 = r^2 - x^2$. Alternatively, use (8) to see that

$$\frac{r-x}{y} = \frac{1+h^2}{2h} - \frac{1-h^2}{2h} = h. \qquad \square$$

In the special case when $y = 0$, the half-slope h is either 0 or ∞, depending on whether x is positive or negative. In a diagram we represent the half-slope h of a vector $\mathbf{v}$ as shown in Figure 1.

5. Examples of half-slopes for unit vectors

Rotor coordinates describe rational vectors in the plane without prior set-up of the full real number system. They provide a useful, simpler and often more powerful alternative to polar coordinates.

The table below gives some examples of unit vectors $\mathbf{v} = (x, y)$, so that $r = 1$, together with their half-slopes $h \equiv h(\mathbf{v})$, and their corresponding angles θ, where we write $\theta \approx h$. We restrict to the cases for which h is positive, so that $y \geq 0$, with corresponding angles θ in the range $0 \leq \theta \leq 180°$. If we negate a half-slope, then the corresponding angle is also negated.

Unit vector $\mathbf{v}$	*Half-slope* h	*Angle* θ
$\left(1/\sqrt{2},\ 1/\sqrt{2}\right)$	$\dfrac{1-1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2}-1$	$45°$
$\left(-1/\sqrt{2},\ 1/\sqrt{2}\right)$	$\dfrac{1+1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2}+1$	$135°$
$\left(\sqrt{3}/2,\ 1/2\right)$	$\dfrac{1-\sqrt{3}/2}{1/2} = 2-\sqrt{3}$	$30°$
$\left(-\sqrt{3}/2,\ 1/2\right)$	$\dfrac{1+\sqrt{3}/2}{1/2} = 2+\sqrt{3}$	$150°$
$\left(1/2,\ \sqrt{3}/2\right)$	$\dfrac{1-1/2}{\sqrt{3}/2} = 1/\sqrt{3}$	$60°$
$\left(-1/2,\ \sqrt{3}/2\right)$	$\dfrac{1+1/2}{\sqrt{3}/2} = \sqrt{3}$	$120°$
$\left(\dfrac{\sqrt{5}-1}{4},\ \dfrac{\sqrt{10+2\sqrt{5}}}{4}\right)$	$\sqrt{5-2\sqrt{5}}$	$72°$
$\left(\dfrac{-\sqrt{5}-1}{4},\ \dfrac{\sqrt{10-2\sqrt{5}}}{4}\right)$	$\sqrt{5+2\sqrt{5}}$	$144°$

6. Projective formulation and the circle sum

While the half-slope h is very convenient for applications, having to treat the special case $h = \infty$ separately becomes an inconvenience for theoretical work. This may be overcome by moving to the more natural *projective parametrization* of the unit circle, which we now explain.

The *projective line* over the rationals consists of proportions

$$\alpha \equiv [t : u]$$

where t and u are rational numbers, not both zero. By scaling these may be taken to be integers. The rational half-slope $h = h(\mathbf{v}) = t/u$ of a vector $\mathbf{v}$ corresponds to the *projective half-slope*

$$\alpha(\mathbf{v}) = [h : 1] = [t : u]$$

while the extended rational half-slope $h = h(\mathbf{w}) = \infty$ of the vector $\mathbf{w} \equiv (-1, 0)$ corresponds to the projective half-slope

$$\alpha(\mathbf{w}) = [1 : 0].$$

In this way both cases can be dealt with uniformly. The bijection between projective half-slopes and the unit circle is

$$\mathbf{e}\left([t:u]\right) \equiv \left[\frac{u^2-t^2}{u^2+t^2}, \frac{2ut}{u^2+t^2}\right].$$

In parallel with (5), for a proportion $\alpha \equiv [t:u]$ define the **rotation matrix**

$$\sigma_\alpha \equiv \frac{1}{(u^2+t^2)}\begin{pmatrix} u^2-t^2 & 2ut \\ -2ut & u^2-t^2 \end{pmatrix}$$

acting on a (row) vector $\mathbf{v}=(x,y)$ on the right by $\mathbf{v} \to \mathbf{v}\sigma_\alpha$. Here is a key theorem.

Theorem 4 (Circle sum). *If $\alpha_1 \equiv [t_1 : u_1]$ and $\alpha_2 \equiv [t_2 : u_2]$ then*

$$\sigma_{\alpha_1}\sigma_{\alpha_2} = \sigma_\alpha$$

where

$$\alpha \equiv [u_1t_2+u_2t_1 : u_1u_2 - t_1t_2] \equiv \alpha_1 \oplus \alpha_2$$

defines the **circle sum** *of the two proportions α_1 and α_2.*

Proof. Note first that the circle sum $\alpha \equiv \alpha_1 \oplus \alpha_2$ is well-defined, in that if we scale the entries in either α_1 or α_2, the proportion α is unchanged, and because Fibonacci's identity

$$(u_1t_2+u_2t_1)^2 + (t_1t_2-u_1u_2)^2 = (t_1^2+u_1^2)(t_2^2+u_2^2) \tag{11}$$

ensures that the entries of α are not both zero. The latter also ensures that we need only check that

$$\begin{aligned}&\begin{pmatrix} u_1^2-t_1^2 & 2u_1t_1 \\ -2u_1t_1 & u_1^2-t_1^2 \end{pmatrix}\begin{pmatrix} u_2^2-t_2^2 & 2u_2t_2 \\ -2u_2t_2 & u_2^2-t_2^2 \end{pmatrix} \\ =&\begin{pmatrix} (u_1u_2-t_1t_2)^2-(u_1t_2+u_2t_1)^2 & 2(u_1u_2-t_1t_2)(u_1t_2+u_2t_1) \\ -2(u_1u_2-t_1t_2)(u_1t_2+u_2t_1) & (u_1u_2-t_1t_2)^2-(u_1t_2+u_2t_1)^2 \end{pmatrix}.\end{aligned}$$

This in turns rests on the identities

$$\left(u_1^2-t_1^2\right)\left(u_2^2-t_2^2\right) - \left(2u_1t_1\right)\left(2u_2t_2\right) = \left(u_1u_2-t_1t_2\right)^2 - \left(u_1t_2+u_2t_1\right)^2 \tag{12}$$

$$\left(u_1^2-t_1^2\right)\left(2u_2t_2\right) + \left(2u_1t_1\right)\left(u_2^2-t_2^2\right) = 2\left(u_1u_2-t_1t_2\right)\left(u_1t_2+u_2t_1\right). \tag{13}$$

□

The circle sum is associative (since it corresponds, by the theorem, to matrix multiplication), commutative, and has identity $[0:1]$. The inverse of $[t:u]$ is $[-t:u]$. The map $\alpha \to \sigma_\alpha$ defines a homomorphism between the group of projective half-slopes under circle sum, and the multiplicative group of rational rotation matrices.

7. Rational circle sums and half-slope functions

When we restate the Circle sum theorem in terms of rational half-slopes h, we find that

$$\sigma_{h_1}\sigma_{h_2} = \sigma_h$$

where

$$h = \frac{h_1 + h_2}{1 - h_1 h_2} \equiv h_1 \oplus h_2. \tag{14}$$

This **rational circle sum** extends to values of ∞ by limiting arguments, or by going back to the projective formulation. The identity is $h = 0$, and the inverse of h is $-h$, so that

$$(-h_1) \oplus (-h_2) = -(h_1 \oplus h_2). \tag{15}$$

Example 1. The half-slope that corresponds to an angle of $45° + 30° = 75°$ is

$$h = \left(\sqrt{2} - 1\right) \oplus \left(2 - \sqrt{3}\right) = \frac{(\sqrt{2} - 1) + (2 - \sqrt{3})}{1 - (\sqrt{2} - 1)(2 - \sqrt{3})} = \sqrt{3} + \sqrt{6} - \sqrt{2} - 2.$$

The addition formulas for C and S are

$$C(h_1 \oplus h_2) = C(h_1) C(h_2) - S(h_1) S(h_2) \tag{16}$$
$$S(h_1 \oplus h_2) = C(h_1) S(h_2) + C(h_2) S(h_1) \tag{17}$$

which are essentially contained in the identities (12) and (13). The addition formula for T relates directly to the circle sum:

$$T(h_1 \oplus h_2) = \frac{T(h_1) + T(h_2)}{1 - T(h_1) T(h_2)} = T(h_1) \oplus T(h_2)$$

and is a consequence of the identity

$$\frac{2\left(\frac{h_1 + h_2}{1 - h_1 h_2}\right)}{1 - \left(\frac{h_1 + h_2}{1 - h_1 h_2}\right)^2} = \frac{\left(\frac{2h_1}{1 - h_1^2}\right) + \left(\frac{2h_2}{1 - h_2^2}\right)}{1 - \left(\frac{2h_1}{1 - h_1^2}\right)\left(\frac{2h_2}{1 - h_2^2}\right)}.$$

The circle sum operation is commutative and also associative, so that

$$(h_1 \oplus h_2) \oplus h_3 = h_1 \oplus (h_2 \oplus h_3) = \frac{h_1 + h_2 + h_3 - h_1 h_2 h_3}{1 - (h_1 h_2 + h_2 h_3 + h_1 h_3)} \tag{18}$$

and similarly

$$h_1 \oplus h_2 \oplus h_3 \oplus h_4 = \frac{h_1 + h_2 + h_3 + h_4 - (h_1 h_2 h_3 + h_1 h_2 h_4 + h_1 h_3 h_4 + h_2 h_3 h_4)}{1 - (h_1 h_3 + h_1 h_4 + h_2 h_3 + h_2 h_4 + h_3 h_4 + h_1 h_2) + h_1 h_2 h_3 h_4}. \tag{19}$$

Theorem 5 (Multiple circle sums)**.** *For any natural number n, any rational numbers $h_1, h_2, \ldots, h_n$, and any natural number k in the range $1 \leq k \leq n$, let*

$$s_k \equiv s_k(h_1, h_2, \ldots, h_n) \equiv \sum_{\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \cdots, n\}} h_{i_1} h_{i_2} \ldots h_{i_k}.$$

If $n = 2m$ is even then

$$h_1 \oplus h_2 \oplus \cdots \oplus h_n = \frac{s_1 - s_3 + \cdots + (-1)^{m-1} s_{2m-1}}{1 - s_2 + s_4 - \cdots + (-1)^m s_{2m}}$$

while if $n = 2m + 1$ is odd then

$$h_1 \oplus h_2 \oplus \cdots \oplus h_n = \frac{s_1 - s_3 + \cdots + (-1)^m s_{2m+1}}{1 - s_2 + s_4 - \cdots + (-1)^m s_{2m}}.$$

Proof. We proceed by induction. We may check that for $n = 1$ and $n = 2$ the formulas are correct. Assume they are true for n, and now to prove the corresponding formulas for $n+1$, for k in the range $1 \leq k \leq n+1$ set

$$\mu_k \equiv \mu_k\left(h_1, h_2, \ldots, h_n, h_{n+1}\right) \equiv \sum_{\{i_1,i_2,\ldots,i_k\}\subseteq\{1,2,\ldots,n+1\}} h_{i_1}h_{i_2}\ldots h_{i_k}.$$

If $n = 2m$ then

$$(h_1 \oplus h_2 \oplus \cdots \oplus h_n) \oplus h_{n+1} = \frac{\left(\dfrac{s_1 - s_3 + \cdots + (-1)^{m-1}s_{2m-1}}{1 - s_2 + \cdots + (-1)^m s_{2m}}\right) + h_{n+1}}{1 - \left(\dfrac{s_1 - s_3 + \cdots + (-1)^{m-1}s_{2m-1}}{1 - s_2 + \cdots + (-1)^m s_{2m}}\right) \times h_{n+1}}$$

while if $n = 2m+1$ then

$$(h_1 \oplus h_2 \oplus \cdots \oplus h_n) \oplus h_{n+1} = \frac{\left(\dfrac{s_1 - s_3 + \cdots + (-1)^m s_{2m+1}}{1 - s_2 + s_4 - \cdots + (-1)\, s_{2m}}\right) + h_{n+1}}{1 - \left(\dfrac{s_1 - s_3 + \cdots + (-1)^m s_{2m+1}}{1 - s_2 + s_4 - \cdots + (-1)^m s_{2m}}\right) \times h_{n+1}}.$$

The induction then rests on two identities, the first when $n = 2m$ being

$$\left(s_1 - s_3 + \cdots + (-1)^{m-1} s_{2m-1}\right) + \left(1 - s_2 + \cdots + (-1)^m s_{2m}\right) h_{n+1} = \mu_1 - \mu_3 + \cdots + (-1)^m \mu_{2m+1}$$

and the second when $n = 2m+1$ being

$$\left(s_1 - s_3 + \cdots + (-1)^m s_{2m+1}\right) + \left(1 - s_2 + s_4 - \cdots + (-1)\, s_{2m}\right) h_{n+1} = \mu_1 - \mu_3 + \cdots + (-1)^m \mu_{2m+1}.$$

□

Taking the circle sum of h with itself yields a rational function of h which we call $U_2(h)$, namely

$$h \oplus h = \frac{2h}{1-h^2} \equiv U_2(h).$$

Continuing, we get a sequence $U_n(h)$ of rational functions, which we call the **half-slope functions**:

$$h \oplus h \oplus h = \frac{3h - h^3}{1 - 3h^2} \equiv U_3(h)$$
$$h \oplus h \oplus h \oplus h = \frac{4h - 4h^3}{1 - 6h^2 + h^4} \equiv U_4(h)$$
$$h \oplus h \oplus h \oplus h \oplus h = \frac{5u - 10u^3 + u^5}{1 - 10u^2 + 5u^4} \equiv U_5(h).$$

The pattern of binomial coefficients follows directly from the Multiple circle sums theorem. These functions have been known for centuries (see [7, p. 155]), although our name for them is new. They warrant more study.

Example 2. If we wish to bisect the sector created by two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with $h \equiv h\left(\mathbf{v}_1, \mathbf{v}_2\right)$, then we need find a half-slope k satisfying

$$U_2\left(k\right) \equiv k \oplus k = \frac{2k}{1-k^2} = h.$$

This quadratic equation $hk^2 + 2k - h = 0$ has discriminant $4(1 + h^2)$, so that we require $1 + h^2$ to be a square.

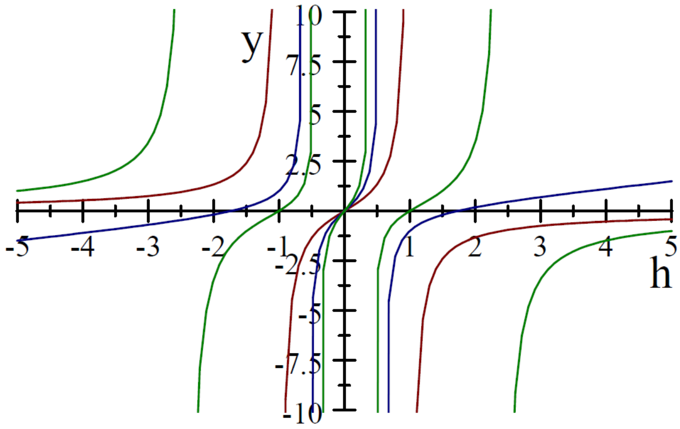

Figure 4: Half-slope functions U_2 (red), U_3 (blue), and U_4 (green)

Example 3. If we wish to trisect the sector created by two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$, then we need find a half-slope k satisfying

$$U_3(k) \equiv k \oplus k \oplus k = \frac{3k - k^3}{1 - 3k^2} = h.$$

This yields the cubic equation $k^3 - 3hk^2 - 3k + h = 0$ which we may transform in the usual way by setting $k = y + h$ to get

$$y^3 = py + q$$

where $p = 3\left(1 + h^2\right)$ and $q = 2h\left(1 + h^2\right)$. The discriminant $4p^2 - 27q^2$ is $108\left(1 + h^2\right)^2$.

Example 4. Suppose we want to verify the half-slopes h associated to the fifth roots of unity. It means solving $U_5(h) = 0$, namely

$$5h - 10h^3 + h^5 = h\left(h^4 - 10h^2 + 5\right) = 0.$$

Besides the obvious solution $h = 0$, we also get $h = \pm\sqrt{5 - 2\sqrt{5}},\ \pm\sqrt{5 + 2\sqrt{5}}$, as in our earlier table.

8. Half-slope transformations

Since this paper is oriented to applications, we will stick with the view of half-slopes as extended rational numbers h, and refer to (14) as simply the circle sum. The reader should have little difficulty in formulating projective versions if required.

Theorem 6 (Half-slope transformations)**.** *Suppose that the vector* $\mathbf{v}$ *has half-slope* h*. Then the reflection of* $\mathbf{v}$ *in the* x*-axis has half-slope* $-h$*, the reflection of* $\mathbf{v}$ *in the* y*-axis has half-slope* h^{-1}*, the vector* $-\mathbf{v}$ *has half-slope* $-h^{-1}$*, while the reflection of* $\mathbf{v}$ *in the line* $y = x$ *and the rotation of* $\mathbf{v}$ *by a one-quarter of the full circle in the positive direction have respective half-slopes*

$$\frac{1-h}{1+h} \quad \textit{and} \quad \frac{1+h}{1-h}.$$

Proof. These are easy calculations, such as

$$1 \oplus (-h) = \frac{1-h}{1+h} \quad \text{and} \quad 1 \oplus h = \frac{1+h}{1-h}. \qquad \square$$

The theorem can also be used to relate angle transformations to half-slope transformations. Denote $\theta \approx h$ the relation between an angle and a half-slope as before. Then $-\theta \approx -h$ and

$$180^\circ - \theta \approx \frac{1}{h} \qquad \text{and} \qquad 180^\circ + \theta \approx -\frac{1}{h}$$
$$90^\circ - \theta \approx \frac{1-h}{1+h} \qquad \text{and} \qquad 90^\circ + \theta \approx \frac{1+h}{1+h}.$$

Figure 5 shows the effect of reflections in the coordinate axes and the lines $y = \pm x$ on the half-slope h.

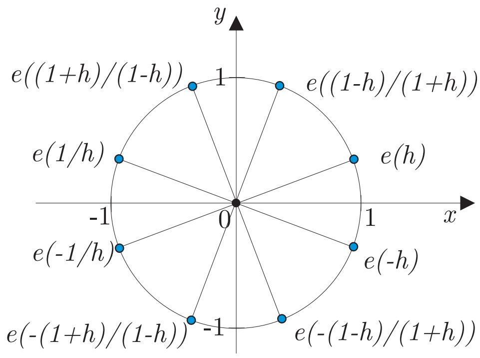

Figure 5: Reflections and half-slopes

One can also easily check that

$$\frac{1}{h_1} \oplus \frac{1}{h_2} = -\left(h_1 \oplus h_2\right).$$

Example 5. Many unit vectors, of interest already to the Pythagoreans, have corresponding angles which do not have tidy values in the radian or degree systems, and so are seldom used in high school examples or tests, despite their simplicity and attractiveness. For example the vector (3/5, 4/5) has half-slope $h = 1/2$, the vector (4/5, 3/5) has $h = 1/3$, the vector (5/13, 12/13) has $h = 2/3$ and the vector (12/13, 5/13) has $h = 1/5$.

Example 6. To find the product of the rotations σ_α corresponding to the unit vectors (3/5, 4/5) and (5/13, 12/13), compute

$$\frac{1}{2} \oplus \frac{2}{3} = \frac{\frac{1}{2}+\frac{2}{3}}{1-\frac{1}{2}\times\frac{2}{3}} = \frac{7}{4},$$

so that

$$\sigma_{1/2}\sigma_{2/3} = \sigma_{7/4} = c\left(\begin{pmatrix} 0 & -7/4 \\ 7/4 & 0 \end{pmatrix}\right) = \frac{1}{65}\begin{pmatrix} -33 & 56 \\ -56 & -33 \end{pmatrix}.$$

Example 7. Here are a few rotor forms for non-unit vectors. If $\mathbf{v} \equiv (1,2)$ then $r = \sqrt{5}$ and

$$h = \frac{\sqrt{5}-1}{2} \approx 0.618\,03$$

is the *Golden ratio*. If $\mathbf{v} \equiv (2,1)$ then

$$h = \sqrt{5} - 2 \approx 0.236\,07.$$

If $\mathbf{v} \equiv (1,3)$ then $r = \sqrt{10}$ and

$$h = \frac{\sqrt{10}-1}{3} \approx 0.720\,76.$$

Clearly once we have found the length r, (10) makes it easy to compute the half-slope h.

9. Relative half-slopes between vectors

Up to now we have defined the half-slope of a single vector, which depends on the choice of positive x-axis. We now define the **half-slope between two vectors** $\mathbf{v}_1 = |\, r_1, h_1\rangle$ and $\mathbf{v}_2 = |\, r_2, h_2\rangle$, or their **relative half-slope**, to be

$$h = h(\mathbf{v}_1, \mathbf{v}_2) \equiv \frac{h_2 - h_1}{1 + h_1 h_2} = h_2 \oplus (-h_1).$$

It follows that

$$h_1 \oplus h = h_2.$$

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in opposite directions, then $h_1 h_2 = -1$, so that $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$ is interpreted as having the value ∞. The relative half-slope is an oriented quantity, in that

$$h(\mathbf{v}_2, \mathbf{v}_1) = -h(\mathbf{v}_1, \mathbf{v}_2).$$

Example 8. If $\mathbf{v}_1 \equiv (3, 2)$ and $\mathbf{v}_2 = (2, 5)$ then

$$h(\mathbf{v}_1, \mathbf{v}_2) = h_2 \oplus (-h_1) = \frac{\left(\frac{\sqrt{29}-2}{5}\right) - \left(\frac{\sqrt{13}-3}{2}\right)}{1 + \left(\frac{\sqrt{29}-2}{5}\right)\left(\frac{\sqrt{13}-3}{2}\right)} = \frac{1}{11}\sqrt{377} - \frac{16}{11}.$$

If we want an *undirected* quantity between $\mathbf{v}_1$ and $\mathbf{v}_2$, we may take the square $H \equiv h^2$ of the half-slope $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$. Note that the spread s between $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$s = \frac{4h^2}{(1 + h^2)^2} = \frac{4H}{(1 + H)^2}.$$

While the half-slope between vectors is unchanged if either is multiplied by a positive number, this is no longer true if we multiply by -1.

Example 9. For any vectors $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$h(-\mathbf{v}_1, \mathbf{v}_2) = -\frac{1}{h(\mathbf{v}_1, \mathbf{v}_2)}.$$

This follows from the half-slope transformation theorem; for if $h_1 \equiv h(\mathbf{v}_1)$ and $h_2 \equiv h(-\mathbf{v}_1)$ then

$$h(-\mathbf{v}_1) = \frac{1}{h_1},$$

so that

$$h(-\mathbf{v}_1, \mathbf{v}_2) = \frac{h_2 - (-1/h_1)}{1 + (-1/h_1)\, h_2} = \frac{1 + h_1 h_2}{h_1 - h_2} = -\frac{1}{h(\mathbf{v}_1, \mathbf{v}_2)}.$$

Example 10. Applying the previous example twice we see that for any vectors $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$h(-\mathbf{v}_1, -\mathbf{v}_2) = h(\mathbf{v}_1, \mathbf{v}_2).$$

Theorem 7 (Relative half-slope formula)**.** *If* $\mathbf{v}_1 \equiv (x_1, y_1)$ *and* $\mathbf{v}_2 \equiv (x_2, y_2)$ *with* $r_1 \equiv r(\mathbf{v}_1)$ *and* $r_2 \equiv r(\mathbf{v}_2)$*, then*

$$h = h(\mathbf{v}_1, \mathbf{v}_2) = \frac{y_1(r_2 - x_2) - y_2(r_1 - x_1)}{y_1 y_2 + (r_1 - x_1)(r_2 - x_2)}.$$

Proof. From the half-slope formula

$$h_1 \equiv h(\mathbf{v}_1) = \frac{r_1 - x_1}{y_1} \quad \text{and} \quad h_2 \equiv h(\mathbf{v}_2) = \frac{r_2 - x_2}{y_2},$$

so that

$$h = h\left(\mathbf{v}_1, \, \mathbf{v}_2\right) \equiv \frac{h_2 - h_1}{1 + h_1 h_2} = \frac{\left(\frac{r_2 - x_2}{y_2}\right) - \left(\frac{r_1 - x_1}{y_1}\right)}{1 + \left(\frac{r_1 - x_1}{y_1}\right)\left(\frac{r_2 - x_2}{y_2}\right)} = \frac{y_1\left(r_2 - x_2\right) - y_2\left(r_1 - x_1\right)}{y_1 y_2 + \left(r_1 - x_1\right)\left(r_2 - x_2\right)}. \qquad \square$$

The next result shows that the relative half-slope is invariant under the rotations σ_h introduced in (5) and (6).

Theorem 8 (Half-slope invariance)**.** *For vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *and any half-slope* h

$$h(\mathbf{v}_1, \, \mathbf{v}_2) = h\left(\mathbf{v}_1 \sigma_h, \mathbf{v}_2 \sigma_h\right).$$

Proof. If $h_1 \equiv h\left(\mathbf{v}_1\right)$ and $h_2 \equiv h\left(\mathbf{v}_2\right)$ then

$$h\left(\mathbf{v}_1 \sigma_h\right) = h_1 \oplus h \quad \text{and} \quad h\left(\mathbf{v}_2 \sigma_h\right) = h_2 \oplus h.$$

Now use (15), and the group properties of the circle sum to get

$$\begin{aligned} h(\mathbf{v}_1 \sigma_h, \, \mathbf{v}_2 \sigma_h) &= \left(h_2 \oplus h\right) \oplus \left(-\left(h_1 \oplus h\right)\right) = h_2 \oplus h \oplus \left(-h_1\right) \oplus \left(-h\right) \\ &= h_2 \oplus \left(-h_1\right) \oplus h \oplus \left(-h\right) = h_2 \oplus \left(-h_1\right) \oplus 0 \\ &= h_2 \oplus \left(-h_1\right) = h\left(\mathbf{v}_1, \, \mathbf{v}_2\right). \end{aligned} \qquad \square$$

Theorem 9 (Triple C formula)**.** *If* $h_1 \oplus h_2 \equiv h_3$ *and* $C_1 \equiv C(h_1)$, $C_2 \equiv C(h_2)$ *and* $C_3 \equiv C(h_3)$, *then*

$$C_1^2 + C_2^2 + C_3^2 = 1 + 2 C_1 C_2 C_3.$$

Proof. Combine (16) with (7) to obtain

$$\left(C_3 - C_1 C_2\right)^2 = (1 - C_1^2)(1 - C_2^2).$$

Now expand to get the result. $\square$

There is no such simple relation between the three values $S_1 \equiv S(h_1)$, $S_2 \equiv S(h_2)$ and $S_3 \equiv S(h_3)$. However their squares, the spreads $s_1 \equiv S_1^2$, $s_2 \equiv S_2^2$ and $s_3 \equiv S_3^2$, satisfy the *Triple spread formula*

$$\left(s_1 + s_2 + s_3\right)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4 s_1 s_2 s_3, \tag{20}$$

one of the main laws of rational trigonometry. This can be derived directly from the Triple C formula by rewriting and squaring it to obtain

$$\left(2 - \left(s_1 + s_2 + s_3\right)\right)^2 = 4\left(1 - s_1\right)\left(1 - s_2\right)\left(1 - s_3\right),$$

and then rearranging.

Theorem 10 (Three half-slopes)**.** *If* $\mathbf{v}_1, \mathbf{v}_2$ *and* $\mathbf{v}_3$ *are three vectors with* $h_{12} \equiv h\left(\mathbf{v}_1, \mathbf{v}_2\right)$, $h_{23} \equiv h\left(\mathbf{v}_2, \mathbf{v}_3\right)$ *and* $h_{13} \equiv h\left(\mathbf{v}_1, \mathbf{v}_3\right)$ *then*

$$h_{13} = h_{12} \oplus h_{23}.$$

Proof. If $h_1 \equiv h\left(\mathbf{v}_1\right)$, $h_2 \equiv h\left(\mathbf{v}_2\right)$ and $h_3 \equiv h\left(\mathbf{v}_3\right)$, then

$$h_{12} \oplus h_{23} = \left(h_2 \oplus \left(-h_1\right)\right) \oplus \left(h_3 \oplus \left(-h_2\right)\right) = h_3 \oplus h_2 \oplus \left(-h_2\right) \oplus \left(-h_1\right) = h_3 \oplus \left(-h_1\right) = h_{13}. \ \square$$

10. The Cross law and vector trigonometry

In this section we establish formulas of vector trigonometry relating to an oriented triangle $\overrightarrow{A_1A_2A_3}$ with respective side lengths r_1, r_2 and r_3, and relative half-slopes $h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right)$, $h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right)$ and $h_3 \equiv h\left(\overrightarrow{A_3A_1}, \overrightarrow{A_3A_2}\right)$.

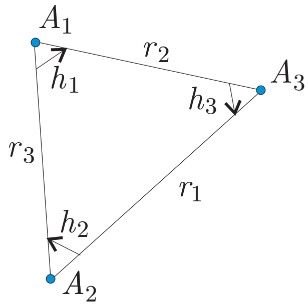

Figure 6: Lengths and relative half-slopes of an oriented triangle $\overrightarrow{A_1A_2A_3}$

Throughout we work in the realm of extended rational numbers and quadratic extensions. We will state the main result in terms of two vectors and the half-slope between them.

Theorem 11 (Cross law–rotor form)**.** *If vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *have respective lengths* r_1 *and* r_2*, and half-slope* $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$*, then* $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$ *has length* r_3*, where*

$$r_3^2 = r_1^2 + r_2^2 - 2r_1r_2C(h).$$

Proof. Suppose that $\mathbf{v}_1 \equiv |\, r_1, h_1\rangle$ and $\mathbf{v}_2 \equiv |\, r_1, h_1\rangle$ so that

$$\mathbf{v}_1 = (r_1C(h_1), r_1S(h_1)) \quad \text{and} \quad \mathbf{v}_2 = (r_2C(h_2), r_2S(h_2))$$

and

$$h \equiv h(\mathbf{v}_1, \mathbf{v}_2) = \frac{h_2 - h_1}{1 + h_1h_2}.$$

Then

$$\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 = (r_2C(h_2) - r_1C(h_1), r_2S(h_2) - r_1S(h_1)).$$

Now compute that

$$\begin{aligned} r_3^2 &= (r_2C(h_2) - r_1C(h_1))^2 + (r_2S(h_2) - r_1S(h_1))^2 \\ &= r_1^2\left(C(h_1)^2 + S(h_1)^2\right) + r_2^2\left(C(h_2)^2 + S(h_2)^2\right) - 2r_1r_2(C(h_1)C(h_2) + S(h_1)S(h_2)) \\ &= r_1^2 + r_2^2 - 2r_1r_2C(h_2 \oplus (-h_1)) = r_1^2 + r_2^2 - 2r_1r_2C(h) \end{aligned}$$

where we have used (7) and the addition formula (16) for $C(h)$. □

Recall that the *triangle inequalities* for a triangle with side lengths r_1, r_2 and r_3 are

$$(r_1 - r_2)^2 \leq r_3^2 \leq (r_1 + r_2)^2.$$

So r_3^2 is a convex combination of $(r_1 - r_2)^2$ and $(r_1 + r_2)^2$, and the Cross law above makes this explicit, as it may be rewritten in the form

$$r_3^2 = \frac{1}{1+h^2}(r_1 - r_2)^2 + \frac{h^2}{1+h^2}(r_1 + r_2)^2 = \frac{1}{1+H}(r_1 - r_2)^2 + \frac{H}{1+H}(r_1 + r_2)^2$$

where $H \equiv h^2$. The next result provides an alternative to the Relative half-slope formula.

Theorem 12 (Vectors half-slope)**.** *If* $\mathbf{v}_1 \equiv (x_1, y_1)$ *and* $\mathbf{v}_2 \equiv (x_2, y_2)$ *are vectors with respective lengths* r_1 *and* r_2*, and relative half-slope* $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$*, then*

$$h^2 = \frac{r_1 r_2 - (x_1 x_2 + y_1 y_2)}{r_1 r_2 + (x_1 x_2 + y_1 y_2)} = \frac{(x_1^2 + y_1^2)(x_2^2 + y_2^2) - 2 r_1 r_2 (x_1 x_2 + y_1 y_2) + (x_1 x_2 + y_1 y_2)^2}{(x_1 y_2 - x_2 y_1)^2}.$$

Proof. Apply the Cross law to the triangle formed from the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, with side lengths r_1, r_2 and $r_3 \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2$, to get

$$C(h) = \frac{1 - h^2}{1 + h^2} = \frac{r_1^2 + r_2^2 - r_3^2}{2 r_1 r_2} = \frac{x_1 x_2 + y_1 y_2}{r_1 r_2}$$

and solve for h^2 to get

$$h^2 = \frac{r_1 r_2 - (x_1 x_2 + y_1 y_2)}{r_1 r_2 + (x_1 x_2 + y_1 y_2)}.$$

Now multiply numerator and denominator by the numerator, and use Fibonacci's identity (11). □

Example 11. For $\mathbf{v}_1 \equiv (3, 2)$ and $\mathbf{v}_2 = (2, 5)$ we get

$$h^2 = \frac{r_1 r_2 - (x_1 x_2 + y_1 y_2)}{r_1 r_2 + (x_1 x_2 + y_1 y_2)} = \frac{\sqrt{13}\sqrt{29} - 16}{\sqrt{13}\sqrt{29} + 16} = \frac{633}{121} - \frac{32}{121}\sqrt{377}.$$

Comparing with Example 8, you may check that this is indeed $\left(\frac{1}{11}\sqrt{377} - \frac{16}{11}\right)^2$.

Theorem 13 (Triangle half-slope)**.** *If an oriented triangle* $\overrightarrow{A_1 A_2 A_3}$ *has respective side lengths* r_1*,* r_2 *and* r_3 *and half-slope* $h_3 \equiv h\left(\overrightarrow{A_3 A_1}, \overrightarrow{A_3 A_2}\right)$*, then*

$$h_3^2 = \frac{r_3^2 - (r_1 - r_2)^2}{(r_1 + r_2)^2 - r_3^2} = \frac{(r_1 - r_2 - r_3)(r_2 - r_1 - r_3)}{(r_1 + r_2 + r_3)(r_1 + r_2 - r_3)}. \tag{21}$$

Proof. We know from the Cross law that $r_3^2 = r_1^2 + r_2^2 - 2 r_1 r_2 C(h_3)$ so that

$$C(h_3) = \frac{r_1^2 + r_2^2 - r_3^2}{2 r_1 r_2}.$$

It follows that

$$h_3^2 = \frac{1 - \dfrac{r_1^2 + r_2^2 - r_3^2}{2 r_1 r_2}}{1 + \dfrac{r_1^2 + r_2^2 - r_3^2}{2 r_1 r_2}} = \frac{r_3^2 - (r_1 - r_2)^2}{(r_1 + r_2)^2 - r_3^2}.$$

Now rewrite this as

$$h_3^2 = \frac{(r_1 - r_2 - r_3)(r_2 - r_1 - r_3)}{(r_1 + r_2 + r_3)(r_1 + r_2 - r_3)}. \tag{22}$$

□

If the quadrances of the triangle $\overline{A_1 A_2 A_3}$ are denoted $Q_1 \equiv r_1^2$, $Q_2 \equiv r_2^2$, and $Q_3 \equiv r_3^2$, then by a rational version of Heron's formula, which we call *Archimedes' formula* (see [8, Theorem 29, page 70]), the *quadrea* of the triangle

$$\begin{aligned} \mathcal{A} &\equiv (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right) \\ &= (r_1 + r_2 + r_3)(-r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3) \end{aligned}$$

is 16 times the square of the triangle's area.

Theorem 14 (Sine law–rotor form)**.** *If an oriented triangle* $\overrightarrow{A_1A_2A_3}$ *has respective side lengths* r_1, r_2 *and* r_3*, half-slopes* $h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right)$*,* $h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right)$ *and* $h_3 \equiv h\left(\overrightarrow{A_3A_1}, \overrightarrow{A_3A_2}\right)$*, and quadrea* $\mathcal{A}$*, then*

$$\frac{S(h_1)}{r_1} = \frac{S(h_2)}{r_2} = \frac{S(h_3)}{r_3} = \frac{\sqrt{\mathcal{A}}}{2r_1r_2r_3}.$$

Proof. Given h_3^2 as in (21),

$$1 + h_3^2 = 1 + \frac{r_3^2 - (r_1 - r_2)^2}{(r_1 + r_2)^2 - r_3^2} = \frac{4r_1r_2}{(r_1 + r_2 + r_3)(r_1 + r_2 - r_3)}.$$

Now combine this with (21) to get

$$s_3 = \frac{4h_3^2}{(1 + h_3^2)^2} = \frac{(r_1 + r_2 + r_3)(-r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3)}{4r_1^2r_2^2}.$$

So

$$\frac{(S(h_3))^2}{r_3^2} = \frac{4h_3^2}{(1 + h_3^2)^2 r_3^2} = \frac{(r_1 + r_2 + r_3)(-r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3)}{4r_1^2r_2^2r_3^2}.$$

But this is symmetric in the three indices, so that

$$\frac{(S(h_1))^2}{r_1^2} = \frac{(S(h_2))^2}{r_2^2} = \frac{(S(h_3))^2}{r_3^2} = \frac{\mathcal{A}}{4r_1^2r_2^2r_3^2}.$$

Now take square roots to get the result, since if one relative half-slope is positive, the others are also. □

Theorem 15 (Triple half-slope formula)**.** *For any three vectors* $\mathbf{v}_1$*,* $\mathbf{v}_2$ *and* $\mathbf{v}_3$*, suppose that*

$$h_{12} \equiv h(\mathbf{v}_1, \mathbf{v}_2), \quad h_{23} \equiv h(\mathbf{v}_2, \mathbf{v}_3) \quad \text{and} \quad h_{31} \equiv h(\mathbf{v}_3, \mathbf{v}_1).$$

Then

$$h_{12} + h_{23} + h_{31} = h_{12}h_{23}h_{31}.$$

Proof. If $h_1 \equiv h(\mathbf{v}_1)$, $h_2 \equiv h(\mathbf{v}_2)$ and $h_3 \equiv h(\mathbf{v}_3)$ then the result follows from the identity

$$\frac{h_3 - h_2}{1 + h_2h_3} + \frac{h_1 - h_3}{1 + h_3h_1} + \frac{h_2 - h_1}{1 + h_1h_2} = \left(\frac{h_3 - h_2}{1 + h_2h_3}\right)\left(\frac{h_1 - h_3}{1 + h_3h_1}\right)\left(\frac{h_2 - h_1}{1 + h_1h_2}\right).$$ □

As a consequence, if two of the half-slopes h_{12}, h_{23}, h_{31} are known, we get *a linear equation for the third*. The next result is the rotor analog of the fact that the angles of a triangle add to π, using the notation of Figure 6.

Theorem 16 (Triangle half-slope formula)**.** *Suppose that* $\overrightarrow{A_1A_2A_3}$ *is an oriented triangle with half-slopes*

$$h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right), \quad h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right) \quad \text{and} \quad h_3 \equiv h\left(\overrightarrow{A_3A_1}, \overrightarrow{A_3A_2}\right).$$

Then

$$h_1h_2 + h_1h_3 + h_2h_3 = 1.$$

Proof. Apply the previous result to the vectors $\mathbf{v}_1 \equiv \overrightarrow{A_1A_2}$, $\mathbf{v}_2 \equiv \overrightarrow{A_2A_3}$ and $\mathbf{v}_3 \equiv \overrightarrow{A_3A_1}$, so that $h_{12} = -1/h_2$, $h_{23} = -1/h_3$ and $h_{31} = -1/h_1$. Then

$$-\frac{1}{h_2} - \frac{1}{h_3} - \frac{1}{h_1} = -\frac{1}{h_1h_2h_3}.$$

After clearing denominators, this becomes

$$h_1h_2 + h_1h_3 + h_2h_3 = 1. \qquad \square$$

We may now *analyze more general triangles more accurately,* without relying either on the usual $30°, 45°, 60°$ or $90°$ formulas, or approximate values obtained for the circular functions by our calculators. In a future paper we will show how this technology clarifies considerably aspects of the metrical geometry of quadrilaterals, and in another paper we will apply vector trigonometry to explaining the Kepler-Newton resolution of the planetary motions as conic sections.

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