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Enumeration of Flat-Foldable Crease Patterns in the Square/Diagonal Grid and Their Folded Shapes

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Abstract. The crease patterns of several basic origami shapes fall within the 45-degree grid system, i.e., the square/diagonal grid of a 4×4 size. This grid system is easily creased on a square, and its flexibility allows to make a variety of shapes by folding along the edges. However, until now, the number of shapes made from a grid has not been known. We enumerated all possible formal crease patterns that are locally flat-foldable, along with their folded shapes, and found that we could enumerate 259,650,300 and 13,452 respectively. Further, we verified that all the shapes can be folded flat without any self-intersections. Some formal crease patterns that are not flat-foldable due to self-intersections were also found.

Key Words: Origami, crease pattern, flat-foldable, square/diagonal grid pattern

MSC 2010: 52B70, 68W05

1. Introduction

Origami is a traditional Japanese art in which various shapes are created by folding a single sheet of paper. The folding technique has been studied in many fields including mathematics, engineering, and art. In addition, the technique of origami has been applied in transformable robots [5]. By switching between the fold and unfold states of hinges on a sheet, the whole shape undergoes different shapes. The square/diagonal grid (Figure 1), which consists of arrayed squares and their diagonals, is commonly used for origami design and the origami robot because 45- and 90-degree angles are easily made by simple folds of a sheet. In addition, a method for designing cube structures by folding the grid has been studied [2]. We refer to the square/diagonal grid pattern simply as a *grid* in this paper. Designing shapes made from a grid is sometimes called *box-pleating* [1].

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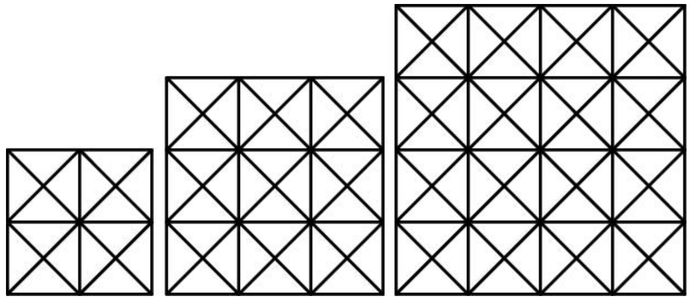

Figure 1: 2×2, 3×3, and 4×4 square/diagonal grid patterns.

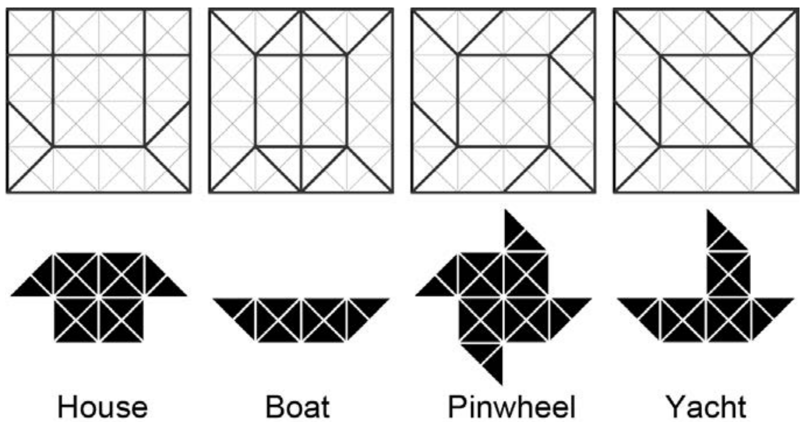

Figure 2: Basic origami design made from 4×4 grid (crease patterns (top) and their folded shapes (bottom)).

Although the 4×4 grid is the most common pattern used in basic origami design (as shown in Figure 2), the number of shapes made by folding subsets of the grid edges is not known as far as we have surveyed. Because it is difficult to find this number mathematically, we enumerated all possible flat-foldable crease patterns in the 4×4 grid as well as 2×2 and 3×3 grids and then enumerated their folded shapes by using a computer. After that, we verified that all the shapes are realizable, i.e., actually flat-foldable with a physical sheet without self-intersections.

In this paper, we present our methods of enumerating crease patterns and their folded shapes. We then present the verification results of the flat-foldability of the shapes.

2. Enumeration of formal crease pattern

The crease patterns we examine in this section are flat-foldable ones without considering self-intersection or mountain/valley parity of folds. These crease patterns are called *formal crease patterns* (FCPs) [8] or *phantom patterns* [4]. The FCPs satisfy the condition of Kawasaki's theorem at any inner vertex; in brief, this theorem states that *the alternate angles between adjacent creases must sum up to* π. Details can be found in [6]. If we see a local area around a vertex in an FCP, it is flat-foldable without intersection. Because of this, we say that FCPs are *locally flat-foldable*. On the other hand, when we look at a whole FCP, it is possible that the FCP is not flat-foldable with a physical sheet of paper due to self-intersections. If a FCP is flat-foldable without any self-intersections, we say that the FCP is *globally flat-foldable*.

2.1. Our approach

The 4×4 grid contains 12 horizontal edges, 12 vertical edges, not including the outer boundary edges, and 32 diagonal edges. Since there are two possible states—to be or not to be a crease line—for each edge, there are 2^{56} combinations for each of the states. Although the search space is huge, we were able to enumerate FCPs in a practical amount of time by constructing an FCP made up of several single-vertex FCPs (namely, FCPs that have just one inner vertex). There are two types of vertices in the grid: four-degree vertices and eight-degree ones. According to Kawasaki's Theorem, we can enumerate four and 36 single-vertex FCPs, respectively, as shown in Figure 3.

By using the depth-first search algorithm as follows, FCPs are quickly constructed. First, we order the inner vertices of the grid in a left-to-right and top-to-bottom manner. Then, we place a single vertex FCP at the first (top-left) inner vertex on the grid and place a single vertex FCP at the next vertex. If the pattern has any inconsistency in terms of edge states, we backtrack. If there are no inconsistencies, we progress to the next vertex until all vertices are assigned a single-vertex FCP. If this process ends without any inconsistencies, we can assume that the pattern is an FCP. We then backtrack to construct another FCP. With this approach, all possible FCPs are constructed.

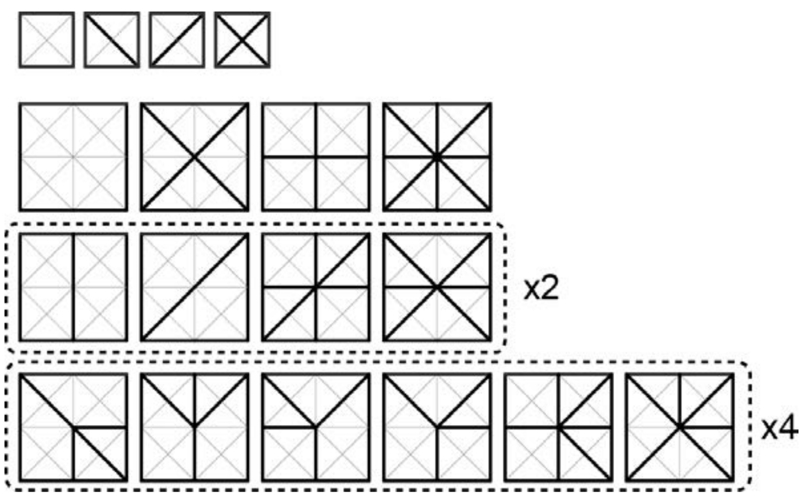

Figure 3: Single-vertex FCPs for a four-degree vertex (top) and an eight-degree vertex (bottom) in the grid. The label '×2' ('×4') means that the patterns have one (three) other pattern(s) coinciding with it by rotation.

2.2. Results

We enumerated 259,650,300 FCPs in the 4×4 grid after five hours of calculation with a PC equipped with Core i7 @2.00GHz and RAM8GB. The duplicated patterns by mirror or rotation operations are not counted.

Among the enumerated FCPs, there is only one FCP that has 0, 1, 55, and 56 edges (Figure 4, bottom), while there are 25,566,613 FCPs that have 33 edges. This is the maximum number if we count them by the number of edges. Figure 4 (top) shows two examples that have 33 edges.

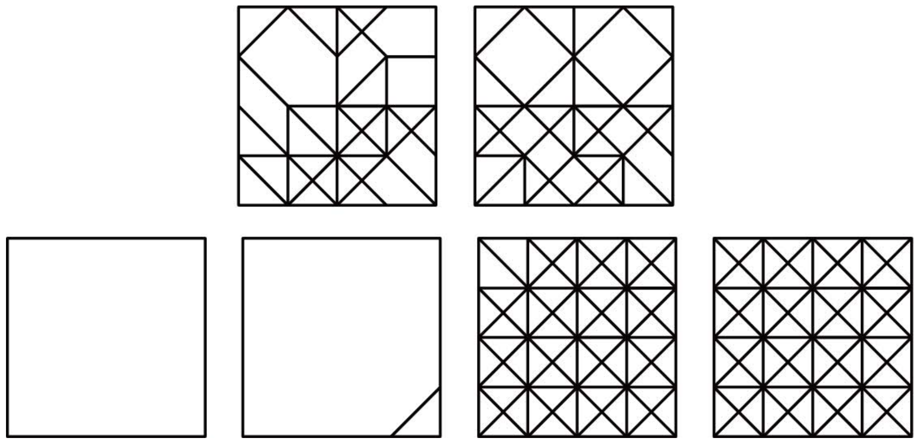

Figure 4: Two examples of FCPs having 33 edges (top). FCPs having 0, 1, 55, and 56 edges (bottom).

3. Enumeration of folded shapes

It is possible to fold different FCPs into the same shape. Figure 5 shows an example of such a case. Here, we do not consider the covering order of layers or the shapes of layers when we discuss shape, opting to focus only on the silhouette. The objects at the bottom of Figure 5 were calculated using ORIPA [9] (developed by Mitani), which is able to output the folded figure from an FCP. In this section, we discuss how many different shapes are made from the FCPs in a 4×4 grid.

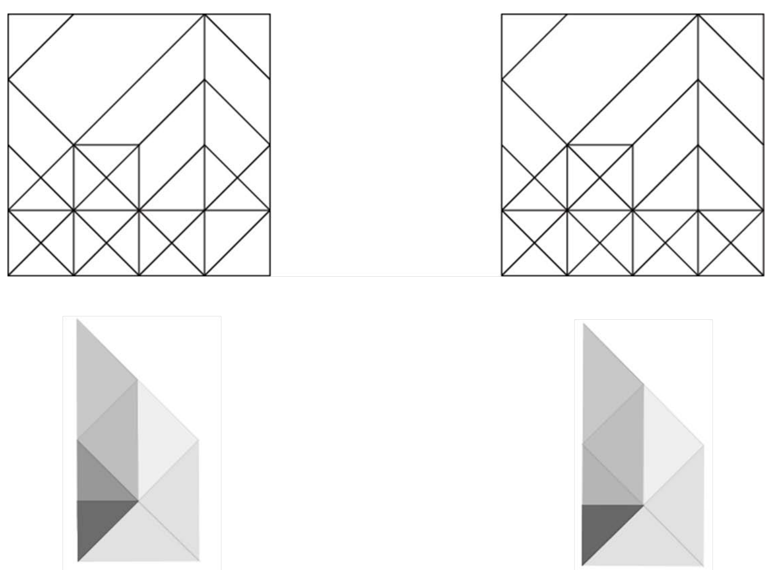

Figure 5: Two different FCPs (top) are folded into the same shape (bottom).

3.1. Our approach

ORIPA uses floating-point numbers for the coordinates of vertices, and numerical calculation errors are unavoidable when a crease pattern is folded by operating the geometric calculation. Therefore, ORIPA is neither robust nor reliable enough to compare folded shapes. We modified

ORIPA so that coordinates are represented by 32-bit integers and folded shapes are represented as a set of unit triangles in the grid. The unit triangle is a quarter of a square grid cell, divided by its diagonals. The 4×4 grid pattern has 64 unit triangles. Since a folded shape consists of unit triangles, any shape can be encoded into a 64-bit integer by giving an ID to each unit triangle and setting a bit flag at the ID-th bit of the 64-bit integer. We calculated the folded shapes of all formal crease patterns and encoded them into 64-bit integers.

3.2. Results

We applied the approach discussed above and were able to enumerate 13,452 shapes after 68 hours of calculation. Duplicated shapes by mirror, rotation, or translation are eliminated. Figure 6 shows the five most popular shapes. A total of 7% of all FCPs are folded into the shape shown in the leftmost side of Figure 6, while there are 782 shapes in which only one crease pattern is folded. Figure 7 shows examples of these.

Figure 6: The five most popular shapes among all possible shapes and, below, the respective number of FCPs that are folded into the shape.

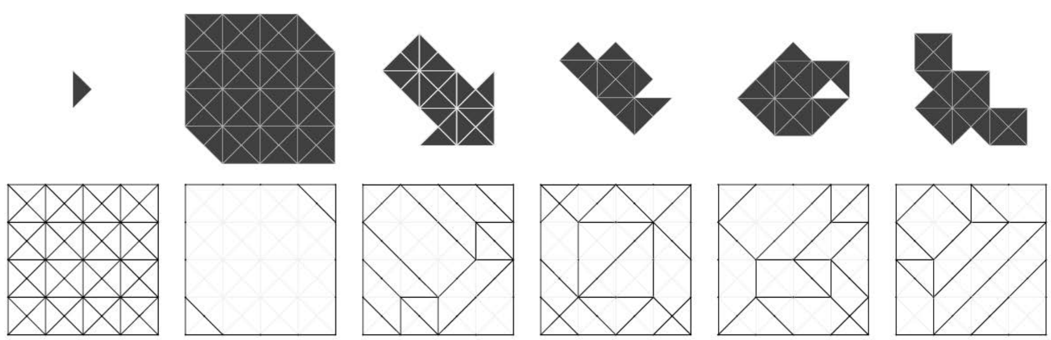

Figure 7: Examples of shapes (top) in which only one crease pattern (bottom) is folded.

3.3. CP finder

We developed an application to search for an FCP that is folded into a shape specified by the user. We call this application *CP finder* and its window is shown in Figure 8. When a user specifies a shape in the left side by composing unit triangles, this tool shows a crease pattern that can be folded into the input shape. Since the data of all FCPs is huge (over 6 GB), our CP finder only keeps a single FCP for each shape. If no FCP can be folded into the shape, the CP finder simply shows the message ‘Not Found’. By using this tool, we can explore a

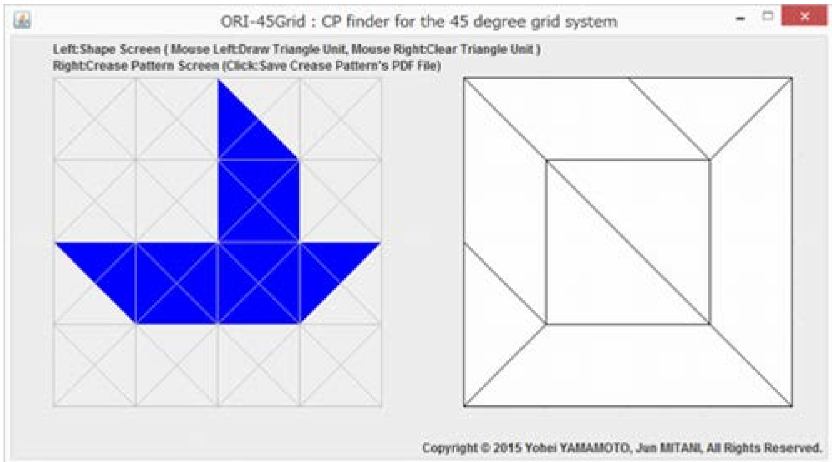

Figure 8: Application window of CP finder.

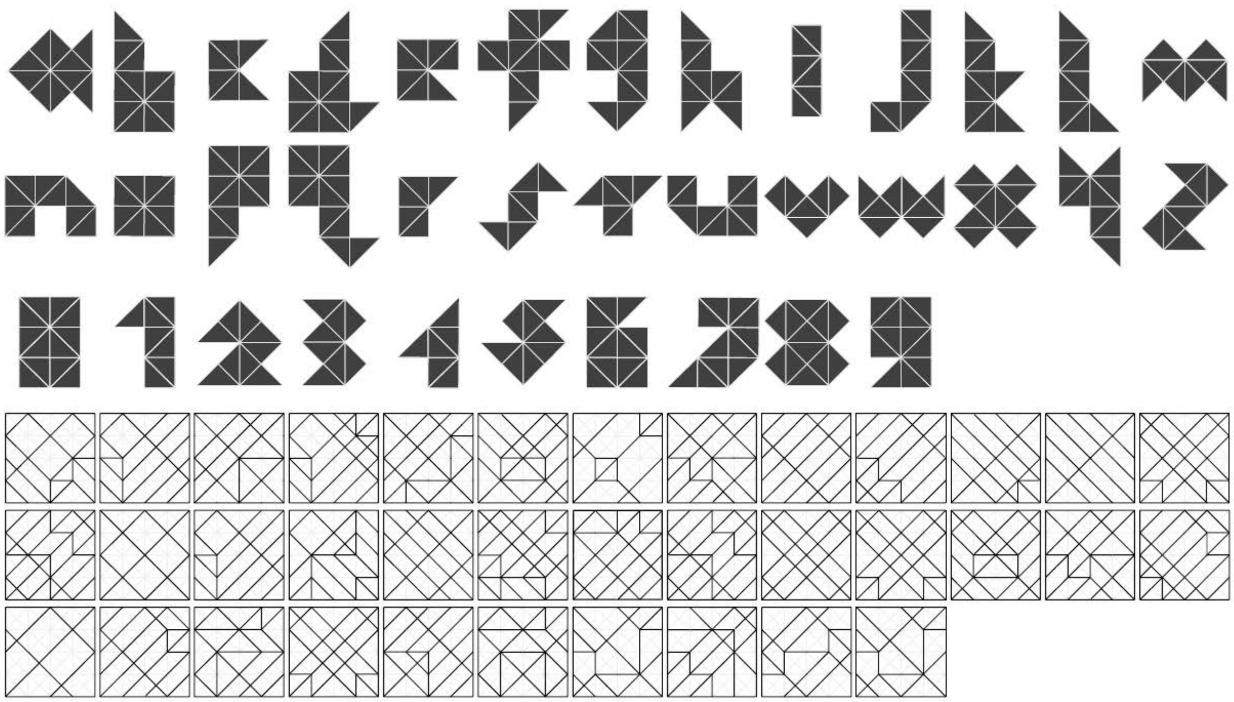

Figure 9: Origami Fonts (top) and crease patterns folded into them (bottom).

variety of shapes that can be made by folding a 4×4 grid. The shapes of the alphabet and numbers we found are shown in Figure 9. We call them *Origami Fonts*.

3.4. 2×2, 3×3, and 3×5 grids

We also enumerated FCPs in the 2×2, 3×3, and 3×5 grids with our approach. The 2×2 and 3×3 grids are for finding the smallest globally not-flat-foldable FCPs (discussed in Section 4). We include the 3×5 grid as an example of a non-square grid. Overall, we found 116 FCPs and 27 shapes in the 2×2 grid, 58,580 and 366 respectively in the 3×3 grid, and 212,030,844 and 18,184, respectively, in the 3×5 grid.

4. Verification of flat-foldability

None of the enumerated shapes are guaranteed to be *realizable*, i.e., actually flat-foldable with a physical sheet of paper without intersection. As discussed in Section 2, an FCP might not be folded globally. Figure 10 shows two examples of such FCPs [6]. The left one does not have any valid mountain/valley assignments. This is confirmed by the *Big-Little-Big Theorem* described in [7]. This theorem states that, if there are at a vertex, in order, a large angle and then a small angle and then another large angle, the two folding lines in between these three angles must have different mountain/valley parities. In the case shown on the left-hand side of Figure 10, there are two 90-degree angles surrounding one less than 90-degree angle at all three vertices. Thus, the three lines of the triangle in the center of the paper are predicted to have alternating mountain/valley parities, but this is impossible. In contrast, the case shown on the right-hand side of Figure 10 has valid mountain/valley assignments, but it cannot be folded flat due to self-intersection.

It is difficult to find such globally non-flat-foldable patterns in a grid. Even so, Kawasaki and Tachi found two such FCPs, examples of which are given in Figure 11. None of them has valid mountain/valley assignments.

It is known that determining whether an FCP is globally flat foldable or not is NP-hard [1, 3].

4.1. Our approach

As described in [1, 3], there are no efficient approaches to determine whether a given FCP is globally flat-foldable or not. We verified not all 259,650,300 FCPs but just one or a few FCPs

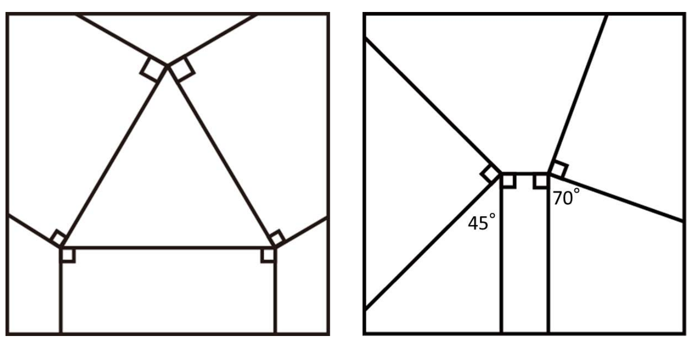

Figure 10: Two examples of FCPs that are not globally flat foldable (as specified by Hull).

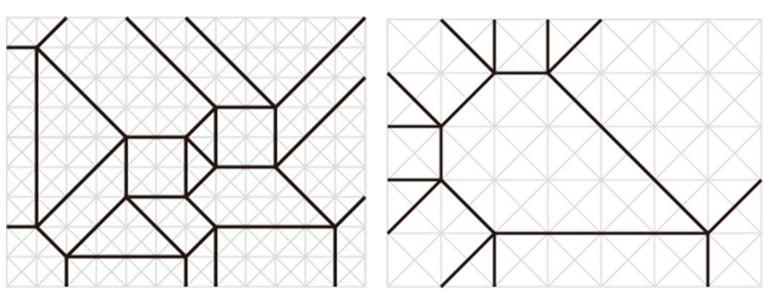

Figure 11: Two examples of FCPs on a grid that do not have any valid mountain/valley assignments (specified by Kawasaki (left) and Tachi (right)).

for each shape, since what we want to know is whether a shape is realizable or not. If an FCP is at least globally flat-foldable into a shape, we can say that the shape is realizable. We do not need to check other FCPs that are to be folded into the same shape. If the FCP we check is not globally flat-foldable, we check another one that is folded into the same shape.

We first divide the 4×4 grid into polygons that we call *faces* along the edges of an FCP. Each face consists of unit triangles. We then calculate the location and orientation of the faces after folding, while maintaining a connection between neighboring faces.

As described in Section 3, a folded shape consists of unit triangles. Here, we can say that if a face-covering order is defined at every unit triangle without any intersections, the FCP is globally flat-foldable. There are three possible cases of intersection, as illustrated in Figure 12. Since all faces consist of unit triangles, all folding lines are exactly located on the grid edges. In case (a), two pairs of connected faces have alternate covering order, and thus have an intersection at the folding. In case (b), a face crosses a pair of connected faces at the folding. Cases (a) and (b) can be detected by observing the face-covering order at a unit triangle. In case (c), there is an inconsistency of the covering order of two faces among different unit triangles.

We use a simple depth-first search algorithm here for finding intersection-free covering orders. The covering orders are searched at each unit triangle in the target shape one by one. If an intersection is found in the process, we simply backtrack. If a face-covering order is defined at every unit triangle without any intersections, we conclude that the shape is realizable.

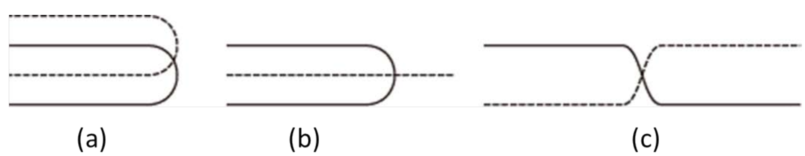

Figure 12: Cases that have an intersection due to invalid face covering order.

4.2. Results and discussion

With the method described above we were able to confirm that all 13,452 shapes are realizable. In the verification process, we found more than 800 non-flat-foldable FCPs. Figure 13 shows five examples of these. Four patterns in Figure 14 consist of 16 edges and are the simplest non-flat-foldable FCPs (i.e., consist of the smallest number of edges) in the 4×4 grid. We also verified all FCPs in the 2×2 and 3×3 grids and confirmed that all FCPs in the 2×2 grid are globally flat-foldable. On the other hand, we found 10 non-flat-foldable FCPs in the 3×3 grid, as shown in Figure 15.

We confirmed that the two FCPs in Figure 16 having 12 edges are the simplest non-flat-foldable FCPs in the 3×3 grid. The left pattern in Figure 16 has valid mountain/valley assignments but cannot be folded due to self-intersection. The right one in Figure 16 does not have any valid mountain/valley assignments.

We consider the question of whether the FCPs in Figure 16 are the simplest and the smallest at the same time. The smallest means that the area of the grid containing the FCP is the smallest. To determine this, we enumerated FCPs in 2×2, 2×3, and 2×4 grids and

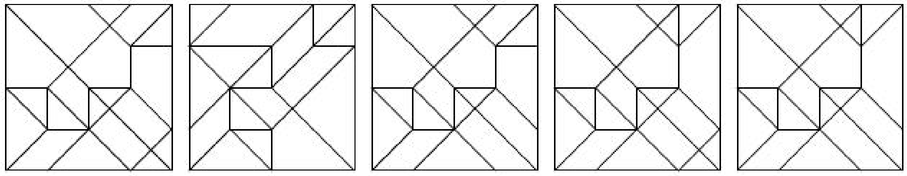

Figure 13: Examples of non-flat-foldable FCPs in the 4×4 grid pattern.

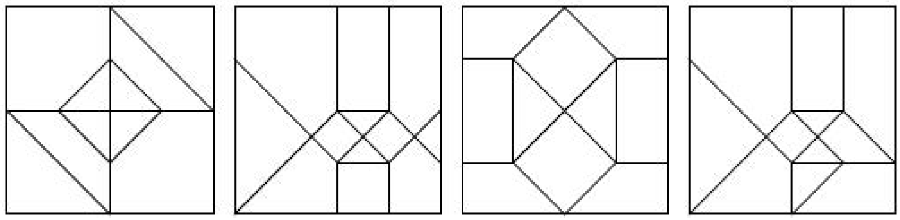

Figure 14: The simplest non-flat-foldable FCPs in the 4×4 grid pattern.

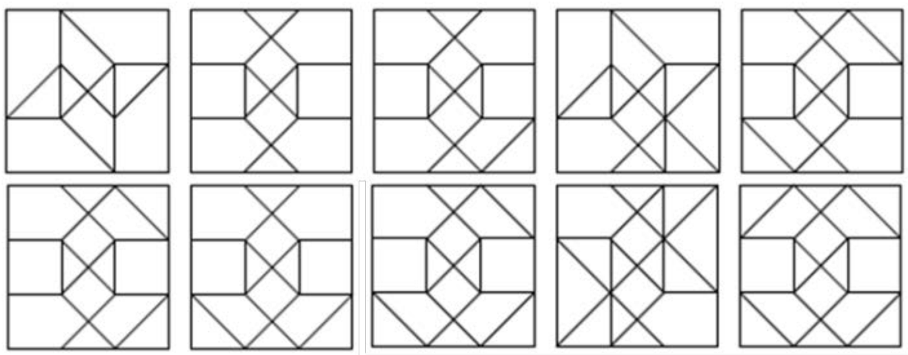

Figure 15: Non-flat-foldable FCPs in the 3×3 grid pattern.

confirmed that all of them are flat-foldable. Consequently, we conclude that the two FCPs shown in Figure 16 are the simplest and smallest non-flat-foldable ones.

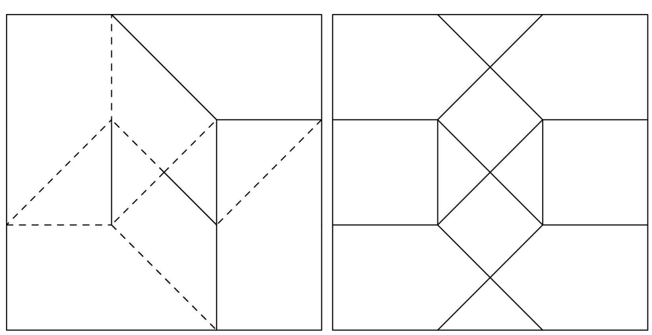

Figure 16: The simplest and smallest non-flat-foldable crease patterns in the square/ diagonal grid pattern. The left one has valid mountain/valley assignments while the right one has none.

5. Summary and future work

We enumerated FCPs and their folded shapes made from 2×2, 3×3, and 4×4 grids. The numbers we obtained are listed in Table 1. We verified that all enumerated shapes are realizable. Through the verification process, we found hundreds of non-flat-foldable FCPs. Additionally, we identified which FCPs are the simplest and smallest non-flat-foldable ones. Further, we enumerated rectangular FCPs in the 3×5 grid and the shapes made from them. However, we did not verify whether the enumerated shapes are realizable. This remains our future work.

Table 1: Number of FCPs and shapes made from the grid patterns.

Grid size	*No. of FCPs*	*No. of shapes*
2×2	116	27
3×3	58,530	366
4×4	259,650,300	13,452

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