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Curved Folding with Pairs of Cylinders

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Abstract. On a sheet of paper we consider a curve $\mathbf{c}^*(s)$. 'Curved paper folding' (or 'curved Origami') along $\mathbf{c}^*(s)$ folded from the planar sheet yields a (spatial) curve $\mathbf{c}(s)$ and two developable strips $\mathbf{f}_{1,2}$ through that curve. We examine the very special case of a configuration where the two surfaces $\mathbf{f}_{1,2}$ are cylinders with generators given by direction vectors $\mathbf{e}_{1,2}$. Such a triple $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{c}(s))$ will be termed *triple for curved folding with cylinders (CFC-triple).*
In this paper we prove the following properties and statements on CFC-triples:
(a) The spherical image $\mathbf{c}'(s)$ of the tangent vectors of $\mathbf{c}(s)$ is, in general, contained in a spherical conic with two of its foci in the directions of $\mathbf{e}_1$ and $\mathbf{e}_2$.
(b) Any curve $\mathbf{c}(s)$ of this triplet is affinely related to a curve of constant slope.
The results are also transferred to the discrete case where $\mathbf{c}(s)$ is replaced by a spatial polygon while the cylinders turn into prisms.

Key Words: curved Origami, curved folding, geodesic curvature, Origami with pairs of cylinders

MSC 2010: 53A05, 51N05, 68U07

1. Introduction

We consider a planar curve $\mathbf{c}^*(s) \in C^2$ parametrized by its arc-length $s \in I \subset \mathbb{R}$. *'Curved paper folding'* (or 'curved Origami') along $\mathbf{c}^*(s)$ yields a (spatial) curve $\mathbf{c}(s)$ and two developable strips $\mathbf{f}_{1,2}$ through that curve which, in turn, forms a sharp crease on this object. Curved Origami was studied by several authors in the last few years — see [2, 3, 4, 5, 8, 9] and the literature cited there. We study the special case where the two surfaces $\mathbf{f}_{1,2}$ are cylinders. The generators of the two cylinders through $\mathbf{c}(s)$ are given by the two different direction vectors $\mathbf{e}_{1,2} \neq \mathbf{o}$.

In terms of differential geometry this procedure can be interpreted as follows: Whenever we roll out the two cylinders $\mathbf{f}_{1,2}$ into a plane π we think of two isometries γ_i of $\mathbf{f}_i$ into π. Additionally, we get two direction vectors $\mathbf{e}^*_{1,2} = \gamma_{1,2}(\mathbf{e}_{1,2})$ parallel to π and two curves $\mathbf{c}^*_{1,2}(s) = \gamma_{1,2}(\mathbf{c}(s)) \subset \pi$ which are related in a direct isometry β including the parameterizations:

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$\beta(\mathbf{c}_2^*(s)) = \mathbf{c}_1^*(s)\,\forall s \in I$. In order to facilitate that, the spatial curve $\mathbf{c}(s)$ must have the same geodesic curvature $\kappa_{g,1}(s) = \pm\kappa_{g,2}(s)$ on the two cylinders $\mathbf{f}_1$ and $\mathbf{f}_2$.

In this paper we characterize triples $(\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{c}(s))$ each consisting of two direction vectors $\mathbf{e}_1, \mathbf{e}_2$ and a spatial curve $\mathbf{c}(s)$ such that $\mathbf{c}(s)$ has the same geodesic curvature $\kappa_{g,1}(s) = \pm\kappa_{g,2}(s)$ with respect to the two cylinders $\mathbf{f}_{1,2}$ with generators parallel to $\mathbf{e}_{1,2}$. Such a triple will be termed *triple for curved folding with cylinders (CFC-triple)*. In Chapter 2 we determine the geodesic curvature of $\mathbf{c}(s)$ with respect to a cylinder, Chapter 3 contains the characterisation of CFC-triples. In Chapter 4 we study the possibility of the existence of a continous motion describing the Origami folding of the spatial situation of the two cylinders to the planar configuration. Chapter 5 is devoted to the discrete version of these results.

2. Spatial curves on cylinders

We start with a spatial curve $\mathbf{c}(s) \in C^3$ parametrized by its arc length $s \in I \subset \mathbb{R}$.[1] The corresponding Frenet-frame is denoted by $\{\mathbf{t}, \mathbf{h}, \mathbf{b}\}$. Additionally we are given two different unit vectors $\mathbf{e}_{1,2}$ which, together with $\mathbf{c}$, define two cylinders $\mathbf{f}_{1,2}(s,v) := \mathbf{c}(s) + v\,\mathbf{e}_{1,2},\ (s,v) \in I \times J \subset \mathbb{R}^2$. We assume that $\{\mathbf{e}_i, \mathbf{c}'\}$ are linearly independent for general $s \in I$.

We determine the geodesic curvature $\kappa_{g,i}$ of the curve $\mathbf{c}$ with respect to the cylinder $\mathbf{f}_i$ $(i = 1, 2)$. The unit vector $\mathbf{n}_i := \mathbf{c}' \times \mathbf{e}_i / \|\mathbf{c}' \times \mathbf{e}_i\|$ is orthogonal to the tangent plane of $\mathbf{f}_i$. Together with the unit *side vector* $\mathbf{s}_i := \mathbf{c}' \times \mathbf{n}_i$ we have defined an orthonormal frame $\{\mathbf{c}', \mathbf{n}_i, \mathbf{s}_i\}$ associated with the curve and the corresponding cylinder. In order to determine the geodesic curvature $\kappa_{g,i}$ of $\mathbf{c}$ with respect to the cylinder $\mathbf{f}_i$ we split the vector $\mathbf{c}''$ into a tangential component $\mathbf{c}''_{t,i}$ parallel to $[\mathbf{c}', \mathbf{s}_i]$ and a normal component $\mathbf{c}''_{n,i}$ parallel to $\mathbf{n}_i$ (see [1] or [6]). We get

$$\mathbf{c}''_{t,i} = \mathbf{s}_i \,\langle \mathbf{s}_i, \mathbf{c}'' \rangle = -\mathbf{s}_i\, \langle \mathbf{c}'', \mathbf{e}_i \rangle \,/\, \|\mathbf{c}' \times \mathbf{e}_i\| \quad \text{and} \quad \mathbf{c}''_{n,i} = \langle \mathbf{n}_i, \mathbf{c}'' \rangle\; \mathbf{n}_i. \tag{1}$$

Therefore the geodesic curvature $\kappa_{g,i}$ with respect to the cylinder $\mathbf{f}_i$ is given by

$$\kappa_{g,i} = -\,\langle \mathbf{c}'', \mathbf{e}_i \rangle \,/\, \|\mathbf{c}' \times \mathbf{e}_i\|\,. \tag{2}$$

We put $\phi_i(s) := \angle(\mathbf{c}'(s), \mathbf{e}_i)$ and get $\cos\phi_i = \langle \mathbf{c}', \mathbf{e}_i \rangle$ and $\sin\phi_i = \|\mathbf{c}' \times \mathbf{e}_i\|$. In the case of cylinders differentiation yields $\langle \mathbf{c}'', \mathbf{e}_i \rangle = -\phi_i' \sin\phi_i$ and therefore

$$\kappa_{g,i}(s) = \phi_i'(s). \tag{3}$$

3. Curves with the same geodesic curvature on two cylinders

In this section we will characterize space curves $\mathbf{c}(s)$ having the same geodesic curvature $\kappa_{g,i}(s)$ with respect to the two cylinders $\mathbf{f}_1$, and $\mathbf{f}_2$ in each of their points.

Remark. Our special case has also the following interpretation: If we roll out the two cylinders into one plane π the corresponding isometries $\gamma_{1,2}\colon\ \mathbf{f}_{1,2} \to \pi$ map the curve $\mathbf{c}(s)$ to curves $\gamma_{1,2}(\mathbf{c})$ that are congruent by a planar isometry β. The geodesic curvature $\kappa_{g,i} = \phi_i'$ is the curvature of the planar curves $\gamma_{1,2}(\mathbf{c})$. Parts of our spatial configuration thus can be viewed as generated by curved folding of a planar piece of paper ('curved Origami') with $\gamma_1(\mathbf{c}(s)) = \gamma_2(\mathbf{c}(s)) := \mathbf{c}^*(s)$ as the common curve. This special case is characterized by[2] $\kappa_{g,1}(s) \equiv \pm\kappa_{g,2}(s)\ \forall s \in I$. The

[1] Derivatives with respect to arc length s of $\mathbf{c}$ will be denoted by primes.

[2] As the two planar parts can be arranged in two different ways we have to admit the negative sign too.

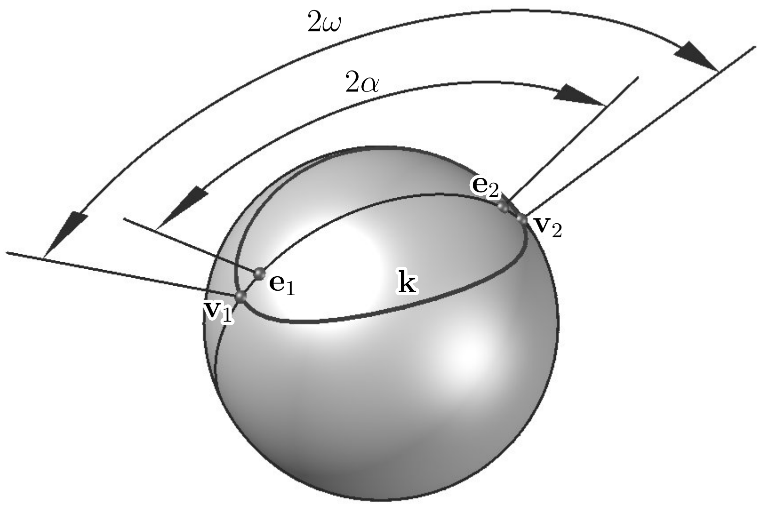


Figure 1: The spherical conic $\mathbf{k}$ containing $\mathbf{c}'(s)$ with real foci $\mathbf{e}_{1,2}$ and two vertices $\mathbf{v}_{1,2}$.

curve $\mathbf{c}(s)$ together with the generators $\mathbf{e}_{1,2}$ of the two cylinders is called *triple for curved folding with cylinders (CFC-triple)* exactly in the cases $\kappa_{g,1}(s) \equiv \pm\kappa_{g,2}(s)\ \forall s \in I$.

According to (3) the characterizing condition $\kappa_{g,1}(s) \equiv \pm\kappa_{g,2}(s)\ \forall s \in I$ is equivalent to

$$|\phi_1(s) \pm \phi_2(s)| = 2\omega = \text{const. } \forall s \in I \tag{4}$$

with some constant angle ω which can be restricted to $[0, \pi/2]$. The sign in (4) is determined by the orientation of the vectors $\mathbf{e}_i$. This orientation can be chosen such that we can use the sum in (4). The spherical image $\mathbf{c}'(s)$ of the tangents of the curve $\mathbf{c}$ then has to be part of a spherical conic $\mathbf{k}$ with two of its foci determined by the direction vectors $\mathbf{e}_i$. The angle 2ω determines the spherical length of one axis of this spherical conic. The corresponding vertices will be denoted by $\mathbf{v}_i$. The angle between $\mathbf{e}_1$ and $\mathbf{e}_2$ shall be denoted by 2α with $0 \le \alpha < \omega$ (see Figure 1).

Then we have

$$\cos\phi_1(s) \ = \ \cos\phi_2(s)\cos 2\omega \mp \sin\phi_2(s)\sin 2\omega \tag{5}$$

for all $s \in I$. This yields the following condition on the tangent vectors $\mathbf{c}'$

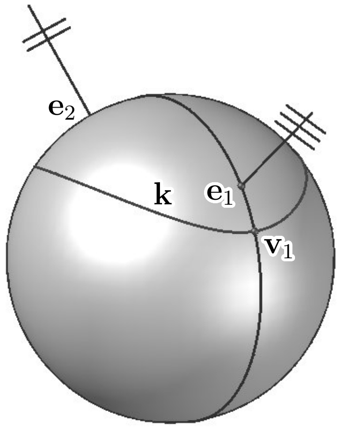
$$(\langle \mathbf{c}', \mathbf{e}_1\rangle - \langle \mathbf{c}', \mathbf{e}_2\rangle \cos 2\omega)^2 \ = \ \left(1 - \langle \mathbf{c}', \mathbf{e}_2\rangle^2\right)\sin^2 2\omega. \tag{6}$$

With the help of $\langle \mathbf{c}', \mathbf{c}'\rangle = 1\ \forall s \in I$ (6) can be rewritten as

$$\langle \mathbf{c}', \mathbf{e}_1\rangle^2 - 2\,\langle \mathbf{c}', \mathbf{e}_1\rangle\,\langle \mathbf{c}', \mathbf{e}_2\rangle\, \cos 2\omega + \langle \mathbf{c}', \mathbf{e}_2\rangle^2 \ = \ \langle \mathbf{c}', \mathbf{c}'\rangle\ \sin^2 2\omega. \tag{7}$$

This is a homogeneous quadratic equation for the unit tangent vectors $\mathbf{c}'$ to our curve. The tangents of the given space curve $\mathbf{c}(s)$ have to be parallel to the generators of a quadratic cone[3] Γ. According to (4) the direction vectors $\mathbf{e}_{1,2}$ of the two cylinders $\mathbf{f}_{1,2}$ define the real focal lines of the quadratic cone Γ.

[3]That cone has the vertex O and contains the spherical conic $\mathbf{k}$.



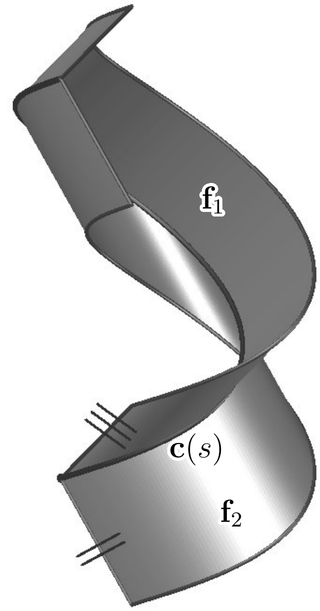


Figure 2: The spherical conic $\mathbf{k}$ of the tangent vectors $\mathbf{c}'(s)$ of an affinely transformed helix $\mathbf{c}(s)$ and the two cylinders $\mathbf{f}_{1,2}$ given by the corresponding CFC-triple $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{c}(s))$.

Remarks. a) Spatial curves $\mathbf{c}(s)$ with tangents parallel to a quadratic cone Γ (but no cone of rotation) determine a spherical conic $\mathbf{k}$ containing the spherical image of the tangent vectors $\mathbf{c}'(s)$. This spherical conic $\mathbf{k}$, in general, has six foci - exactly two of them are real, if the conic is no circle. They determine the directions $\mathbf{e}_{1,2}$ of the generators of the two cylinders $\mathbf{f}_{1,2}$ through $\mathbf{c}$. Through any curve $\mathbf{c}$ of that type there exist two different cylinders $\mathbf{f}_{1,2}$ such that $\mathbf{c}(s)$ has the same geodesic curvature with respect to these two cylinders.
b) Any regular affine mapping α transforming Γ into a cone of revolution transforms $\mathbf{k}$ into a circle on the unit sphere. The tangents of $\alpha(\mathbf{c})$ then are parallel to the cone of revolution $\alpha(\Gamma)$; $\alpha(\mathbf{c})$ is a *curve of constant slope*. Thus the original curve $\mathbf{c}$ by α^{-1} is *affinely equivalent to a curve of constant slope*. If this quadratic cone was a cone of revolution, the space curve would be a curve of constant slope. Therefore the curves with tangents parallel to a general quadratic cone are affinely equivalent to so-called curves of constant slope. Each curve affinely equivalent to a curve of constant slope admits a representation equivalent to (7) and therefore has the desired property.
c) If the curve $\mathbf{c}$ really is a curve of constant slope the spherical image of its tangents covers a part of a circle — the real foci of this special spherical conic collapse. In this special case the two cylinders $\mathbf{f}_{1,2}$ through $\mathbf{c}$ coincide. So we will have to exclude this trivial case in further considerations.

A very special and exceptional case occours for $\omega = k\,\pi/2$, $k \in \{1, 2, 3\}$: Then we have $\cos 2\omega = \cos(k\,\pi) = (-1)^k$ for $\omega = k\,\pi/2$ and equation (7) yields

$$\langle \mathbf{c}', \mathbf{e}_1 \rangle \mp \langle \mathbf{c}', \mathbf{e}_2 \rangle = 0. \tag{8}$$

This is a homogenous linear equation for the coordinates of the tangent vectors $\mathbf{c}'(s)$; the corresponding curves $\mathbf{c}(s)$ are planar. The two cylinders are gained by reflection in the plane of $\mathbf{c}(s)$.

This can be summarized in the following characterisations of CFC-triples $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{c}(s))$ (see the example in Figure 2):

Theorem 1. *Let $\mathbf{c}(s)$ be a C^3-curve with the same geodesic curvature with respect to two different cylinders (generators with directions $\mathbf{e}_{1,2}$) at any of its points. Then the tangents of $\mathbf{c}(s)$ have to be parallel to the generators of a cone Γ of degree 2 which is not a cone of revolution. The directions $\mathbf{e}_{1,2}$ determine the two real focal lines of Γ.*
The cone Γ can degenerate into a plane. In this case the curve $\mathbf{c}(s)$ is planar, and the two directions $\mathbf{e}_{1,2}$ are symmetric with respect to that plane.

Theorem 2. *Through any curve $\mathbf{c}(s)$, either planar or with tangents parallel to a cone Γ of degree 2 (which is not a cone of revolution) there exist two different cylinders $\mathbf{f}_{1,2}$ such that $\mathbf{c}(s)$ has the same geodesic curvature with respect to these two cylinders. The directions of the generators of these two cylinders are determined by the real focal lines of Γ. If $\mathbf{c}(s)$ is planar, any pair of cylinders through $\mathbf{c}$ symmetric with respect to the plane of $\mathbf{c}$ has the desired property.*
Exactly the curves $\mathbf{c}(s)$ of these two special classes and the directions $\mathbf{e}_{1,2}$ corresponding to the real focal lines of the cone Γ form CFC-triples for curved Origami with pairs of cylinders.

4. From the spatial configuration to the planar Origami

We start with a spatial curve $\mathbf{c}(s)$ constructed according to Theorems 1 and 2. The corresponding cylinders $\mathbf{f}_{1,2}$ have generators parallel to $\mathbf{e}_{1,2}$. The planar Origami configuration shall be generated by two series of isometries $\gamma_{1,2}(t)$ of $\mathbf{f}_1$ and $\mathbf{f}_2$ into a common plane π. The real $t \in [0,1]$ shall parametrize these two continous sets of transformations: $t = 0$ shall give the identity map, $t = 1$ shall give the result in π. All these isometries shall be Minding isometries, which keep the generators of the cylinders during these isometries. $\gamma_{1,2}(t)$ transform the curve $\mathbf{c}(s)$ into two series $\mathbf{c}_{1,2}(s,t)$. The generators $\mathbf{e}_{1,2}$ are mapped into $\mathbf{e}_1(t) := \gamma_1(t)(\mathbf{e}_1)$ and $\mathbf{e}_2(t) := \gamma_2(t)(\mathbf{e}_2)$. If possible, we would like to choose the two series $\gamma_{1,2}(t)$ of isometries such that $\mathbf{c}_1(s,t) = \mathbf{c}_2(s,t)$ for all $t \in [0,1]$, $s \in I$.

If this is possible we will call $\gamma_{1,2}(t)$ *two coupled Origami foldings* for the spatial configuration of the two cylinders. In this special case all intermediate stages $\mathbf{c}_1(s,t) = \mathbf{c}_2(s,t)$ for fixed t together with $\mathbf{e}_{1,2}(t)$ again have to define a CFC-triple. Theorems 1 and 2 have to be valid for all fixed $t \in [0,t^*]$. As an isometry does not change angles on the surface the constant 2ω in formula (4) is valid for all possible intermediate stages. The angle between $\mathbf{e}_1(t)$ and $\mathbf{e}_2(t)$ has to change from 2α for the initial state to 2ω for the final planar arrangement (see Figures 1 and 3a).

Now we want to work out a necessary condition for this possible case: We use a Cartesian coordinate frame, put $\mathbf{e}_{1,2}(u(t)) := (0, \pm\sin u(t),\ \cos u(t))^t$ with $u(t) = (1-t)\alpha + t\omega$ and $\mathbf{c}'(s) := (x(s),\ y(s),\ z(s))^t$. Then (7) yields the equation of the corresponding cones of degree two

$$\Gamma(u)\ldots\ x^2\sin^2\omega\cos^2\omega + (\sin^2\omega - \sin^2 u)(y^2\cos^2\omega - z^2\sin^2\omega) = 0. \tag{9}$$

The starting configuration relates to $u(0) = \alpha$, the corresponding planar Origami is reached for $u(1) = \omega$. Equation (9) defines a pencil of cones of degree two, all with symmetry with respect to the three coordinate planes and two real vertex generators in the directions

$$\mathbf{v}_{1,2} = (0,\ \pm\sin\omega,\ \cos\omega)^t. \tag{10}$$

The vectors $\mathbf{e}_{1,2}(u)$ determine the two real focal lines of $\Gamma(u)$. The generators of $\Gamma(u)$ can be

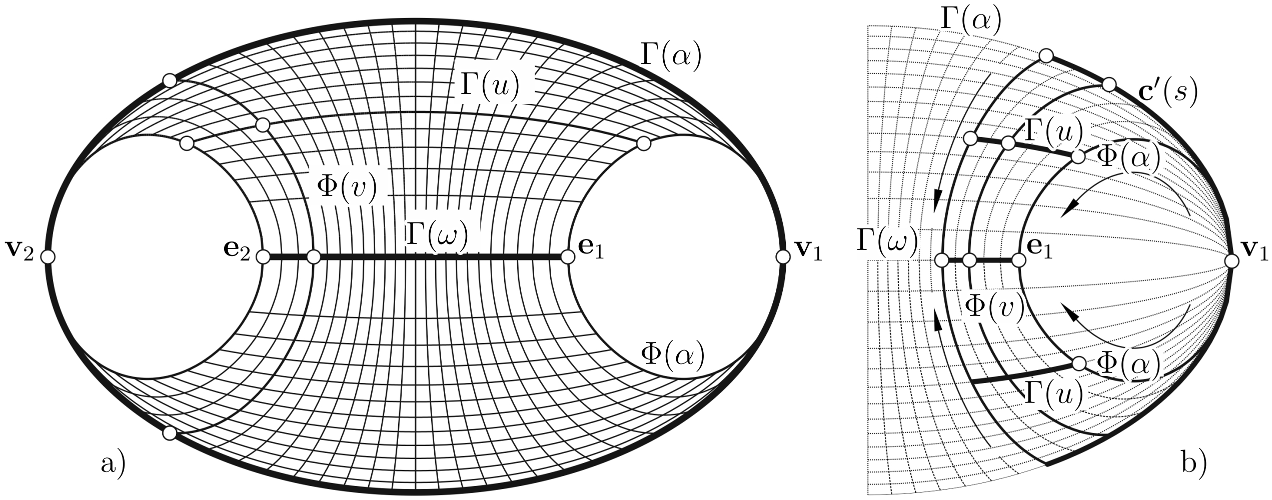


Figure 3: a) Parts of the intersections of the quadratic cones $\Gamma(u)$ and the quartic cones $\Phi(v)$ with the plane $z = 1$ for some values of $u \in [\alpha, \omega]$ and $v \in [\omega - \alpha, 2\omega - \alpha]$.
b) $\mathbf{c}'(s)$ (bold on $\Gamma(\alpha)$) is running across $\mathbf{v}_1$ - the continous Origami folding has to split.

parametrized by

$$\begin{aligned} y \sin u + z \cos u &= \sqrt{x^2 + y^2 + z^2}\,\cos v, \\ -y \sin u + z \cos u &= \sqrt{x^2 + y^2 + z^2}\,\cos(2\omega - v) \end{aligned} \tag{11}$$

with the angle $v \in [\omega - \alpha, 2\omega - \alpha]$ of $\mathbf{e}_{1,2}(u)$ and $\mathbf{c}'(s,t)$. The values $v =$ const. in (11) yield the homogeneous equation of degree four for the corresponding cones $\Phi(v)$ of tangent vectors

$$\begin{aligned} (x^2 + y^2 + z^2)[a(v)z^2 + b(v)y^2], \quad -y^2 z^2 = 0 \quad \text{with} \\ a(v) := \sin^2 \omega \sin^2(\omega - v), \quad b(v) := \cos^2 \omega \cos^2(\omega - v). \end{aligned} \tag{12}$$

For coupled Origami foldings the tangent vectors of any stage $t \in [0, 1]$ of the curve $\mathbf{c}(s,t)$ have to be parallel to generators of the cone $\Gamma(u(t))$. These isometries keep the angles between the generators $\mathbf{e}_{1,2}(u(t))$ and the tangents $\mathbf{c}'(s, u(t))$. Thus, $v = v(s)$ in (11) can be used to parametrize the corresponding generators on the cones $\Gamma(u(t))$ during the Origami folding. Figure 3a displays the intersections of the cones $\Gamma(u)$ and $\Phi(v)$ for some values of u and v for $\alpha = \pi/4$ and $\omega = 3\pi/4$.

The coupled Origami folding from the spatial to the planar configuration yields a continous deformation of the spherical image $\mathbf{c}'(v(s), t)$ from $t = 0$ to $t = 1$. There the point $\mathbf{v}_1$ on the starting configuration $\Gamma(\alpha = u(0))$ has to move on the possible path on $\Phi(\alpha)$ towards $\mathbf{e}_1$ (see Figure 3). This can either be done on the 'upper' or the lower half of $\Phi(\alpha)$. The same situation comes up for $\mathbf{v}_2$ and $\mathbf{e}_2$. Thus, if $\mathbf{c}'(s)$ parametrizes a part of $\mathbf{k} \subset \Gamma(\alpha)$ (see the bold parts in Figure 3b) running across one of the two vertices $\mathbf{v}_{1,2}$ the coupled Origami folding from space to plane will split. Such a folding will not be possible without damaging the configuration.

This yields the following necessary condition for the existence of a coupled Origami folding for pairs of cylinders.

Theorem 3. *For a spatial C^2-curve $\mathbf{c}(s)$ with tangents parallel to a cone $\Gamma(\alpha)$ of degree two (but not a cone of revolution), the real focal lines of Γ determine direction vectors $\mathbf{e}_{1,2}$ which — together with the curve $\mathbf{c}(s)$ — determine a CFC-triple. The two vertex generators in the plane of the real foci $\mathbf{e}_{1,2}$ shall be denoted by $\mathbf{v}_{1,2}$. The coupled Origami folding of the corresponding CFC-triple cannot be performed without damage if the spherical image $\mathbf{c}'(s)$ covers parts on $\Gamma(\alpha)$ running over $\mathbf{v}_1$ or $\mathbf{v}_2$.*

Remark. This is only a necessary but not a sufficient condition for the existence of a coupled Origami folding from the spatial to the planar arrangement with cylinders. The folding can be physically impossible even if the configuration does not fall into the realm of Theorem 3.

5. The discrete version

The spatial C^2-curve $\mathbf{c}(s)$ can be replaced by a spatial polygon $\mathbf{p} := \{\mathbf{p}_j,\ j = 0, \ldots, n\}$ with vertices $\mathbf{p}_j$. The cylinders $\mathbf{f}_{1,2}$ then are replaced by two prisms with generators parallel to unit vectors $\mathbf{e}_{1,2}$. The problem of *curved Origami* with pairs of cylinders then turns into a problem of *polygonal Origami with pairs of prisms*: If there are two isometries $\gamma_i\ (i = 1, 2)$ of the two prisms $\mathbf{f}_i$ into a plane π such that the two planar polygons $\gamma_1(\mathbf{p})$ and $\gamma_2(\mathbf{p})$ are related in a planar displacement we will speak of a discrete curved Origami folding with pairs of prisms. In this special case the corresponding triple $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{p})$ consisting of two unit vectors $\mathbf{e}_{1,2}$ and the spatial polygon $\mathbf{p}$ will be called *triple for polygonal Origami folding with pairs of prisms* or for short *PFP-triple*.

In this case the angles $\phi_{i,j}$ $(i = 1, 2;\ j = 0, \ldots, n-1)$ between the polygon's segments on the lines $[\mathbf{p}_j, \mathbf{p}_{j+1}]$ and the generators of the prisms (parallel to $\mathbf{e}_i$) are kept under the two isometries γ_i $(i = 1, 2)$. The two isometries map the directions $\mathbf{e}_i$ into $\mathbf{e}_i^* := \gamma_i(\mathbf{e}_i)$. In the plane π for a given PFP-triple $(\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{p}_i)$ we get

$$|\phi_1(j) \pm \phi_2(j)| = 2\omega = \text{const.} \tag{13}$$

for all $j = 0, \ldots, n-1$. This is the counterpart to (4) for the discrete case. Thus, our considerations for the continous case can easily be transferred to the discrete case. Theorems 1 and 2 have the following discrete counterpart

Theorem 4. *A polygon* $\mathbf{p}$ *and two direction vectors* $\mathbf{e}_{1,2}$ *for prisms through the polygon make up a PFP-triple for discrete curved Origami with pairs of prims exactly in one of the following two cases:*
Either $\mathbf{p}$ *is a planar polygon or the segments* $[\mathbf{p}_j, \mathbf{p}_{j+1}]$ *of the polygon* $\mathbf{p}$ *are parallel to the generators of a cone* Γ *of degree* 2 *which is not a cone of revolution. In the first case the directions* $\mathbf{e}_{1,2}$ *are symmetric with respect to the plane of the polygon, in the second case they determine the two real focal lines of* Γ.

Theorem 3 can be transferred to the discrete case in the same way:

Theorem 5. *We start with a polygon* $\mathbf{p}$ *and two direction vectors* $\mathbf{e}_{1,2}$ *forming a PFP-triple. The segments* $[\mathbf{p}_j, \mathbf{p}_{j+1}]$ *of the polygon* $\mathbf{p}$ *are parallel to the generators of a cone* $\Gamma(\alpha)$ *of degree two which is not a cone of revolution. The two vertex generators in the plane of the real foci* $\mathbf{e}_{1,2}$ *shall be denoted by* $\mathbf{v}_{1,2}$.
The coupled Origami folding of the corresponding PFP-triplet cannot be performed without damage if the spherical image of the polygon's segments $[\mathbf{p}_j, \mathbf{p}_{j+1}]$ *covers parts on* $\Gamma(\alpha)$ *running over* $\mathbf{v}_1$ *or* $\mathbf{v}_2$.

6. Conclusions

We studied the problem of curved Origami folding with pairs of cylinders (generators parallel to $\mathbf{e}_{1,2}$) meeting in curve $\mathbf{c}(s)$. We speak of a CFC-triple if the configuration can be isometrically transformed into a planar Origami. We were able to characterize these curves as affine images of curves of constant slope. The spherical image $\mathbf{c}'(s)$ of their tangents must be contained in a spherical conic (but not in a circle). The real foci of this spherical conic uniquely determine the directions $\mathbf{e}_{1,2}$ of the generators of the two possible cylinders through such curves $\mathbf{c}(s)$. We worked out a necessary condition for possible Origami folds from the spatial to the planar arrangement. Fortunately, the results can be transferred to the discrete case without any reservations.

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