Journal for Geometry and Graphics
Volume 22 (2018), No. 1, 15–20.

Ruled Surfaces of Finite Chen-Type

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Abstract. In this paper, we study ruled surfaces in the 3-dimensional Euclidean space which are of finite *III*-type, that is, they are of finite type, in the sense of B.-Y. Chen, with respect to the third fundamental form. We show that helicoids are the only ruled surfaces of finite *III*-type.

Key Words: Surfaces in Euclidean space, Surfaces of finite type, Beltrami operator.

MSC 2010: 53A05, 47A75

1. Introduction

To set the stage for our work, we present briefly some elements of the theory of surfaces of finite type in the Euclidean space $\mathbb{E}^3$. We follow the notations and definitions of [5].

In the three-dimensional Euclidean space $\mathbb{E}^3$ let S be a C^r-surface, $r \geq 3$, defined on a region U of $\mathbb{R}^2$, by an injective C^r-immersion $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$, whose Gaussian curvature K never vanishes. We denote by

$$I = g_{ij}\, du^i du^j, \quad II = h_{ij}\, du^i du^j, \quad III = e_{ij}\, du^i du^j, \quad i, j = 1, 2,$$

the first, second and third fundamental forms of S and by g^{ij}, h^{ij} and e^{ij} the inverses of the tensors g_{ij}, h_{ij} and e_{ij}.

The notion of Euclidean immersions of finite type was introduced in 1983 by B.-Y. Chen, see [4]. In terms of B.-Y. Chen theory, a surface S is said to be *of finite type* if its coordinate

ISSN 1433-8157/\$ 2.50

functions are a finite sum of eigenfunctions of its second Beltrami operator $\triangle^I$ with respect to the first fundamental form I.

In the thematic circle of the surfaces of finite type in the Euclidean space $\mathbb{E}^3$, S. Stamatakis and H. Al-Zoubi in [8] introduced the notion of surfaces of finite type with respect to the second or the third fundamental form of S in the following way: A surface S is said to be of finite type with respect to the fundamental form J, or briefly *of finite J-type*, where $J = II$, III, if the position vector $\boldsymbol{x}$ of S can be written as a finite sum of nonconstant eigenvectors of the second Beltrami operator $\triangle^J$ with respect to the fundamental form J, that is, if

$$\boldsymbol{x} = \boldsymbol{x}_0 + \sum_{i=1}^{k} \boldsymbol{x}_i, \quad \triangle^J \boldsymbol{x}_i = \lambda_i \, \boldsymbol{x}_i, \quad i = 1, \dots, k, \tag{1}$$

where $\boldsymbol{x}_0$ is a constant vector and $\lambda_1, \lambda_2, \dots, \lambda_k$ are eigenvalues of the operator $\triangle^J$. In particular if all eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ are mutually distinct, then S is said to be of finite J-type k. When $\lambda_i = 0$ for some $i = 1, \dots, k$, then S is said to be *of finite null J-type k*. Otherwise S is said to be *of infinite type*.

In the sequel we focus on surfaces which are of finite III-type. Up to now, the only known surfaces of finite III-type are

(a) the spheres, which actually are of finite III-type 1,

(b) the minimal surfaces, which are of finite null III-type 1, and

(c) the parallel surfaces to the minimal surfaces, which in fact are of finite null III-type 2 (see [8]).

So the following question seems to be interesting:

Problem 1. *Are there other surfaces of finite III-type in $\mathbb{E}^3$, besides the above mentioned?*

In order to give an answer to the above problem, one can study important families of surfaces. More precisely, in [1] H. Al-Zoubi, K. Jaber and S. Stamatakis studied tubes in $\mathbb{E}^3$ and they proved that all tubes in $\mathbb{E}^3$ are of infinite type. On the other hand, classical families of surfaces, such as surfaces of revolution, translation surfaces, quadrics, cyclides of Dupin, spiral surfaces as well as helicoidal surfaces, the classification of its finite type surfaces with respect to the third fundamental form is not known yet.

Another generalization of the above problem is to study surfaces in $\mathbb{E}^3$ with the position vector $\boldsymbol{x}$ satisfying

$$\Delta^{III} \boldsymbol{x} = A \, \boldsymbol{x} \tag{2}$$

where $A \in \mathbb{R}^{3\times 3}$.

From this point of view, we also pose the following problem:

Problem 2. *Classify all surfaces in $\mathbb{E}^3$ with the position vector $\boldsymbol{x}$ satisfying relation* (2).

Concerning this problem, in [9] S. Stamatakis and H. Al-Zoubi studied the class of surfaces of revolution and they proved: A surface of revolution S satisfies (2) if and only if S is a catenoid or part of a sphere. In [2] the same authors studied the class of ruled surfaces and the class of quadric surfaces. In particular, they proved that helicoids and spheres are the only ruled and quadric surfaces satisfying (2), respectively. Recently, in [3] the same authors and others studied the class of translation surfaces and they proved that the only translation surface in the 3-dimensional Euclidean space which satisfies (2) is Scherk's surface.

In this paper we contribute to the solution of the first problem by investigating the ruled surfaces in $\mathbb{E}^3$. Our main result is the following

Theorem 1. *The only ruled surfaces of finite III-type in the three-dimensional Euclidean space are the helicoids.*

2. Proof of Theorem 1

In the three-dimensional Euclidean space $\mathbb{E}^3$ let S be a ruled C^r-surface[1], $r \geq 3$, of non-vanishing Gaussian curvature defined by an injective C^r-immersion $\boldsymbol{x} = \boldsymbol{x}(s,t)$ on a region $U := I \times \mathbb{R}$ ($I \subset \mathbb{R}$ open interval) of $\mathbb{R}^2$. The surface S can be expressed in terms of a directrix curve $\mathfrak{F}\colon \boldsymbol{a} = \boldsymbol{a}(s)$ and a unit vector field $\boldsymbol{b}(s)$ pointing along the rulings as follows

$$S\colon\ \boldsymbol{x}(s,t) = \boldsymbol{a}(s) + t\,\boldsymbol{b}(s), \quad s \in I, \quad t \in \mathbb{R}. \tag{3}$$

Moreover, we can take the parameter s to be the arc length along the spherical curve $\boldsymbol{b}(s)$. Then we have

$$\langle \boldsymbol{a}', \boldsymbol{b}\rangle = 0, \quad \langle \boldsymbol{b}, \boldsymbol{b}\rangle = 1, \quad \langle \boldsymbol{b}', \boldsymbol{b}'\rangle = 1,$$

where the differentiation with respect to s is denoted by a prime and $\langle\,,\rangle$ denotes the standard scalar product in $\mathbb{E}^3$. It is easily verified that the first and the second fundamental forms of S are given by

$$\begin{aligned} I &= q\,ds^2 + dt^2, \\ II &= \frac{p}{\sqrt{q}}\,ds^2 + \frac{2A}{\sqrt{q}}\,ds\,dt, \end{aligned}$$

where

$$\begin{aligned} q &:= \langle \boldsymbol{a}', \boldsymbol{a}'\rangle + 2\langle \boldsymbol{a}', \boldsymbol{b}'\rangle t + t^2, \\ p &:= (\boldsymbol{a}', \boldsymbol{b}, \boldsymbol{a}'') + [(\boldsymbol{a}', \boldsymbol{b}, \boldsymbol{b}'') + (\boldsymbol{b}', \boldsymbol{b}, \boldsymbol{a}'')]\,t + (\boldsymbol{b}', \boldsymbol{b}, \boldsymbol{b}'')\,t^2, \\ A &:= (\boldsymbol{a}', \boldsymbol{b}, \boldsymbol{b}'). \end{aligned}$$

If, for simplicity, we put

$$\begin{aligned} \kappa &:= \langle \boldsymbol{a}', \boldsymbol{a}'\rangle, & \lambda &:= \langle \boldsymbol{a}', \boldsymbol{b}'\rangle, \\ \mu &:= (\boldsymbol{b}', \boldsymbol{b}, \boldsymbol{b}''), & \nu &:= (\boldsymbol{a}', \boldsymbol{b}, \boldsymbol{b}'') + (\boldsymbol{b}', \boldsymbol{b}, \boldsymbol{a}''), \\ \rho &:= (\boldsymbol{a}', \boldsymbol{b}, \boldsymbol{a}''), \end{aligned}$$

we have

$$q = t^2 + 2\lambda t + \kappa, \qquad p = \mu t^2 + \nu t + \rho.$$

For the Gauss curvature K of S we find

$$K = -\frac{A^2}{q^2}. \tag{4}$$

The second Beltrami differential operator with respect to the third fundamental form is defined by

$$\triangle^{III} f = \frac{-1}{\sqrt{e}}\,\frac{\partial\left(\sqrt{e}\,e^{ij}\,\dfrac{\partial f}{\partial u^i}\right)}{\partial u^j},$$

[1] The reader is referred to [7] for definitions and formulae on ruled surfaces.

where f is a sufficiently differentiable function on S and $e := \det(e_{ij})$. After a long computation it can be expressed as follows (see [2]):

$$\begin{aligned}
\Delta^{III} &= -\frac{q}{A^2}\frac{\partial^2}{\partial s^2} + \frac{2qp}{A^3}\frac{\partial^2}{\partial s \partial t} - \left(\frac{q^2}{A^2} + \frac{qp^2}{A^4}\right)\frac{\partial^2}{\partial t^2} + \left(\frac{q_s}{2A^2} + \frac{qp_t}{A^3} - \frac{pq_t}{2A^3}\right)\frac{\partial}{\partial s} \\
&\quad + \left(\frac{qp_s}{A^3} - \frac{pq_s}{2A^3} - \frac{pqA'}{A^4} - \frac{qq_t}{2A^2} + \frac{p^2q_t}{2A^4} - \frac{2qpp_t}{A^4}\right)\frac{\partial}{\partial t} \\
&= Q_1\frac{\partial^2}{\partial s^2} + Q_2\frac{\partial^2}{\partial s \partial t} + Q_3\frac{\partial}{\partial s} + Q_4\frac{\partial}{\partial t} + Q_5\frac{\partial^2}{\partial t^2}
\end{aligned} \tag{5}$$

where

$$q_t := \frac{\partial q}{\partial t}, \quad q_s := \frac{\partial q}{\partial s}, \quad p_t := \frac{\partial p}{\partial t}, \quad p_s := \frac{\partial p}{\partial s}$$

and $Q_1, Q_2, \ldots, Q_5$ are polynomials in t with functions in s as coefficients, and $\deg(Q_i) \leq 6$. More precisely, we have

$$\begin{aligned}
Q_1 &= -\frac{1}{A^2}\left[t^2 + 2\lambda t + \kappa\right], \\
Q_2 &= \frac{2}{A^3}\left[\mu t^4 + (2\lambda\mu + \nu)\,t^3 + (2\lambda\nu + \rho + \kappa\mu)\,t^2 + (2\lambda\rho + \kappa\nu)\,t + \kappa\rho\right], \\
Q_3 &= \frac{1}{A^3}\left[\mu t^3 + 3\lambda\mu t^2 + (\lambda\nu - \rho + 2\kappa\mu + \lambda' A)\,t + \frac{1}{2}\kappa' A - \lambda\rho + \kappa\nu\right], \\
Q_4 &= \frac{1}{A^4}\Big[-3\mu^2 t^5 + (\mu' A - \mu A' - 4\mu\nu - 7\lambda\mu^2)\,t^4 \\
&\quad + (\nu' A - \nu A' + 2\lambda\mu' A - 2\lambda\mu A' - \lambda'\mu A - A^2 - 10\lambda\mu\nu - 2\mu\rho - \nu^2 - 4\kappa\mu^2)\,t^3 \\
&\quad + \Big(\kappa\mu' A - \kappa\mu A' - \frac{1}{2}\kappa'\mu A + 2\lambda\nu' A - 2\lambda\nu A' - \lambda'\nu A - \rho A' + \ rho' A - 3\lambda A^2 \\
&\quad - 3\lambda\nu^2 - 6\lambda\mu\rho - 6\kappa\mu\nu\Big)\,t^2 + \Big(\kappa\nu' A - \kappa\nu A' - \frac{1}{2}\kappa'\nu A + 2\lambda\rho' A \\
&\quad - 2\lambda\rho A' - \lambda'\rho A - \kappa A^2 - 2\lambda^2 A^2 - 2\kappa\nu^2 + \rho^2 - 2\lambda\nu\rho - 4\kappa\mu\rho\Big)\,t \\
&\quad + \Big(\kappa\rho' A - \kappa\rho A' - \frac{1}{2}\kappa'\rho A + \lambda\rho^2 - \kappa\lambda A^2 - 2\kappa\nu\rho\Big)\Big], \\
Q_5 &= -\frac{1}{A^4}\big[\mu^2 t^6 + (2\mu\nu + 2\lambda\mu^2)\,t^5 + (2\mu\rho + \nu^2 + 4\lambda\mu\nu + \kappa\mu^2 + A^2)\,t^4 \\
&\quad + (2\nu\rho + 4\lambda\mu\rho + 2\lambda\nu^2 + 2\kappa\mu\nu + 4\lambda A^2)\,t^3 + (\rho^2 + 4\lambda\nu\rho + 2\kappa\mu\rho \\
&\quad + \kappa\nu^2 + 4\lambda^2 A^2 + 2\kappa A^2)\,t^2 + (2\lambda\rho^2 + 2\kappa\nu\rho + 4\lambda\kappa A^2)\,t + (\kappa\rho^2 + \kappa^2 A^2)\big].
\end{aligned}$$

Applying (5) on the position vector (3) of the ruled surface S we find

$$\Delta^{III}\boldsymbol{x} = Q_1\boldsymbol{a}'' + Q_2\boldsymbol{b}' + Q_3\boldsymbol{a}' + Q_4\boldsymbol{b} + (Q_1\boldsymbol{b}'' + Q_3\boldsymbol{b}')\,t. \tag{6}$$

We write this last expression of $\triangle^{III}\boldsymbol{x}$ as a vector $\boldsymbol{P}_1(t)$ whose components are polynomials in t with functions in s as coefficients as follows:

$$\begin{aligned}\boldsymbol{P}_1(t) \;=\; & \frac{1}{A^4}\Big[-3\mu^2\boldsymbol{b}t^5 + [(\mu' A - \mu A' - 4\mu\nu - 7\lambda\mu^2)\,\boldsymbol{b} + 3\mu A\boldsymbol{b}']\,t^4 \\ & + [\mu A\boldsymbol{a}' - A^2\boldsymbol{b}'' + (2\nu A + 7\lambda\mu A)\boldsymbol{b}' + (\nu' A - \nu A' + 2\lambda\mu' A - 2\lambda\mu A' - \lambda'\mu A \\ & - A^2 - 10\lambda\mu\nu - 2\mu\rho - \nu^2 - 4\kappa\mu^2)\boldsymbol{b}]\,t^3 + \big[\big(\kappa\mu' A - \kappa\mu A' - \frac{1}{2}\kappa'\mu A \\ & + 2\lambda\nu' A - 2\lambda\nu A' - \lambda'\nu A - \rho A' + \rho' A - 3\lambda A^2 - 3\lambda\nu^2 - 6\lambda\mu\rho - 6\kappa\mu\nu\big)\boldsymbol{b} \\ & + 3\lambda\mu A\boldsymbol{a}' - 2\lambda A^2\boldsymbol{b}'' - A^2\boldsymbol{a}'' + (\lambda' A + 5\lambda\nu + 4\kappa\mu + \rho)A\boldsymbol{b}'\big]\,t^2 \\ & + \big[\big(\frac{1}{2}\kappa' A + 3\kappa\nu + 3\lambda\rho\big)A\boldsymbol{b}' + \big(\kappa\nu' A - \kappa\nu A' - \frac{1}{2}\kappa'\nu A + 2\lambda\rho' A - 2\lambda\rho A' \\ & - \lambda'\rho A - \kappa A^2 - 2\lambda^2 A^2 - 2\kappa\nu^2 + \rho^2 - 2\lambda\nu\rho - 4\kappa\mu\rho\big)\boldsymbol{b} \\ & - 2\lambda A^2\boldsymbol{a}'' - \kappa A^2\boldsymbol{b}'' + (\lambda\nu - \rho + 2\kappa\mu + \lambda' A)\,A\boldsymbol{a}'\big]\,t \\ & + \big(\kappa\rho' A - \kappa\rho A' - \frac{1}{2}\kappa'\rho A + \lambda\rho^2 - \kappa\lambda A^2 - 2\kappa\nu\rho\big)\boldsymbol{b} \\ & -\kappa A^2\boldsymbol{a}'' + 2\kappa\rho A\boldsymbol{b}' + \left(\frac{1}{2}\kappa' A - \lambda\rho + \kappa\nu\right)A\boldsymbol{a}'\Big].\end{aligned} \tag{7}$$

Notice that $\deg(\boldsymbol{P}_1) \leq 5$. Furthermore, we have $\deg(\boldsymbol{P}_1) = 5$ if and only if $\mu \neq 0$, otherwise $\deg(\boldsymbol{P}_1) \leq 3$.

Before we start the proof of the first theorem we give the following Lemma which can be proved by a straightforward computation.

Lemma 1. *Let g be a polynomial in t with functions in s as coefficients and* $\deg(g) = d$. *Then* $\triangle^{III} g = \widehat{g}$, *where $\widehat{g}$ is a polynomial in t with functions in s as coefficients and* $\deg(\widehat{g}) \leq d+4$.

We suppose that S is of finite III-type k. It is well known that there exist real numbers $c_1, \dots, c_k$ such that

$$\left(\triangle^{III}\right)^{k+1}\boldsymbol{x} + c_1\left(\triangle^{III}\right)^{k}\boldsymbol{x} + \cdots + c_k\,\triangle^{III}\boldsymbol{x} = \boldsymbol{0} \tag{8}$$

(see [4]). By applying Lemma 1, we conclude that there is an $\mathbb{E}^3$-vector-valued function $\boldsymbol{P}_k$ in the variable t with some functions in s as coefficients, such that

$$\left(\triangle^{III}\right)^{k}\boldsymbol{x} = \boldsymbol{P}_k(t),$$

where $\deg(\boldsymbol{P}_k) \leq 4k+1$. Now, if k goes up by one, the degree of each component of $\boldsymbol{P}_k$ goes up at most by 4. Hence the sum (8) can never be zero, unless of course [6]

$$\triangle^{III}\boldsymbol{x} = \boldsymbol{P}_1 = \boldsymbol{0}. \tag{9}$$

On account of the well known relation

$$\triangle^{III}\boldsymbol{x} = \nabla^{III}\left(\frac{2H}{K}, \boldsymbol{n}\right) - \frac{2H}{K}\boldsymbol{n}$$

where H, $\boldsymbol{n}$ and ∇^{III} denote the mean curvature, the unit normal vector field and the first Beltrami-operator with respect to III (see [8]), from (9) we result that S is minimal, and that S is a helicoid.

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Received December 16, 2017; final form April 12, 2018