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A Synthetic Way to Geometrize the Method of Coordinates*

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Abstract. In this paper we propose a synthetic way (ensuing from Euclid's *Elements*) to geometrize the *method of coordinates* and thus to reformulate analytic geometry using a synthetic, axiomatic approach. In the theory that we will develop, the *segment arithmetic* (Streckenrechnung) introduced by David Hilbert in his *Grundlagen der Geometrie* plays a crucial role. Analytic geometry has fundamental scientific and mathematical significance since, e.g., it is essential for the application of mathematics to physical and natural sciences. Our synthetic approach is certainly useful for a theoretical understanding of hierarchical structures of axiomatic theories, it can stimulate problem solving in the spirit of undergraduate mathematics, and it can even help to enhance classroom learning, all this being very important in modern times.

Key Words: Coordinate system, Euclidean geometry, analytic geometry, synthetic geometry, segment arithmetic, Theorem of Desargues, Theorem of Pappus

MSC 2010: 51M05, 51N20, 03A05

1. Introduction

The first step of the transition process from synthetic to analytic geometry was taken with the birth of the *coordinate method*. Around 1630, Pierre de Fermat and René Descartes independently discovered the advantages of using numbers in geometry as coordinates.

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DESCARTES was the first who published a detailed account in his book *La géométrie* from 1637. This program, carried out with the help of coordinates, is called the *arithmetization of geometry*, but according to [6]

> *the truth is that Descartes had no intention of arithmetizing geometry. In fact, the purpose of "La géométrie" might, with equal validity, be described as the translation of algebraic operations into the language of geometry.*

When David HILBERT gave in 1891 his first course on geometry, he proposed a division of geometry into three different branches: intuitive geometry, axiomatic geometry, and analytic geometry. And he decided to take a purely synthetic approach (ensuing from EUCLID's *Elements*). Recently, based on a set of unpublished notes used by HILBERT for courses on geometry, the author of [14] gives an interpretation of internal arithmetization of geometry in [15], i.e., in HILBERT's *Grundlagen der Geometrie* (Foundations of Geometry):

> *A central concern that motivated* HILBERT*'s axiomatic investigations from very early on was the aim of providing an independent basis for geometry. Accordingly, these concerns about an independent grounding for elementary geometry determined very clear methodological constraints in the process of embedding it into a formal axiomatic system.*

Still in [14] it is said that HILBERT did not only try to show that geometry could be considered as a pure mathematical theory, when presented as a formal axiomatic system; he also aimed to show that in the construction of such an axiomatic system one could proceed purely geometrically, avoiding concepts borrowed from other mathematical disciplines like arithmetic or analysis.

In his *Elements*, EUCLID interrupted his geometrical expositions after the first four books in order to expound a theory of proportion; the application to plane figures then follows in Book VI. With *segment arithmetic* (Streckenrechnung), defined in his *Grundlagen der Geometrie* on a purely synthetic basis, HILBERT developed a suitable proportion theory and accomplished a unification of two theories that, from the time of EUCLID, had always stood on separate foundations (cf. [17]). What none saw before HILBERT was, however, the full scope of possibilities for arithmetizing geometry from inside. This was meant in the sense of building new bridges from purely synthetic, axiomatic geometry to analytic geometry that operate over various number fields (see again [17]). In [20] J. STILLWELL wrote that

> *Hilbert transformed our view of the Pappus and Desargues theorems by showing that they express the underlying algebraic structure of projective geometry* [...]*. Hilbert's treatments of projective and hyperbolic geometry have another important common element: construction of real numbers. To achieve this, Hilbert has to add an axiom of continuity to the geometry axioms, but he evidently wants to show that the real numbers can be put on a geometric foundation.*

As it was noted by VAILATI (see [21]), it seems that already EUCLID wanted to look for alternate demonstrations demanding the use of the theory of proportions, for example as if he supplies also a demonstration based on the 'equivalence theory' of the Pythagorean theorem. Moreover, he highlighted that during the Renaissance there was a new interest in the geometric speculations as well as the tendency to replace the theory of proportions exposed in EUCLID's *Elements* by one that was 'more' geometric, and he also described this theoretical way of the Italian and the German school until the beginning of the 19th century citing V. GIORDANO, H. GRASSMANN, L. RAJOLA-PESCARINI, R. HOPPE, and G. BIAGI. In the development of a theory of proportions that is not depending on the equivalence theory, the theorems of Pappus and Desargues play an essential role. This strict connection was also highlighted more recently in [5]. Using a synthetic, axiom based approach, we will reformulate analytic geometry on a purely synthetic basis. In the theory that we will develop a crucial role is played by the segment arithmetic of HILBERT (see also [1]).

The introduction of coordinates in affine space via geometric (but not synthetic) axioms is presented in the book *Projective and Polar Spaces* by P. CAMERON (see [7, Chap 3]), or in the monograph *Geometric Algebra* by E. ARTIN (see [2, Chap 2]). ARTIN wrote:

> *We are all familiar with analytic geometry where a point in a plane is described by a pair* (x, y) *of real numbers, a straight line by a linear equation* [...]. *A much more fascinating problem is, however, the converse. Given a plane geometry* [...] *assume that certain axioms of geometric nature are true. Is it possible to find a field* $\mathbb{K}$ *such that the points of our geometry can be described by coordinates from* $\mathbb{K}$ *and the lines by linear equations?*

It is known that analytic geometry has a scientific and mathematical significance since it is essential for the application of mathematics to physical and natural sciences. A synthetic approach to analytic geometry could be useful also to remove the evident gap that exists between the manner in which vector calculus is usually taught by mathematicians and the way in which it is used by other scientists, in particular physicists (cf. [12]).

Moreover, a synthetic approach is justified by the fact that nowadays geometry, as it is represented in EUCLID's *Elements*, is somehow recovered and takes into account also developments in contemporary mathematics (see [1] and several references given there). One interesting consequence is that recent representations of geometric fields tend to recover the contribution of diagrammatic reasoning, also and in particular within axiomatics (see [3] and [16]). We think that this approach could be useful to stimulate problem solving, also in the framework of undergraduate mathematics, and even to enhance classroom learning.

In this work, renouncing the minimality of the classical Euclidean axioms of the plane $\mathcal{E}^2$ and the space $\mathcal{E}^3$, we will add as axioms the configurations of Pappus and Desargues (see Figure 2). This will allow us to develop in Section 2 a fluently and natural theory of proportions in $\mathcal{E}^3$ just based on geometric objects, that overcome the problems that we have mentioned above. Using this theory in Sections 3 and 4, we identify the natural structure of normed vector spaces in Euclidean space and, within this, the parametric equations of straight lines, planes, circles, and spheres. Moreover, we will introduce the fundamental notions of scalar product and vector product from a synthetic point of view. In Section 5 the transition from synthetic geometry to analytic geometry is completed by introducing the notion of Cartesian coordinate system, which allows the usual identification of the Euclidean plane with $\mathbb{R}^2$, and of the Euclidean space with $\mathbb{R}^3$.

2. Geometric proportions in $\mathcal{E}^3$

The theory of proportion is not easy to follow in EUCLID's book, and it represents also a critical aspect in teaching. The mathematician and philosopher H. FREUDENTHAL states that HILBERT, with his Proportion Theory based on 'Streckenrechnung', has been able to make something that can be considered as an ideal for classical Greek thought: an integrated geometric treatment of mathematics [13]. In this section we will introduce a notion of geometric proportion in Euclidean space $\mathcal{E}^3$, that extends the notion of geometric proportion in the Euclidean plane $\mathcal{E}^2$ given in [1].

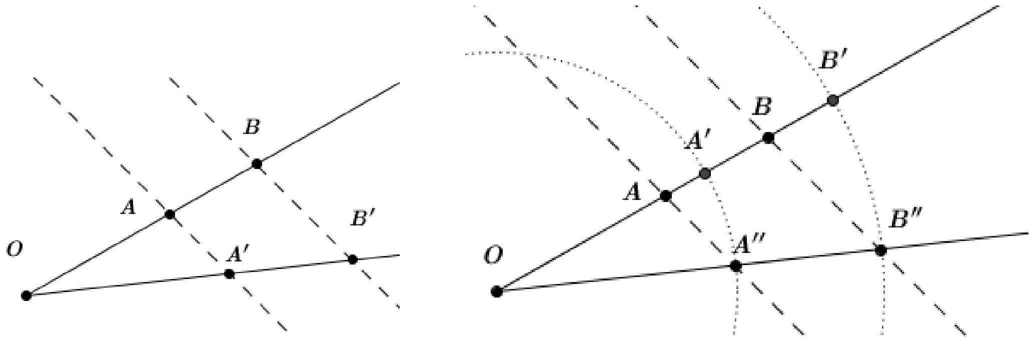

Figure 1: Geometric proportions

In the spirit of synthetic geometry, we will say that two segments (resp. angles) are *equal* if they can be transported one into the other by means of the classical constructions in Euclidean space.

In a Euclidean space $\mathcal{E}^3$ with fixed point O, we consider two points A and A', distinct from O and such that the *rays* OA and OA' with apex O do not coincide. Let $B \in OA$ and $B' \in OA'$. We shall write $A : B =_O A' : B'$, or more briefly $A : B = A' : B'$, if and only if $AA' \| BB'$ (meaning parallelity; see Figure 1 on the left).

Obviously, if $A = B$, then $A' = B'$, and if A and A' belong to the same circle with center O, the same applies for B and B'. The definition and properties also can be given when the rays OA and OA' coincide. In this case

$$A : B = A' : B'$$

means that $A : B = A'' : B''$, with A'' not belonging to the straight line OA, and belonging to the sphere with center O through A', and B'' is the intersection between the ray OA'' and the sphere with center O through B' (see Figure 1 on the right). The formula $A : B = A' : B'$ gets the name of a *(geometric) proportion.*

As in [1] we assume the following statements I and II as *axioms*.

I (Pappus): If H, C belong to the ray OB and K, L belong to the ray OA, then

$$(A : L = B : H,\ K : A = H : C) \implies K : L = B : C\,.$$

II (Desargues): If A' belongs to the ray OA, B' belongs to the ray OB, and C' belongs to ray OC, then the following implication holds:

$$(A : A' = B : B',\ B : B' = C : C') \implies A : A' = C : C'.$$

With the obvious changes in the formulation, all the classical properties of numerical proportions remain valid.

3. The natural structure of a vector space in $\mathcal{E}^2$ and $\mathcal{E}^3$

By proportion theory introduced in the above section, following [1], we have that it is possible to define a structure of a (complete ordered) field over a straight line OU.

We denote this structure as $\mathbb{R}_{OU}$. The elements of $\mathbb{R}_{OU}$ are called *scalars* and denoted by Latin lowercase letters. Obviously, the same field $\mathbb{R}_{OU}$ is defined in each plane of the Euclidean space to which O and U belong.

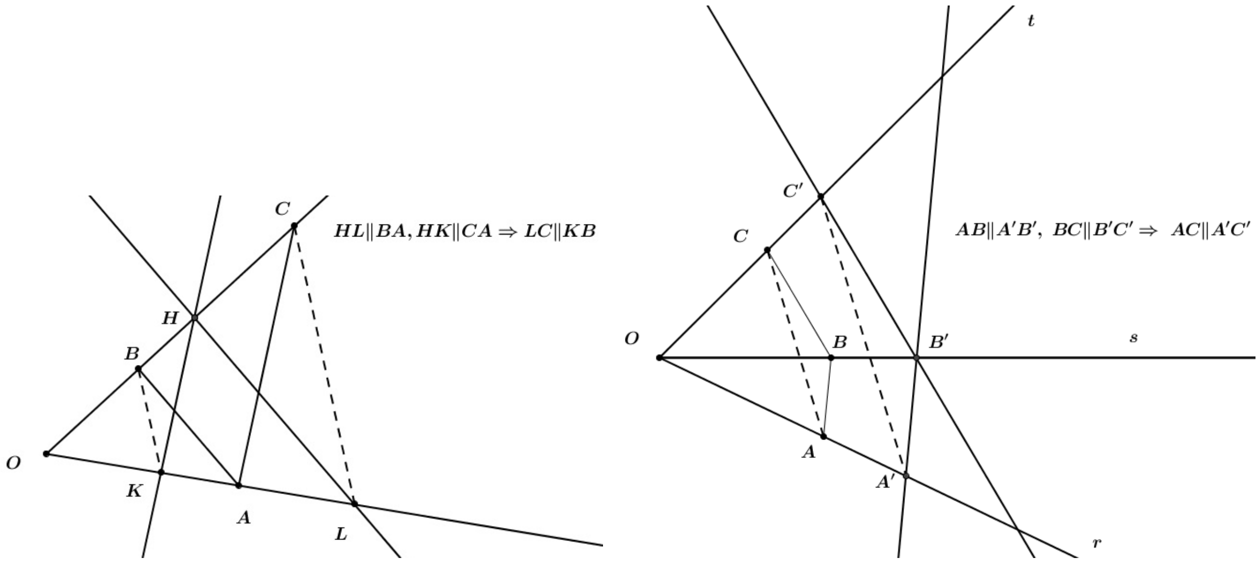

Figure 2: Configurations of Pappus (on the left) and Deasargues (on the right) in $\mathcal{E}^3$

$\mathcal{E}^2$ and $\mathcal{E}^3$ can be structured as vector spaces with the following operations:

Sum of points.
$P + Q = K$, where K is the point such that the ray PK is parallel and with the same orientation of the ray OQ, and the segment $\overline{PK}$ is equal to $\overline{OQ}$. The point K can be obtained via construction, by moving the angle and of the segment, and it is uniquely determined (see Figure 3).

If O, P and Q are aligned, $P+Q$ is on the straight line passing through them, and the sum is reduced to the sum of lengths of segments of ordinary Euclidean geometry. If the points O, P and Q are not aligned, the point $P+Q$ that is obtained can be seen as the vertex opposite to O in the parallelogram whose ordered vertices are P, O, Q, $P+Q$, and the triangle $\triangle(UOQ)$ is translated in the triangle with vertices U', P and $P+Q$. The construction does not lose its meaning if $P = O$; in fact, in this case it is $P + O = P$ and $O + P = P$.

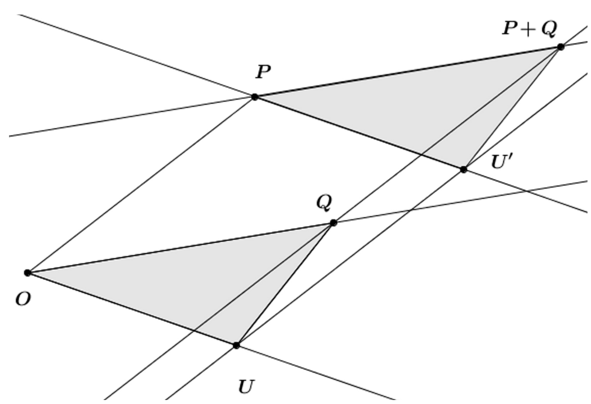

Figure 3: Sum of two points

Multiplication of a scalar of $\mathbb{R}_{OU}$ by a point of $\mathcal{E}^2$ [resp. of $\mathcal{E}^3$].
Let $t \in \mathbb{R}_{OU}$ and $P \in \mathcal{E}^2$ [resp. of $\mathcal{E}^3$]. We set: $t \cdot P = tP = O$ if either $t = O$ or $P = O$, and

if $t \neq O$ and $P \neq O$, then $t \cdot P = tP = K$, where K is the point satisfying the proportion

$$U : t = P : K \quad \text{when } t \text{ belongs to the ray } OU,$$
$$-U : t = -P : K \quad \text{when } t \text{ belongs to the ray } O(-U).$$

Remark 1. *In Euclidean space, consider the two scalar fields $\mathbb{R}_{OU}$ and $\mathbb{R}_{OU'}$, with U and U' belonging to the same sphere of center O. Let $t \in \mathbb{R}_{OU}$, $t' \in \mathbb{R}_{OU'}$ and t' belong to the sphere with center O and passing through t. If t' belongs to the ray OU' and t belongs to the ray OU, or if t' belongs to the ray $O(-U')$ and t belongs to the ray $O(-U)$, then $t'A = tA$. If t' belongs to the plane AOB, then $t'(A + B) = t'A + t'B$, and hence we have $t(A + B) = tA + tB$ for each t (by $t'(A + B) = t(A + B)$ and $t'A + t'B = tA + tB$).*

Using the notion of *geometric proportion* and replacing the Theorem of Thales (intercept theorem) by axioms I and II, one can easily check that the argument that Stillwell used in [19], to highlight the intrinsic algebraic structure on a straight line, also shows that the set of points of the Euclidean plane and, by Remark 1, the set of the points of the Euclidean space can be seen as a vector space over the scalar field $\mathbb{R}_{OU}$.

We call these structures the *Euclidean plane OU*, denoted by $\mathcal{E}^2_{OU}$ [resp. the *Euclidean space OU*, denoted by $\mathcal{E}^3_{OU}$]. Here the points of the straight line OU can play indifferently the role of scalars or vectors.

Remark 2. *Let $Q \neq O$. Then we have:*

1. *For each $t \in \mathbb{R}_{OU}$, $P = tQ$ belongs to the straight line OQ. Conversely, for each P belonging to the straight line OQ there exists a unique scalar t such that $P = tQ$. Then $P = tQ$ is an equation of the straight line OQ.*
2. *Let P_0 be a point of the $\mathcal{E}^3_{OU}$. For each $t \in \mathbb{R}_{OU}$, the points $P = P_0 + tQ$, $t \in \mathbb{R}_{OU}$, belong to the straight line r passing through P_0 and being parallel to OQ. Conversely, for each point P of r there exists a unique scalar t such that $P = P_0 + tQ$. Then $P = P_0 + tQ$ is an equation of the straight line r.*

Remark 3. *Let Q_1, Q_2 be non-collinear with O. Then we have:*

1. *For each ordered pair $(r, s) \in \mathbb{R}^2_{OU}$, $P = rQ_1 + sQ_2$ belongs to the plane OQ_1Q_2. Conversely, for each P belonging to the plane OQ_1Q_2 there exists a unique ordered pair $(r, s) \in \mathbb{R}^2_{OU}$ such that $P = rQ_1 + sQ_2$. Then $P = rQ_1 + sQ_2$ is an equation of the plane OQ_1Q_2.*
2. *Let P_0 be a point of the $\mathcal{E}^3_{OU}$. For each pair $(r, s) \in \mathbb{R}^2_{OU}$, $P = P_0 + rQ_1 + sQ_2$ belongs to the plane π passing through P_0 and being parallel to the plane OQ_1Q_2. Conversely, for each P belonging to the plane π there exists a unique ordered pair $(r, s) \in \mathbb{R}^2_{OU}$ such that $P = P_0 + rQ_1 + sQ_2$. Then $P = P_0 + rQ_1 + sQ_2$ is an equation of the plane π.*

4. The natural norm in $\mathcal{E}^3_{OU}$

In this section we will define a natural norm in $\mathcal{E}^3_{OU}$ and then introduce the 'scalar product' and the 'vector product'. If we consider a point P of $\mathcal{E}^3_{OU}$ distinct from O, we use the symbol $\|P\|$ to denote the intersection point of the ray OU with the sphere having the center O and passing through P (see Figure 4); $\|P\|$ is called the *modulus* of P. We set $\|O\| = O$. This definition is an extension of what was introduced for the plane (see [1]).

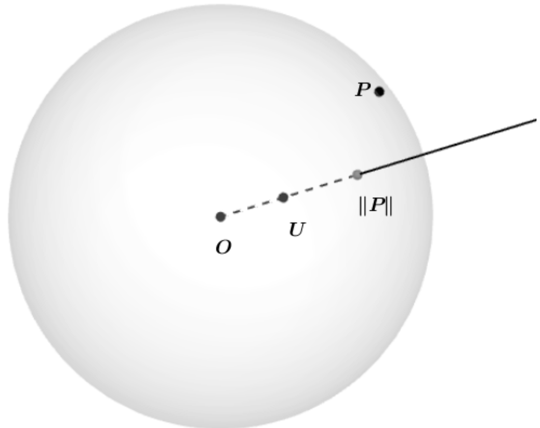

Figure 4: Modulus of P

Given the segment $\overline{AB}$, let $K = \|A - B\|$. Then we have $\overline{OK} = \overline{AB}$. We note that all distances are reproducible on the ray OU; summing up, the distances between points or multiplying them can be presented by summing or multiplying the corresponding points on the ray OU. Therefore we can express Euclid's theorem in terms of relations between moduli, and thus we get the Pythagorean theorem (see [1]).

Let A belong to the $\mathcal{E}^3_{OU}$. Then, evidently, the following properties hold:

- $\|A\| = O$ iff $A = O$;
- $\|A\| = A$ iff A belongs to the ray OU;
- $\|A\| = -A$ iff A belongs to the ray $O(-U)$;
- $\|kA\| = k\|A\|$ if k belongs to the ray OU;
- $\|A\| = \|-A\|$;
- $\|A + B\| \leq \|A\| + \|B\|$, and equality holds if and only if A and B belong to the same ray with initial point O (*triangle inequality*).

Remark 4. *Let r_0 be distinct from O and belong to the ray OU.*

- *The equality $\|P\| = r_0$ is satisfied in $\mathcal{E}^2_{OU}$ by points of the circle with center O and passing through r_0, and in $\mathcal{E}^3_{OU}$ by points of the sphere with center O and passing through r_0.*
- *The equality $\|P - P_0\| = r_0$ is satisfied in $\mathcal{E}^2_{OU}$ by points of the circle with center P_0 and radius r_0, and in $\mathcal{E}^3_{OU}$ by points of the sphere with center P_0 and passing through r_0.*

4.1. Scalar product in $\mathcal{E}^3_{OU}$

In the language of linear algebra (see [18]), $\|A\|^2$ is a quadratic form, and its associated bilinear form is

$$\frac{1}{2}\left(\|A + B\|^2 - (\|A\|^2 + \|B\|^2)\right).$$

Following [1], we set

$$\langle A, B\rangle = \frac{1}{2}\left(\|A + B\|^2 - (\|A\|^2 + \|B\|^2)\right)$$

and call $\langle A, B\rangle$ the *scalar product* between A and B. We have

$$\frac{1}{2}\left(\|A + B\|^2 - (\|A\|^2 + \|B\|^2)\right) = \pm\|A\|\|B'\|,$$

where B' is the perpendicular foot point of B on the straight line OA (see Figure 5), and we have *plus* if the angle $A\widehat{O}B$ is acute, otherwise *minus* (see [1]).

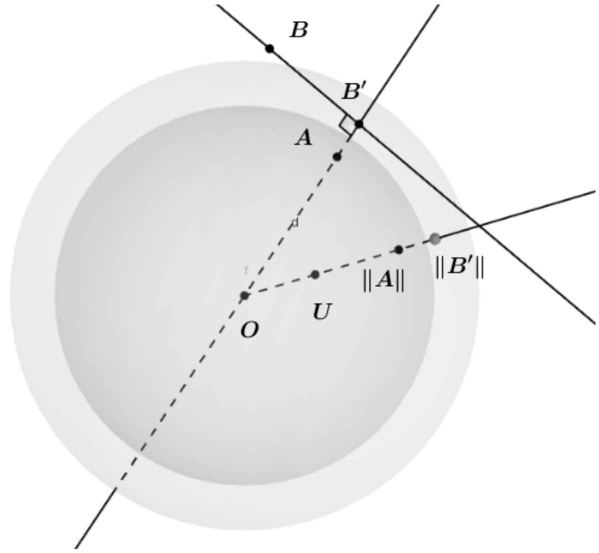

Figure 5: Modulus of B'

The following properties can be easily and directly proved in the context of synthetic geometry (for linearity see, e.g., [12]):

- $\langle A, B\rangle = \langle B, A\rangle$;
- $\langle kA, B\rangle = k\langle A, B\rangle$, for all k belonging to the straight line OU;
- $-\|B\|\|A\| \leq \langle A, B\rangle \leq \|B\|\|A\|$, where equality holds on the left side if and only if the rays OA and OB are opposite, and equality on the right side holds if and only if the rays OA and OB are coincident (*Cauchy-Schwarz inequality*);
- $\langle A + B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$ (*linearity*).

Moreover, we have $\langle A, A\rangle = \|A\|^2$ and, by the Pythagorean theorem, that if A, B are distinct from O, then $\langle A, B\rangle = O$ if and only if the angle $A\widehat{O}B$ is a right angle.

Now we set

$$\cos(A, B) = \frac{\langle A, B\rangle}{\|A\|\|B\|} \quad \text{and} \quad \sin(A, B) = \sqrt{U^2 - \cos(A, B)^2}\,.$$

If the oriented (counter-clockwise) angle $A\widehat{O}B$ is convex, we set

$$\cos(A\widehat{O}B) = \cos(A, B) \quad \text{and} \quad \sin(A\widehat{O}B) = \sin(A, B)\,.$$

If the oriented angle $A\widehat{O}B$ is concave, we set

$$\sin(A\widehat{O}B) = -\sin(A, B) \quad \text{and} \quad \cos(A\widehat{O}B) = \cos(A, B).$$

Remark 5. *Let $Q \neq O$. Then we have:*

- *For each P_0 belonging to $\mathcal{E}^2_{OU}$ the equality*

$$\langle P - P_0, Q\rangle = O \tag{1}$$

is satisfied by $P = P_0$ and by points P such that the angle $(P - P_0)\widehat{O}Q$ is right, i.e., by the points of the straight line passing through P_0 and being perpendicular to OQ.

- *For each P_0 belonging to $\mathcal{E}^3_{OU}$ the equality* (1) *is satisfied by $P = P_0$ and by points P such that the angle $(P - P_0)\widehat{O}Q$ is right, i.e., by the points of the plane passing through P_0 and being perpendicular to OQ.*

4.2. The vector product in $\mathcal{E}^3_{OU}$

In the Euclidean space OU, let A, B be distinct from O. By the Cauchy-Schwarz inequality, A and B are collinear with O if and only if $\|A\|^2\|B\|^2 - \langle A, B\rangle^2 = O$. Now let A and B be non-collinear with O. Then there exists a unique point K such that the orientation of the ordered triple (A, B, K) is right-handed, the ray OK is perpendicular to the plane AOB, and

$$\langle A, B\rangle^2 + \|K\|^2 = \|A\|^2\|B\|^2 \ (Lagrange's\, identity).$$

We call K the *vector product* between A and B.

From $\langle A, B\rangle = \|A\|\|B\|\cos(A, B)$ we get

$$\|K\| = \|A\|\|B\|\sin(A, B).$$

The vector product between A and B is denoted by $A \wedge B$. If A and B are collinear with O, we set $A \wedge B = O$.

If $k \in \mathbb{R}_{OU}$ and R, S, T belong to $\mathcal{E}^3_{OU}$, then the following properties of the vector product can be proved in the context of synthetic geometry (for linearity see, e.g., [12] and [4]):

- $k(R \wedge S) = (kR) \wedge S = R \wedge (kS)$;
- $(R + S) \wedge T = R \wedge T + S \wedge T$ (*linearity*);
- $R \wedge S = O$ if $R \in OS$ or $S \in OR$.

5. The cartesian coordinate system

Now we introduce the notion of Cartesian coordinate system which allows identification of $\mathcal{E}^2_{OU}$ with $\mathbb{R}^2_{OU}$, and of $\mathcal{E}^3_{OU}$ with $\mathbb{R}^3_{OU}$.

In $\mathcal{E}^2_{OU}$, let U_1, U_2 be distinct from O and non-collinear with O. For each P, we have $P = p_1U_1 + p_2U_2$, p_1U_1 being the projection of P onto the straight line OU_1 in the direction of OU_2, and p_2U_2 being the projection of P onto the straight line OU_2 in the direction of OU_1. The scalars p_1, p_2 are uniquely identified and called the *coordinates of P in the (coordinate) system* (U_1, U_2) (see Figure 6).

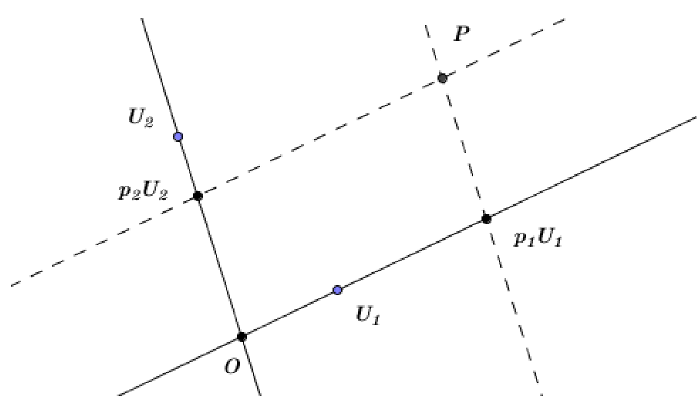

Figure 6: Coordinate system in $\mathcal{E}^2_{OU}$

5.1. The coordinate system in $\mathcal{E}^2_{OU}$

Clearly, if $P = p_1U_1+p_2U_2$, $Q = q_1U_1+q_2U_2$ and $t \in \mathbb{R}_{OU}$, then $P+Q = (p_1+q_1)U_1+(p_2+q_2)U_2$ and $tP = tp_1U_1 + tp_2U_2$.

The structure of the vector space of $\mathcal{E}^2_{OU}$ yields $\mathbb{R}^2_{OU}$, with scalar field $\mathbb{R}_{OU}$, putting

$$\begin{aligned} t(a_1, a_2) &= (ta_1, ta_2) \quad \text{and} \\ (a_1, a_2) + (b_1, b_2) &= (a_1 + b_1, a_2 + b_2)\,, \end{aligned}$$

for all $t \in \mathbb{R}_{OU}$, $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2_{OU}$.

In this context the equation of the straight line $P = P_0 + tQ$, $t \in \mathbb{R}_{OU}$(see Remark 2) is equivalent to the representation $(x, y) = (x_0 + t\lambda_1,\ y_0 + t\lambda_2)$, $t \in \mathbb{R}_{OU}$, where (x, y) are the coordinates of P, (x_0, y_0) the coordinates of P_0, and (λ_1, λ_2) the coordinates of Q.

In $\mathcal{E}^2_{OU}$, let Y belong to the circle with center O passing trough U such that the oriented angle $U\widehat{O}Y$ is right. We call the coordinate system (U, Y) the *standard Cartesian coordinate system* of $\mathcal{E}^2_{OU}$.

Remark 6. *In $\mathcal{E}^2_{OU}$, let (U, Y) be the standard Cartesian coordinate system. We have:*

- *If $P = p_iU+, p_2Y$, then $P = \langle P, U\rangle U + \langle P, Y\rangle Y = \|P\| \cos(P, U)U + \|P\| \cos(P, Y)Y = \|P\| \cos(P\widehat{O}U)U + \|P\| \sin(P\widehat{O}U)Y$.*
- *If $A = a_1U + a_2Y$, then $\|A\| = \sqrt{a_1^2 + a_2^2}$, and if $B = b_1U + b_2Y$, then $\langle A, B\rangle = a_1b_1 + a_2b_2$.*
- *If $P_0 = x_0U + y_0Y$ and $Q = aU + bY$, by (1) in Remark 5 the equality*

  $$ax + by = x_0a + y_0b$$

  is satisfied by the coordinates (x, y) of points of the straight line passing through P_0 and being perpendicular to OQ.

5.2. The coordinate system in $\mathcal{E}^3_{OU}$

In $\mathcal{E}^3_{OU}$, let U_1, U_2 and U_3 be three points not coplanar with O. A point P that does not belong to any of the planes OU_1U_2, OU_1U_3, OU_2U_3 is an extreme of the diagonal of a parallelepiped with vertex O and with three faces, respectively on OU_1U_2, OU_1U_3, OU_2U_3, and three edges, respectively on OU_1, OU_2, and OU_3. The other faces are located on the planes parallel to the planes OU_1U_2, OU_1U_3, OU_2U_3 passing through P. Then we have $P = p_1U_1 + p_2U_2 + p_3U_3$, with $(p_1, p_2, p_3) \in \mathbb{R}^3_{OU}$ uniquely determined (see Figure 7).

In the planes OU_1U_2, OU_1U_3, OU_2U_3 we can consider the coordinate systems (U_1, U_2), $(U_3, U_1,)$, (U_2, U_3), respectively. Hence, if P belongs to one of these planes, we also have

$$P = p_1U_1 + p_2U_2 + p_3U_3$$

with $(p_1, p_2, p_3) \in \mathbb{R}^3_{OU}$ uniquely determined.

We call p_1, p_2, p_3 the *coordinates of P in the (coordinate) system* (U_1, U_2, U_3). If $P = p_1U_1 + p_2U_2 + p_3U_3$, $Q = q_1U_1 + q_2U_2 + q_3U_3$ and $t \in \mathbb{R}_{OU}$, evidently,

$$P + Q = (p_1 + q_1)U_1 + (p_2 + q_2)U_2 + (p_3 + q_3)U_3 \quad \text{and} \quad tP = tp_1U_1 + tp_2U_2 + tp_3U_3.$$

Then the structure of the vector space with scalar field $\mathbb{R}_{OU}$ identified with the Euclidean space can be carried over to $\mathbb{R}^3_{OU}$ by putting for all $t \in \mathbb{R}_{OU}$, $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{R}^3_{OU}$

$$\begin{aligned} t(a_1, a_2, a_3) &= (ta_1, ta_2, ta_3) \quad \text{and} \\ (a_1, a_2.a_3) + (b_1, b_2, b_3) &= (a_1 + b_1, a_2 + b_2, a_3 + b_3). \end{aligned}$$

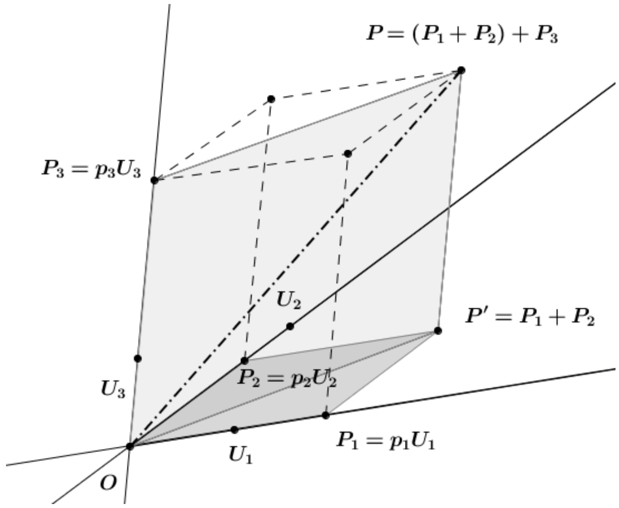

Figure 7: Coordinate system in $\mathcal{E}^3_{OU}$

Hence, the equality $P = P_0 + tQ$, $t \in \mathbb{R}_{OU}$, (see Remark 2) is equivalent to $(x, y, z) = (x_0 + t\lambda_1,\ y_0 + t\lambda_2,\ z_0 + t\lambda_3)$, $t \in \mathbb{R}_{OU}$, where (x, y, z) are the coordinates of P, (x_0, y_0, z_0) the coordinates of P_0, and $(\lambda_1, \lambda_2, \lambda_3)$ the coordinates of Q.

Moreover, the equality $P = P_0 + rQ_1 + sQ_2$, $(r, s) \in \mathbb{R}^2_{OU}$, (see Remark 3) is equivalent to the equality $(x, y, z) = (x_0 + r\lambda_1 + s\mu_1,\ y_0 + r\lambda_2 + s\mu_2,\ z_0 + r\lambda_3 + s\mu_3)$, $(r, s) \in \mathbb{R}^2_{OU}$, where (x, y, z) are the coordinates of P, (x_0, y_0, z_0) the coordinates of P_0 , $(\lambda_1, \lambda_2, \lambda_3)$ the coordinates of Q_1, and (μ_1, μ_2, μ_3) the coordinates of Q_2.

In $\mathcal{E}^3_{OU}$ the coordinate system (U, Y, Z) is called a *standard Cartesian coordinate system* (of $\mathcal{E}^3_{OU}$) if Y, Z belong to the sphere with center O and passing through U, and (U, Y), (Y, Z) and (Z, U) are the standard Cartesian coordinate systems in Euclidean planes OU containing Y, OY containing Z, and OZ containing U, respectively.

Remark 7. *In $\mathcal{E}^3_{OU}$, let (U, Y, Z) be a standard Cartesian coordinate system.*

- *If $P = p_iU + p_2Y + p_3Z$, then $P = \langle P, U\rangle U + \langle P, Y\rangle Y + \langle P, Z\rangle Z$ and*

$$P = (\|P\| \cos P\widehat{O}U)U + (\|P\| \cos P\widehat{O}Y)Y + (\|P\| \cos P\widehat{O}Z)Z.$$

- *If $A = a_1U + a_2Y + a_3Z$, then $\|A\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, and if $B = b_1U + b_2Y + b_3Z$, then $\langle A, B\rangle = a_1b_1 + a_2b_2 + a_3b_3$.*
- *If in (1) $P_0 = x_0U + y_0Y + z_0Z$ and $Q = aU + bY + cZ$ holds, then (by Remark 5) the equation*

$$ax + by + cz = x_0a + y_0b + z_0c$$

is satisfied by (x, y, z), the coordinates of points in the system (U, Y, Z) of the plane through P_0 and perpendicular to OQ.
- *$U \wedge Y = Z, Y \wedge Z = U, Z \wedge U = Y$.*
- *If $R = r_1U + r_2 + Y + r_3Z$ and $S = s_1U + s_2V + s_3Z$, then the coordinates of $R \wedge S$ in the system (U, Y, Z) are*

$$(r_2s_3 - r_3s_2, r_3s_1 - r_1s_3, r_1s_2 - r_2s_1)\,.$$

The transition from synthetic geometry to analytic geometry is completed. The usual identification of the Euclidean plane with $\mathbb{R}^2$ and of the Euclidean space with $\mathbb{R}^3$ is made possible.

6. Conclusions

Our geometric approach to analytic geometry could also be useful for removing the evident gap that exists between the way in which vector calculus is usually taught by mathematicians and how it is used by other scientists (physicists, in particular). For the vector product, H. Azad has said in [4] that

> *the geometric approach gives direct access to geometric problems and enhances geometric and conceptual understanding.*

Concerning the notions of scalar and vector product, T. Dray and C. E. Manogue asserted in [11] that

> *it is easier to derive the algebraic formula from the geometric one than the other way around and, as students tend to remember best first definitions, this should not be an algebraic formula devoid of context. The geometric definitions of scalar and vector products are coordinates independent and therefore convey invariant properties of these products, not just a formula for calculating them.*

Moreover, we note that this approach takes into account the historical development of vector analysis (see [10]).

References

[1] G. Anatriello, F.S. Tortoriello, G. Vincenzi: *On an assumption of geometric foundation of numbers*. Int. J. Math. Educ. Sci. Technol. **47**/3, 395–407 (2016).

[2] E. Artin: *Geometric Algebra*. Reprint of the 1957 original, Wiley Classics Library. John Wiley Sons, Inc., New York 1988.

[3] J. Avigad, E. Dean, J. Mumma: *A formal system for Euclid's Elements*. Rev. Symb. Log. **2**/4, 700–768 (2009).

[4] H. Azad: *On the definition of the cross-product*. Int. J. Math. Educ. Sci. Technol. **32**/4, 585–587 (2001).

[5] J. Banning: *Proportionen: Eine Grundlage der affinen Geometrie*. Math. Semesterber. **35**/1, 64–80 (1988).

[6] C.B. Boyer: *Descartes and the geometrization of Algebra*. Amer. Math. Monthly **66**, 390–393 (1959).

[7] P.J. Cameron: *Projective and Polar Spaces*. Vol. 13, University of London, Queen Mary and Westfield College 1992.

[8] U. Cassina: *Sulla teoria delle grandezze e dell' equivalenza*. In M. Villa (ed.): *Repertorio di Matematiche*, Vol. II, CEDAM, Padova 1971, pp. 195–218.

[9] F. Conforto: *Postulati della geometria euclidea e geometria non euclidea*. In M. Villa (ed.): *Repertorio di Matematiche*, Vol. II, CEDAM, Padova 1971, pp. 45–77.

[10] M.J. Crowe: *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System*. Dover Publications, Mineola (NY) 1994.

[11] T. Dray, C.A. Manogue: *Bridging the gap between mathematics and the physical sciences*. 2004, `http://math.oregonstate.edu/bridge/papers/bridge.pdf`.

[12] T. Dray, C.A. Manogue: *The geometry of the dot and cross products*. Journal of Online Mathematics and its Applications 1156, 1–13 (2006).

[13] H. Freudenthal: *Zur Geschichte der Grundlagen der Geometrie*. Nieuw Archief voor Wiskunde, ser. 3, **5**, 105–142 (1957).

[14] E.N. Giovannini: *Bridging the gap between analytic and synthetic geometry: Hilbert's axiomatic approach*. Synthese **193**, 31–70 (2016).

[15] D. Hilbert: *Grundlagen der Geometrie*. 2nd ed., Springer, Berlin 1943.

[16] N. Miller: *Euclid and his Twentieth Century Rivals: Diagrams in the Logic of Euclidean Geometry*. CSLI Publications, Stanford 2007.

[17] D. Rowe: *The calm before the storm: Hilbert's early views on foundations*. In V.F. Hendricks et al. (eds): *Proof Theory*. Kluwer Acad. Publ., Dordrecht 2000, pp. 55–93.

[18] I.R. Shafarevich, A.O. Remizov: *Linear Algebra and Geometry*. Springer, Heidelberg 2013.

[19] J. Stillwell: *The Four Pillars of Geometry*. Springer, New York 2005.

[20] J. Stillwell: *Ideal elements in Hilbert's Geometry*. Perspectives on Science **22**/1, 35–55 (2014).

[21] G. Vailati: *Sulla teoria delle proporzioni*. In E. Enriques (ed.): *Questioni Riguardanti le Matematiche Elementari – Raccolte e Coordinate da Federigo Enriques*, Vol. I: *Critica dei Principii*, Zanichelli, Bologna 1912, pp. 143–191.

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