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Classification and Normal Forms of Planar 4-Multisets and Quadrangles

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Abstract. This article is devoted to the problem of describing similarity types of geometric objects. We give a solution of this problem for the case of four points in plane. The general case is subdivided into several subcases depending on configurations of points in maximum distance. Similarity types are parametrized by a point or a pair of points in the plane.

Key Words: normal forms, similarity, quadrangle

MSC 2010: 51M04, 51N20, 97G50

1. Introduction

The problem of defining normal forms in Euclidean geometry up to similarity and a solution of this problem for triangles was described in [2].

A next step is to solve this problem for quadrangles or, more generally, for 4-multisets of points in a plane. This means to describe a set $\mathbb{S}$ of mutually non-similar 4-multisets of a plane with a fixed Cartesian system such that any 4-multiset in this plane would be similar to a unique element of $\mathbb{S}$. In Section 3.2 we describe uniquely defined representatives of similarity classes of plane 4-multisets. These representatives can be considered normal forms of 4-multisets. Furthermore, in Section 3.3 we deal with similarity classes of plane quadrangles. Thus we obtain a classification of quadrangles up to similarity.

We assume that fixed Cartesian coordinates are introduced in the plane, $\mathbb{S}$ is designed using the Cartesian coordinates. These normal forms may be useful in solving geometry problems involving similarity and teaching geometry.

2. Notations and review

2.1. Sets and multisets

In this article we use multisets to take into account cases of coinciding points. We denote multisets using \mathbf letters and double curly brackets, e.g. $\mathbf{M} = \{\{a, a, b, c\}\}$. Sets are

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denoted using blackboard bold letters (\mathbb). The set corresponding to the multiset $\mathbf{M}$ is denoted by $Set(\mathbf{M})$. If $Set(\mathbf{M}) \subseteq \mathbb{A}$ where $\mathbb{A}$ is a set, we also use the notation $\mathbf{M} \subseteq \mathbb{A}$. If a multiset happens to be a set, we may also think of and denote it as a set. We use normal letters to denote fixed objects and calligraphic letters (\mathcal) to denote objects as variable elements of some ambient set.

2.2. Graphs

In this article we use undirected graphs $\Gamma = (V, E)$ defined by a vertex set or a multiset V and an edge set E. We denote adjacency of vertices a and b as $a - b$.

2.2.1. Geometry

Consider $\mathbb{R}^2$ with a fixed Cartesian system of coordinates (x, y) with origin O. Our objects of study are multisets of points in $\mathbb{R}^2$ having 4 elements (4-multisets, 4-submultisets of $\mathbb{R}^2$). Additionally we require that no point has multiplicity 4: $|Set(\mathbf{M})| \geq 2$.

It is known that affine transformations generate the dilation group of $\mathbb{R}^2$, denoted by some authors as $IG(2)$ (see Hazewinkel [3]). Two multisets $\mathbf{M}_1$ and $\mathbf{M}_2$ are similar (denoted as $\mathbf{M}_1 \sim \mathbf{M}_2$) if there exists an affine transformation $g \in IG(2)$, such that $g(\mathbf{M}_1) = \mathbf{M}_2$ (as multisets). In this article by mappings we mean affine transformations applied to the plane containing the given multiset. The convex hull of the multiset $\mathbf{M}$ is denoted as $Conv(\mathbf{M})$ (see Audin [1] and Venema [4] for comprehensive modern expositions of Euclidean geometry).

Let $\mathbf{M} \subseteq \mathbb{R}^n$. Let d be the maximal distance between two points in $\mathbf{M}$. Recall that the undirected graph $\Delta_{\mathbf{M}} = (\mathbf{M}, E(\Delta_{\mathbf{M}}))$ such that $\{u, v\} \in E(\Delta_{\mathbf{M}})$ iff $dist(u, v) = d$, is called the *diameter graph* of $\mathbf{M}$.

2.3. Review of a normal form of triangles

We review one of our previous results given in [2], the Theorems 2.2 and 2.3: Each triangle is similar to a unique triangle $\triangle AB\mathcal{C}$ such that $A = (0, 0)$, $B = (1, 0)$, $\mathcal{C} \in \mathbb{S}_C = \{(x, y) \,|\, y \geq 0,\ x \geq \frac{1}{2},\ x^2 + y^2 \leq 1\}$ (see Figure 1). We will use the symbol $\mathbb{S}_C$ in this article.

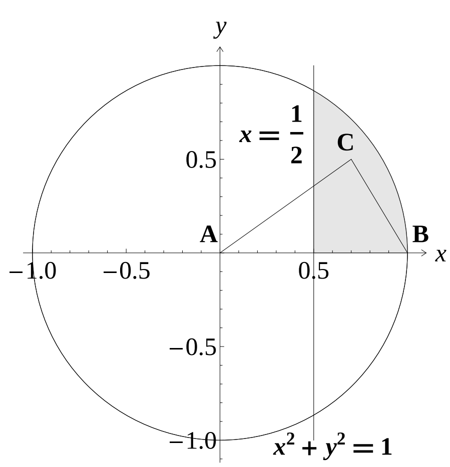

Figure 1: The domain $\mathbb{S}_C$.

3. Main results

3.1. Diameter graph of a planar 4-multiset

We note that affine transformations preserve the isomorphism type of the diameter graph.

Proposition 1. *$\mathbf{M} \subseteq \mathbb{R}^2$, $|\mathbf{M}| = 4$, $|Set(\mathbf{M})| \geq 2$.*

1. $1 \leq |E(\Delta_{\mathbf{M}})| \leq 5$.

2. All isomorphism types for $\Delta_{\mathbf{M}}$, subject to 1., are possible.

Proof. 1. We have to exclude the case $|E(\Delta_{\mathbf{M}})| = 6$. Let $\mathbf{M} = \{\{X, Y, Z, T\}\}$. If $|E(\Delta_{\mathbf{M}})| = 6$ then using $\mathbf{M}$ we construct two equivilateral triangles $\triangle XYZ$ and $\triangle YZT$. Clearly $|XT| \neq |XY|$, thus we get a contradiction.

2. This statement is proved by giving examples for all cases.

• Case $|E(\Delta_{\mathbf{M}})| = 1$ happens for

$$\mathbf{M}_1 = \left\{(0,0),\ (1,0),\ \left(\frac{1}{2},\ \frac{1}{2}\right),\ \left(\frac{1}{2}, \frac{1}{2}\right)\right\}.$$

• There are two possible isomorphism types of graphs with $|E| = 2$ — two disjoint edges or two incident edges and a vertex of degree 0. Disjoint edges happen for vertex sets of squares, incident edges happen for

$$\mathbf{M}_2 = \left\{(0,0),\ (1,0),\ \left(\cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right),\ p\right\},$$

where p belongs to the interior of the convex hull of the first three points.

• There are three possible isomorphism types of graphs with $|E| = 3$ — the triangle and a vertex of degree 0, the path of length 3 and the 4-vertex tree with a vertex of degree 3. The triangle happens for

$$\mathbf{M}_{31} = \left\{(0,0),\ (1,0),\ \left(\cos\frac{\pi}{3},\ \sin\frac{\pi}{3}\right),\ \left(\frac{1}{2}, \frac{1}{2}\right)\right\}.$$

The path of length 3 happens for

$$\mathbf{M}_{32} = \left\{(0,0),\ (1,0),\ \left(\cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right),\ \left(1 - \cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right)\right\}.$$

The tree with a vertex of degree 3 happens for

$$\mathbf{M}_{33} = \left\{(0,0),\ (1,0),\ \left(\cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right),\ \left(\cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right)\right\}.$$

• There are two isomorphism types if $|E| = 4$: complements of 2 incident edges (the triangle with another vertex attached) and 2 disjoint edges (the 4-cycle). The complement of two incident edges happens for

$$\mathbf{M}_{41} = \left\{(0,0),\ (1,0),\ \left(\cos\frac{\pi}{3},\ \sin\frac{\pi}{3}\right),\ \left(\cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right)\right\}.$$

The 4-cycle happens for

$$\mathbf{M}_{42} = \{(0,0),\ (0,0),\ (1,0),\ (1,0)\}$$

or

$$\mathbf{M}_{43} = \left\{(0,0),\ (0,0),\ (1,0),\ \left(\cos\frac{\pi}{6},\ \sin\frac{\pi}{6}\right)\right\}.$$

• $|E(\Delta_{\mathbf{M}})| = 5$ happens if three points of $\mathbf{M}$ are vertices of an equilateral triangle and the fourth point coincides with one of these three points. $\square$

3.2. Normal forms of plane 4-multisets

3.2.1. General algorithm

We define $A = (0,0)$, $B = (1,0)$. Let $\mathbf{M}$ be a 4-multiset. Our general algorithm to find similarity types and normal forms of plane 4-multisets has the following steps.

1. Do a mapping g such that some two points with maximal distance are mapped to $\{A, B\}$.
2. Find a point $X \in g(\mathbf{M})$ maximizing $\max(|XA|, |XB|)$. Map X to $\mathbb{S}_C$, using the reflection in the line $x = \frac{1}{2}$ and the y-axis, if necessary. Denote this third point by $\mathcal{C}$. Find the set of possible positions of $\mathcal{C}$.
3. Having fixed the set $\{A, B, \mathcal{C}\}$, find the set of possible positions of the fourth point, denote it by $\mathcal{D}$. By the previous step we have that $|A\mathcal{C}| \geq |A\mathcal{D}|$ and $|A\mathcal{C}| \geq |B\mathcal{D}|$. Use transformations preserving $\{A, B, \mathcal{C}\}$, if desirable. Find the set of possible positions of $\mathcal{D}$.

3.2.2. Diameter graphs containing a triangle

Proposition 2. *Let $\mathbf{M}$ be a 4-multiset such that $\Delta_{\mathbf{M}}$ contains a triangle. Then $\mathbf{M}$ is similar to exactly one 4-multiset $\{\{A, B, C, \mathcal{D}\}\}$ such that*

$$C = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right), \ \mathcal{D} \in \mathbb{D}_1 = \left\{(x,y) \,\middle|\, x \geq \frac{1}{2}, \ y \geq \left(\tan\frac{\pi}{6}\right)x, \ x^2 + y^2 \leq 1\right\}.$$

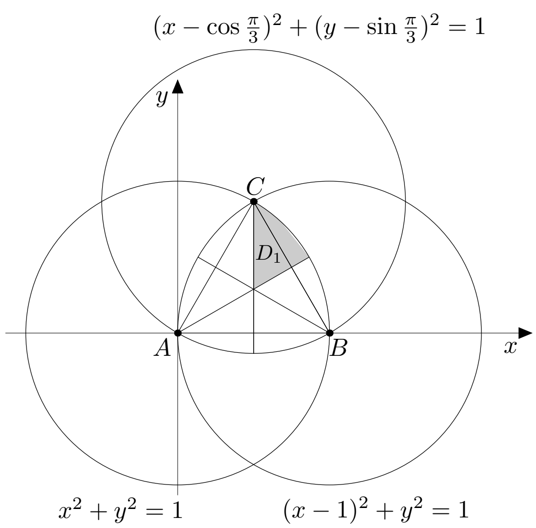

Figure 2: The domain $\mathbb{D}_1$.

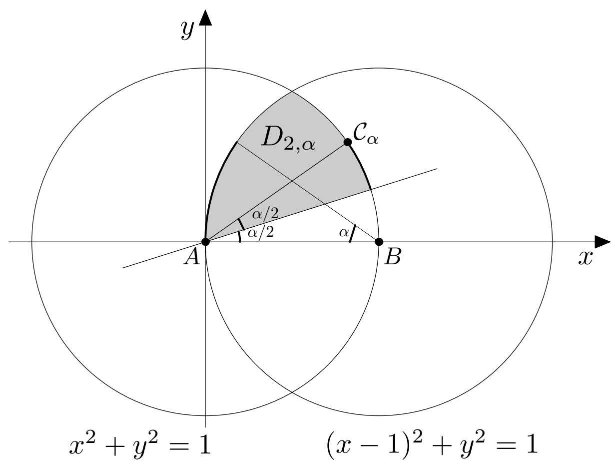

Figure 3: The domain $\mathbb{D}_{2\alpha}$ with $\alpha = \frac{\pi}{5}$.

Proof. We have to show that any 4-multiset $\mathbf{M}$, such that $\Delta_{\mathbf{M}}$ contains a triangle, can be mapped to a multiset in the described form. First we map any three points of $\mathbf{M}$ corresponding to a $\Delta_{\mathbf{M}}$-triangle to $\{A, B, C\}$. Then we map the fourth point to $\mathbb{D}_1$ using reflections in bisectors of $\triangle ABC$ (Figure 2). Note that

1) if $\mathcal{D} = C$ then $|E(\Delta_{\mathbf{M}})| = 5$,
2) if $\mathcal{D}$ belongs to the unit circle then $|E(\Delta_{\mathbf{M}})| = 4$,
3) otherwise $|E(\Delta_{\mathbf{M}})| = 3$.

Now we have to prove that two distinct multisets of the described normal form are not similar. Let $\mathbf{M}_i = \{\{A, B, C, \mathcal{D}_i\}\}$, with $\mathcal{D}_i \in \mathbb{D}_1$, $i \in \{1, 2\}$. Any affine transformation must preserve $\{A, B, C\}$, thus it can only be a rotation by $\pm\frac{2\pi}{3}$ or a reflection in bisectors. If $\mathcal{D}_1 \neq \mathcal{D}_2$, then $g(\mathcal{D}_1) \neq g(\mathcal{D}_2)$ for any such affine transformation, thus $\mathbf{M}_1 \not\sim \mathbf{M}_2$. $\square$

Remark 3. $\mathbf{M}$ is a proper multiset iff $\mathcal{D} = C$.

3.2.3. Trianglefree diameter graphs containing paths of length 2

Proposition 4. *Let* $\mathbf{M}$ *be a* 4*-multiset such that* $\Delta_{\mathbf{M}}$ *contains a path of length* 2*, but does not contain a triangle. Then* $\mathbf{M}$ *is similar to exactly one* 4*-multiset* $\{\{A, B, \mathcal{C}_\alpha, \mathcal{D}\}\}$ *such that*

$$\mathcal{C}_\alpha = (\cos\alpha, \sin\alpha),\ 0 \leq \alpha < \pi/3, \quad \mathcal{D} \in \mathbb{D}_{2\alpha} = \mathbb{D}_{3\alpha} \cup \mathbb{D}_{4\alpha} \cup \mathbb{D}_{5\alpha}$$

(see Figure 3), where

$$\begin{aligned}
\mathbb{D}_{3\alpha} &= \left\{(x, y) \,\middle|\, y \geq \left(\tan\frac{\alpha}{2}\right) x,\ (x-1)^2 + y^2 < 1,\ x^2 + y^2 < 1\right\},\\
\mathbb{D}_{4\alpha} &= \left\{(x, y) \,\middle|\, x > 0,\ x^2 + y^2 = 1,\ \tan\frac{\alpha}{2} \leq \frac{y}{x} \leq \tan\alpha\right\},\\
\mathbb{D}_{5\alpha} &= \left\{(x, y) \,\middle|\, (x-1)^2 + y^2 = 1,\ x - 1 < 0,\ 0 \leq \frac{y}{1-x} \leq \tan\alpha\right\}.
\end{aligned}$$

Proof. The maximal angle between two intervals XY and XZ of maximal length having a common vertex X is less than $\frac{\pi}{3}$. Let this angle be equal to α. We map X to A and $\{Y, Z\}$ to $\{B, \mathcal{C}_\alpha\}$. If necessary we make the reflection in the line $y = \left(\tan\frac{\alpha}{2}\right) x$ so that the fourth point is mapped above this line, we denote its image by $\mathcal{D}$. We check that $\mathcal{D} \in \mathbb{D}_{3\alpha} \cup \mathbb{D}_{4\alpha} \cup \mathbb{D}_{5\alpha}$. Note the following subcases.

1) $\mathcal{D} \in \mathbb{D}_{3\alpha}$ correspond to cases of $\Delta_{\mathbf{M}}$ having two incident edges and an isolated vertex;
2) $\mathcal{D} \in \mathbb{D}_{4\alpha}$ correspond to $\Delta_{\mathbf{M}}$ being the tree with a vertex of degree 3;
3) $\mathcal{D} \in \mathbb{D}_{5\alpha}$ correspond to $\Delta_{\mathbf{M}}$ being the path of length 3.

Cases with any $\mathcal{C}_\alpha$ and $\mathcal{D} = (0, 0)$ correspond to $\Delta_{\mathbf{M}}$ being a 4-cycle.

Let $\mathbf{M}_i = \{\{A, B, \mathcal{C}_{\alpha_i}, \mathcal{D}_i\}\}$, with $\mathcal{D}_i \in \mathbb{D}_{2\alpha_i}$, $i \in \{1, 2\}$, $\mathcal{D}_1 \neq \mathcal{D}_2$. If $\alpha_1 \neq \alpha_2$ then $\mathbf{M}_1 \not\sim \mathbf{M}_2$ since affine transformations preserve the maximal angle between intervals of maximal length. If $\alpha_1 = \alpha_2$ but $\mathcal{D}_1$ and $\mathcal{D}_2$ belong to different sets $\mathbb{D}_{k,\alpha_1}$, $k \in \{3, 4, 5\}$, then $\mathbf{M}_1 \not\sim \mathbf{M}_2$ since affine transformations preserve the isomorphism type of maximal distance graph. If $\alpha_1 = \alpha_2$ and $\mathcal{D}_1$ and $\mathcal{D}_2$ belong to the same set $\mathbb{D}_{k,\alpha_1}$ then any affine transformation must fix $\{A, B, \mathcal{C}_{\alpha_1}\}$ and $\mathcal{D}_i$, thus $\mathbf{M}_1 \not\sim \mathbf{M}_2$. $\square$

Remark 5. $\mathbf{M}$ is a proper multiset in the following cases: 1) $\mathcal{C} = \mathcal{D} = \mathcal{C}_\alpha$, 2) $\mathcal{C} = \mathcal{C}_\alpha$, $\mathcal{D} = A$, 3) $\mathcal{C} = \mathcal{C}_0 = B$, $\mathcal{D} \in \mathbb{D}_{2,0}$.

3.2.4. Diameter graph consisting of two disjoint edges

Proposition 6. *Let* $\mathbf{M}$ *be a* 4*-multiset such that* $\Delta_{\mathbf{M}}$ *has only two disjoint edges. Then* $\mathbf{M}$ *is similar to exactly one* 4*-multiset* $\{\{A, B, \mathcal{C}, \mathcal{D}\}\}$ *from the following list*

1. *(both longest intervals crossing in midpoints)*

$$\mathcal{C} = \mathcal{C}_\alpha = \left(\frac{1}{2} - \frac{1}{2}\cos\alpha,\ \frac{1}{2}\sin\alpha\right), \quad \mathcal{D} = \mathcal{D}_\alpha = \left(\frac{1}{2} + \frac{1}{2}\cos\alpha,\ -\frac{1}{2}\sin\alpha\right),$$

where $0 < \alpha \leq \frac{\pi}{2}$.

2. *(at least one longest interval crossed not in its midpoint)*

$$\mathcal{C} = \mathcal{C}_{\alpha t u} = (t - u\cos\alpha,\ u\sin\alpha)\,, \quad \mathcal{D} = \mathcal{D}_{\alpha t u} = (t + (1-u)\cos\alpha,\ -(1-u)\sin\alpha)\,,$$

where $\frac{1}{2} < t < 1$, $\frac{1}{2} \leq u \leq t$, $\frac{t^2+u^2-1}{2tu} < \cos\alpha < \frac{1-t^2-(1-u)^2}{2t(1-u)}$. *The set* $\mathbb{A}$ *of possible values of* (t,u)*, for which sets of* $\mathcal{C}$*-values and* $\mathcal{D}$*-values are nonempty, satisfies an additional inequality* $t^2 - (u-\frac{1}{2})^2 \leq \frac{3}{4}$ *(see Figure 4).*

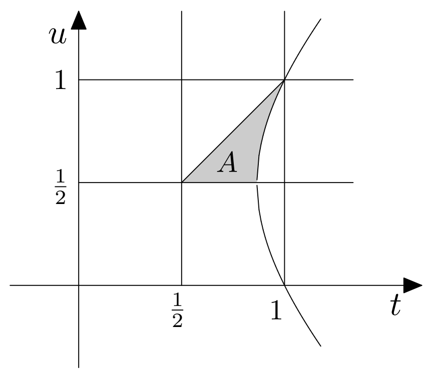

Figure 4: The set $\mathbb{A}$.

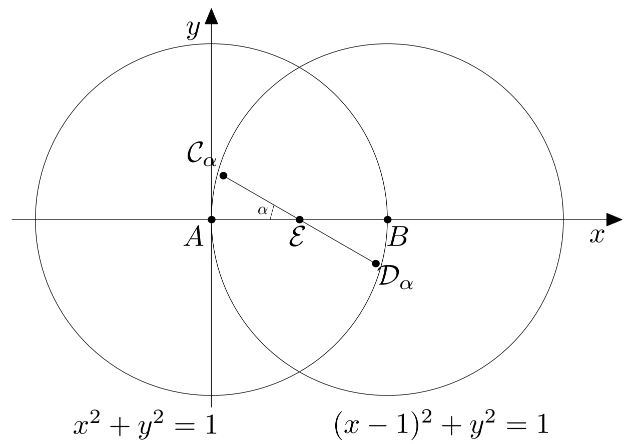

Figure 5: $\mathcal{C}_\alpha$ and $\mathcal{D}_\alpha$ for $\alpha = \frac{\pi}{6}$.

Proof. First we note that the two intervals of largest distance must intersect in interior points. We map $\mathbf{M}$ to $\{\{A, B, \mathcal{C}, \mathcal{D}\}\}$ so that the following conditions and notations hold:

1) the intervals of maximal distance are mapped to AB and $\mathcal{CD}$,
2) denote $AB \cap \mathcal{CD} := \mathcal{E}$,
3) $|A\mathcal{E}| = t \geq |\mathcal{E}B|$,
4) $|\mathcal{CE}| = u \geq |\mathcal{ED}|$, $u \leq t$,
5) $\mathcal{C}$ is above the x-axis,
6) denote $\alpha := \angle A\mathcal{EC}$.

We describe possible positions of $\mathcal{C}$ and $\mathcal{D}$.

(1) Case $t = u = \frac{1}{2}$. In this case both intervals of maximal distance intersect in their midpoints. Applying the reflection in the line $x = \frac{1}{2}$, if necessary, we get that $\mathcal{C} = \mathcal{C}_\alpha$ and $\mathcal{D} = \mathcal{D}_\alpha$, $0 < \alpha \leq \frac{\pi}{2}$ (see Figure 5). Note that this case covers all rectangles.

(2) Case $t > \frac{1}{2}$. For all $\alpha \in [0, \pi]$ we have $|\mathcal{C}B| < 1$ and $|\mathcal{D}B| < 1$. The parameters t, u and α must satisfy the system $\begin{cases} |A\mathcal{C}| < 1 \\ |A\mathcal{D}| < 1 \end{cases}$ which is equivalent to

$$\begin{cases} t^2 + u^2 - 2tu\cos\alpha < 1 \\ t^2 + (1-u)^2 + 2t(1-u)\cos\alpha < 1 \end{cases} \tag{1}$$

It follows that

$$\frac{t^2+u^2-1}{2tu} < \cos\alpha < \frac{1-t^2-(1-u)^2}{2t(1-u)}, \tag{2}$$

which gives the stated condition on α. In this construction we denote $\mathcal{C}_{\alpha tu} := \mathcal{C}$ and $\mathcal{D}_{\alpha tu} := \mathcal{D}$.

The inequality (2) has a solution with respect to α for the given values of t and u, iff

$$\frac{t^2 - u^2 - 1}{2tu} \leq \frac{1 - t^2 - (1-u)^2}{2t(1-u)},$$

which is equivalent to the hyperbole-type inequality

$$t^2 - \left(u - \frac{1}{2}\right)^2 \leq \frac{3}{4}.$$

Consider the two distinct 4-multisets, $\mathbf{M}_i = AB\mathcal{C}_i\mathcal{D}_i$, $i \in \{1, 2\}$. They must differ in at least one of the parameters α, t or u. It implies $\mathbf{M}_1 \not\sim \mathbf{M}_2$. $\square$

Example 7. If $t = 0.8$ and $u = 0.6$, then α satisfies $\arccos(0.3125) \leq \alpha \leq \frac{\pi}{2}$ (see Figure 6).

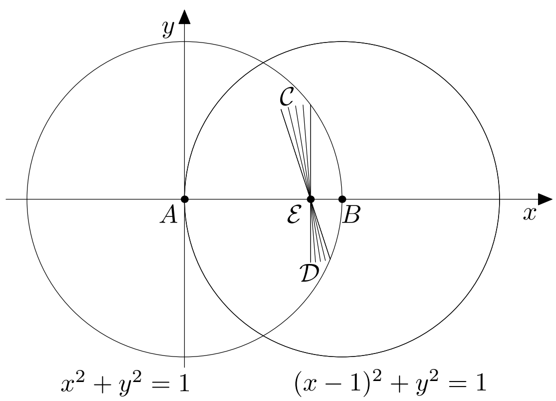

Figure 6: The admissible positions of $\mathcal{C}$ and $\mathcal{D}$ for $t = 0.8$, $u = 0.6$.

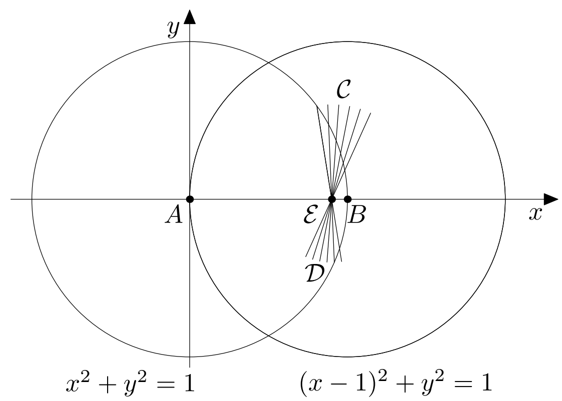

Figure 7: Some nonadmissible positions of $\mathcal{C}$ and $\mathcal{D}$ for $t = 0.9$, $u = 0.6$.

Example 8. If $t = 0.9$ and $u = 0.6$, then there are no admissible points $\mathcal{C}$ and $\mathcal{D}$, since $|A\mathcal{C}| > 1$ or $|A\mathcal{D}| > 1$ (see Figure 7).

3.2.5. Diameter graph having one edge

Proposition 9. *Let* $\mathbf{M}$ *be a 4-multiset such that* $\Delta_{\mathbf{M}}$ *has one edge. Then* $\mathbf{M}$ *is similar to exactly one 4-multiset* $\{\{A, B, \mathcal{C}, \mathcal{D}\}\}$ *from the following list (see Figures 8 and 9).*

1. *($\mathcal{C}$ not on the line $x = \frac{1}{2}$)* $\mathcal{C}_{r\alpha} = (r\cos\alpha, r\sin\alpha)$, $\frac{1}{2} < r < 1$, $0 \leq \tan\alpha < \sqrt{4r^2 - 1}$, $\mathcal{D} \in \mathbb{D}_{6r\alpha} = (\mathbb{D}_{7r} \cup \mathbb{D}_{8r\alpha} \cup \mathbb{D}_{9r\alpha}) \cap \mathbb{D}_{10r\alpha}$, *where*

$$\begin{aligned}
\mathbb{D}_{7r} &= \{(x,y) \,|\, x^2 + y^2 < r^2,\ (x-1)^2 + y^2 < r^2\},\\
\mathbb{D}_{8r\alpha} &= \left\{(x,y) \,|\, x^2 + y^2 = r^2,\ -\tan\alpha \leq \frac{y}{x} \leq \tan\alpha\right\},\\
\mathbb{D}_{9r\alpha} &= \left\{(x,y) \,|\, (x-1)^2 + y^2 = r^2,\ -\tan\alpha \leq \frac{y}{1-x} \leq \tan\alpha\right\},\\
\mathbb{D}_{10r\alpha} &= \{(x,y) \,|\, (x - r\cos\alpha)^2 + (y - r\sin\alpha)^2 < 1\}.
\end{aligned}$$

2. *($\mathcal{C}$ on line $x = \frac{1}{2}$)* $\mathcal{C}_h = \left(\frac{1}{2}, h\right)$, $0 \leq h < \frac{\sqrt{3}}{2}$,

$$\mathcal{D} \in \mathbb{D}_{11h} = \left\{(x,y) \,|\, x \geq \frac{1}{2},\ x^2 + y^2 \leq \frac{1}{4} + h^2,\ \left(x - \frac{1}{2}\right)^2 + (y-h)^2 < 1\right\}.$$

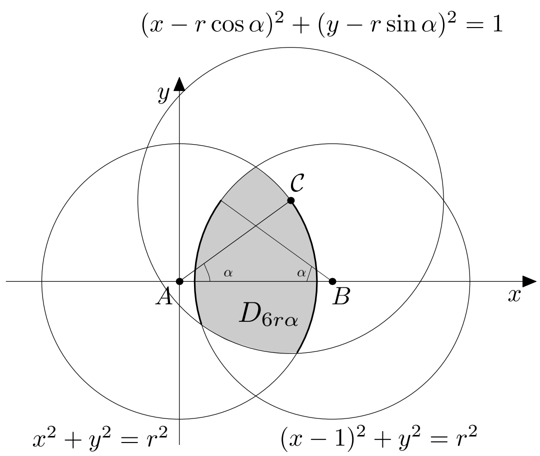

Figure 8: The domain $\mathbb{D}_{6r\alpha}$ with $r = 0.9$, $\alpha = \dfrac{\pi}{5}$.

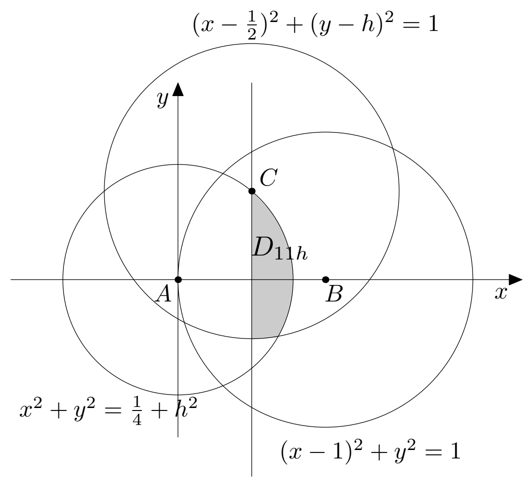

Figure 9: The domain $\mathbb{D}_{11h}$ with $h = 0.6$.

Proof. We apply an affine transformation so that the endpoints of the interval of $\mathbf{M}$ of longest distance are mapped to $\{A, B\}$. Using reflections in the y-axis and the line $x = \frac{1}{2}$ (thus fixing $\{A, B\}$) the remaining two points are mapped to some points $\mathcal{C}, \mathcal{D}$ so that the following conditions are satisfied:

1) $\mathcal{C} \in \mathbb{S}_C$,
2) $|\mathcal{C}A| \geq |\mathcal{D}A|$,
3) $|\mathcal{C}A| \geq |\mathcal{D}B|$,
4) if $|\mathcal{C}A| = |\mathcal{D}A|$ then $\angle \mathcal{C}AB \geq \angle \mathcal{D}AB$,
5) if $|\mathcal{C}A| = |\mathcal{D}B|$ then $\angle \mathcal{C}AB \geq \angle \mathcal{D}BA$.

In other words, we map to $\mathbb{S}_C$ the point p in the image of $\mathbf{M}$ under previous mappings which has the largest distance to A and the largest angle formed by the y-axis and the line through $A = (0, 0)$ and p.

1. If $\mathcal{C}$ does not belong to the line $x = \frac{1}{2}$, then any affine transformation fixing $\{A, B, \mathcal{C}\}$ setwise must fix it pointwise. Two distinct 4-multisets of this type must differ in at least one of the points $\mathcal{C}$ and $\mathcal{D}$ and therefore are nonsimilar.

2. Let $\mathcal{C}$ belong to the line $x = \frac{1}{2}$ and have coordinates $(\frac{1}{2}, h)$. Then we do an additional reflection in the line $x = \frac{1}{2}$ (which fixes $\{A, B, \mathcal{C}\}$) and map the fourth point to the half-plane $x \geq \frac{1}{2}$. Two distinct 4-multisets of this type are nonsimilar for the same reason as in 1. ☐

Remark 10. $\mathbf{M}$ is a proper multiset in the following cases: 1) $\mathcal{D} = \mathcal{C}_{r\alpha}$, 2) $\mathcal{D} = \mathcal{C}_h$.

3.3. Normal forms of plane quadrangles

In this subsection we consider 4-multisets $\mathbf{M}$ which are 4-sets and such that $Conv(\mathbf{M})$ is not an interval.

Given such a 4-set, a quadrangle is associated with an undirected 4-cycle graph having the given points as vertices. We distinguish three classes of quadrangles: convex, nonconvex and self-intersecting. We call a 4-set $\mathbb{M}$ *convex* iff the boundary of $Conv(\mathbb{M})$ is a quadrangle.

For a given 4-set a quadrangle can be constructed in three ways. A nonconvex 4-set generates three (nonconvex) quadrangles. These quadrangles are similar iff the vertices are vertices of an equilateral triangle and its center. A convex 4-set generates one non self-intersecting quadrangle and two self-intersecting quadrangles. These self-intersecting quadrangles are similar iff the vertices are vertices of a rhomboid.

Definition 11. Given two quadrangles R_1 and R_2 constructed using a 4-set $\mathbb{M}$ we call them *similar* ($R_1 \sim R_2$) iff there is an affine transformation $s\colon \mathbb{M}_1 \to \mathbb{M}_2$ such that for any $a, b \in \mathbb{M}_1$ we have $a - b$ iff $s(a) - s(b)$.

Proposition 12. *Rectangles corresponding to nonsimilar* 4*-multisets are not similar.*

Proof. Rectangle similarity implies multiset similarity. The statement follows by contraposition. $\square$

4. Possible uses of normal forms in education

Normal forms of 4-multisets and quadrangles can be used to represent their similarity types. It may be useful to have an example for students showing that the similarity types of any quadrangle can be parametrized by the coordinates of a pair of points.

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