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Combinatorics of Triangulated Polyhedra

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Abstract. We give an algorithm for computing the different combinatorics of a triangulated polyhedron with a fixed number of vertices.

Key Words: polyhedron, combinatorics, triangulation

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1. Introduction

In this paper, we consider compact triangulated polyhedra P of genus 0 (see [1] for the basic definitions on polyhedra). As P is triangulated, its faces are triangles and two adjacent faces can be coplanar. If we denote by F the number of triangles in such a polyhedron, by $V \geq 4$ the number of vertices and by E the number of edges, Euler-Poincaré's formula implies $F - E + V = 2$.

Since each face is a triangle we have $3F = 2E$ and so $V = 2 + \frac{1}{2}F$, implying F is even. Conversely, for every integer $F \geq 4$, there exists at least one triangulated polyhedron with F faces, $E = \frac{3}{2}F$ edges and $V = 2 + \frac{1}{2}F$ vertices: just consider the diamond with a $\frac{1}{2}F$-gon as basis.

We fix a $(n+3)$-tuple $\mathcal{V}_n = (v_0, \dots, v_{n+2})$ of the unit sphere S^2 $(n \geq 1)$ and denote by $\mathcal{T}_n = \{\mathcal{T}_n^{(1)}, \dots\}$ the set of triangulations of S^2 whose vertices are the points of $\mathcal{V}_n$. From a combinatorial point of view, there is a one-to-one correspondence from compact triangulated polyhedra of genus 0 with $n+3$ vertices and $\mathcal{T}_n$.

A useful tool to display them is the so-called "Schlegel diagram" [3]: it can be considered as the view of the triangulation throughout the triangle (v_0, v_{n+1}, v_{n+2}).

Definition 1.

1. The *combinatorics* of a triangulation $\mathcal{T}_n^{(i)}$ of $\mathcal{T}_n$ is the $(n+3)-$tuple

$$\mathcal{C}_n^{(i)} = (deg(v_0), \dots, deg(v_{n+2})).$$

2. Let $\mathcal{C}_n^{(i)}$ and $\mathcal{C}_n^{(j)}$ be the combinatorics of two triangulations $\mathcal{T}_n^{(i)}$ and $\mathcal{T}_n^{(j)}$ respectively. We say that $\mathcal{C}_n^{(i)}$ and $\mathcal{C}_n^{(j)}$ are *equivalent* if there exists a one-to-one map $\varphi : \mathcal{V}_n \to \mathcal{V}_n$ such that for all triangles $T \in \mathcal{T}_n^{(i)}$ we have $\varphi(T) \in \mathcal{T}_n^{(j)}$.

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By definition the sum of the degrees of a combinatorics of $\mathcal{V}_n$ is always equal to $2E$. It is immediate that $\mathcal{V}_1$ has a single triangulation $\mathcal{T}_1$ (and so a single combinatorics) and that $\mathcal{V}_2$ has three triangulations $\mathcal{T}_2^{3,1}, \mathcal{T}_2^{4,1}, \mathcal{T}_2^{4,2}$ with equivalent combinatorics (see Figure 1). Likewise, four triangulations of $\mathcal{V}_3$ are represented in Figure 2 but clearly some of them are equivalent.

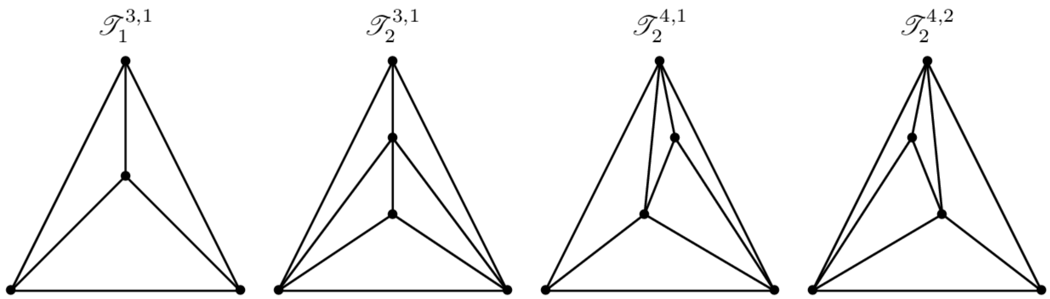

Figure 1: Triangulations of $\mathcal{V}_1$ and $\mathcal{V}_2$

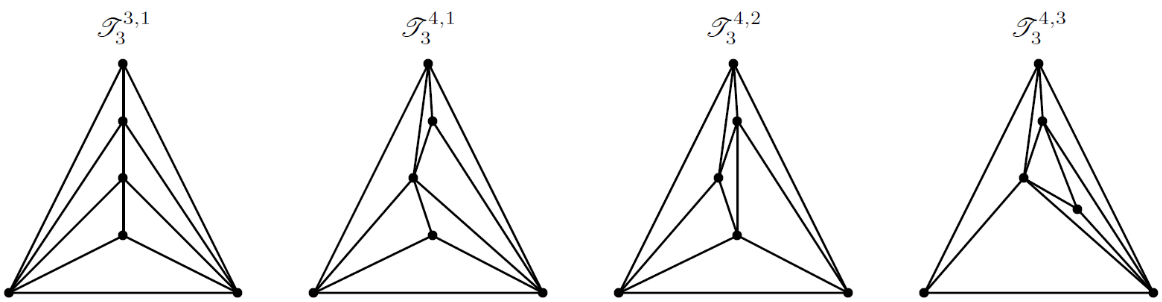

Figure 2: Some triangulations of $\mathcal{V}_3$

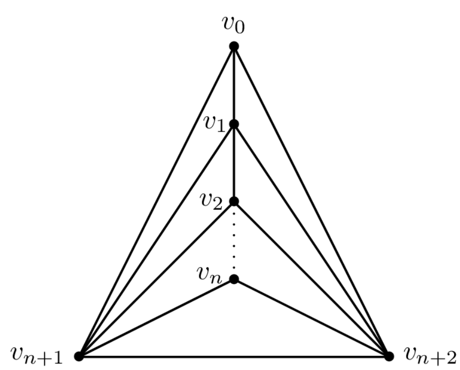

Figure 3: The standard triangulation Δ_n of $\mathcal{V}_n$

Since the works of K. Wagner [2], we know that we can transform any triangulation of $\mathcal{V}_n$ to the standard triangulation Δ_n (see Figure 3) via a well chosen sequence of *flips*. We recall that a flip consists in exchanging two diagonal lines of a convex quadrilateral, the latter being formed by two adjacent triangles of the triangulation. However, Wagner's theorem does not tell us how to find all the triangulations of $\mathcal{V}_n$, neither their combinatorics. For instance (cf. Figures 1 and 2), the combinatorics of $\mathcal{V}_1$ is $(3, 3, 3, 3)$. These of $\mathcal{V}_2$ are $(3, 4, 4, 4, 3)$ and $(4, 3, 4, 3, 4)$, and are equivalent because they are coming from two glued tetrahedra. Another example is the following combinatorics $(3,5,5,4,4,3)$, $(4,4,5,3,5,3)$, $(4,3,5,4,5,3)$ of $\mathcal{V}_3$. There are equivalent, but it is not that easy to see.

2. Triangulation algorithm

For a given $n \geq 1$, we want to determine all the triangulations of $\mathcal{V}_{n+1}$. This will be done by induction on n : we will construct $\mathcal{T}_{n+1}$ from $\mathcal{T}_n$.

We have chosen to use an algorithmic approach. Let $T_n^{d-1,k}$ be a triangulation of $\mathcal{T}_n$ with $deg(v_0) = d-1$. We recall that the triangle (v_0, v_{n+1}, v_{n+2}) always appears in $T_n^{d-1,k}$. We then denote by

$$v_{i_0} = v_{n+1}, v_{i_1}, \ldots, v_{i_{d-3}}, v_{i_{d-2}} = v_{n+2}$$

the neighbors of v_0. We remark that each edge $[v_{i_k}, v_{i_{k+1}}]$ is part of two triangles $(v_0, v_{i_k}, v_{i_{k+1}})$ and $(v_l, v_{i_k}, v_{i_{k+1}})$ of $T_n^{d-1,k}$.

Definition 2. An edge $[v_{i_k}, v_{i_{k+1}}]$ is called *admissible* if the quadrilateral $(v_0, v_{i_k}, v_l, v_{i_{k+1}})$ can be made convex by moving its vertices without changing the combinatorics.

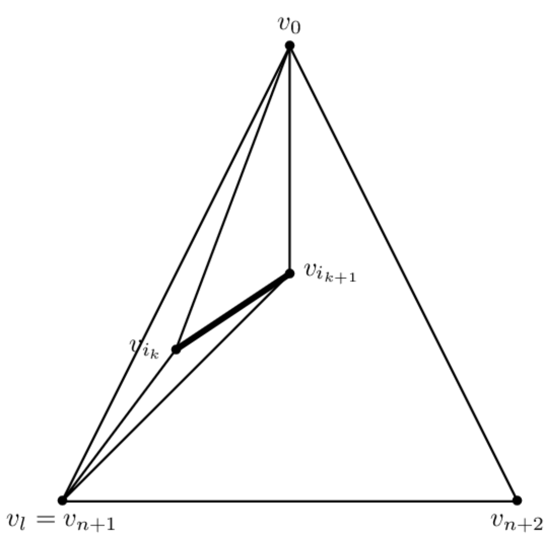

Figure 4: Non-admissible edge vs admissible edge

The idea is to carry out a flip in this quadrilateral if possible, otherwise to move slightly the vertices in order to make it convex (without changing the combinatorics).

Proposition 1. *Let $T_n^{d-1,k}$ be a triangulation of $\mathcal{V}_n$ with $deg(v_0) = d-1$. We denote with $v_{i_0} = v_{n+1}$, v_{i_1}, $\ldots$, $v_{i_{d-3}}$, and $v_{i_{d-2}} = v_{n+2}$ the neighbors of v_0. Then the edge $[v_{i_k}, v_{i_{k+1}}]$ is* admissible *if and only if $l \notin \{i_0, \ldots, i_{d-2}\}$ (we recall that $v_l \neq v_0$ is the vertex such that $(v_l, v_{i_k}, v_{i_{k+1}})$ is a triangle of $T_n^{d-1,k}$).*

2.1. Principle

Let $\mathcal{T}_n^{d,k}$ be a triangulation of $\mathcal{T}_n$ for $n \geq 2$.

- If $d = deg(v_0) = 3$, then the induced triangulation of (v_1, v_{n+1}, v_{n+2}) forms a triangulation of $\mathcal{V}_{n-1} = \{v_1, \ldots, v_{n+2}\}$. The converse is also true.

- If $d = deg(v_0) \geq 4$, let $v_{n+1}, v_{i_1}, \ldots, v_{i_{d-2}}, v_{n+2}$ be the neighbor vertices of v_0. Then we see easily that we can flip one of the edges $[v_0, v_{i_1}], \ldots, [v_0, v_{i_{d-2}}]$ (see also the proof of WAGNER's theorem in [5]). This allows us to decrease the degree of v_0. Convervely, the triangulations of $\mathcal{V}_n$ with $deg(v_0) = d$ come from the triangulations of $\mathcal{V}_n$ with $deg(v_0) = d-1$ by means of flips around the admissible edges $[v_{i_k}, v_{i_{k+1}}]$ (possibly $[v_{n+1}, v_{i_1}]$ and $[v_{i_{d-3}}, v_{n+2}]$ by noting $i_0 = n+1$ and $i_{d-2} = n+2$).

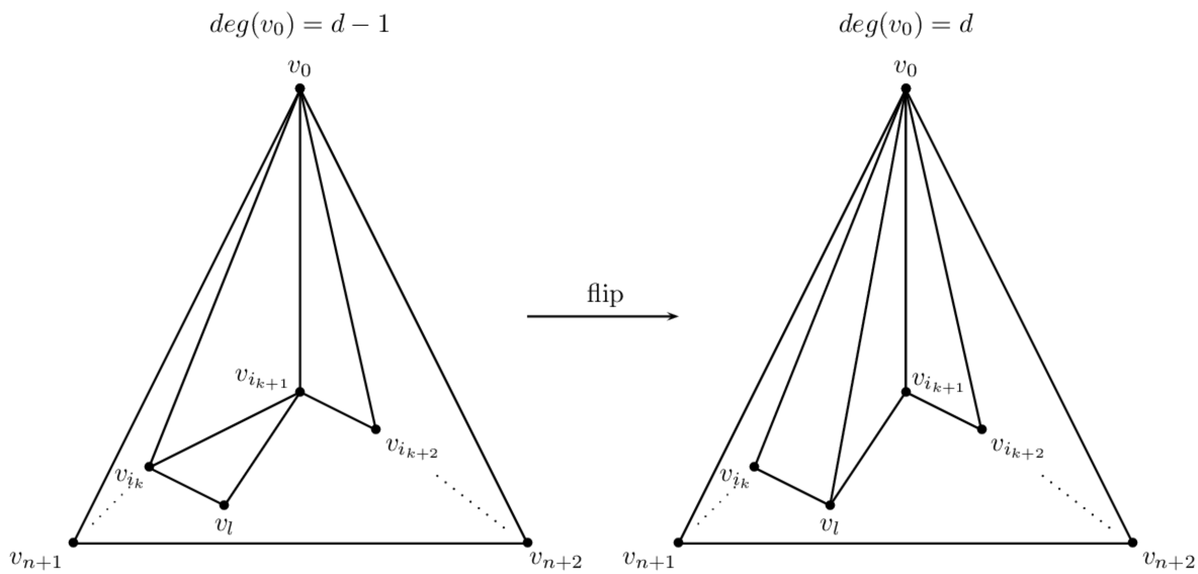

Figure 5: Principle of the iteration step

2.2. Algorithmic aspect

We obtain triangulations of degree 4 to $n+2$ by starting with those of degree 3. In the sequel we will use the variables :

- $I(n,d)$: The number of triangulations of degree d of $\mathcal{V}_n$ $(3 \leq d \leq n+2)$.
- I_n: The total number of triangulations of $\mathcal{V}_n$.
- $T_n^{d,i}$: The triangulations of degree d of $\mathcal{V}_n$ $(1 \leq i \leq I(n,d))$.
- $C_n^{d,i}$: The corresponding combinatorics.

We start with the following initialization (see Figure 1):

- $I(1,3) = I(1) = 1$.
- $T_1^{3,1} = \{(v_0,v_1,v_2),(v_0,v_1,v_3),(v_1,v_2,v_3)\}$.
- $C_1^{3,1} = (3,3,3,3)$.

Iteration step: we assume that $\mathcal{T}_n$ is known.

– Triangulations of degree 3 of $\mathcal{T}_{n+1}$: it suffices to triangulate $\mathcal{V}'_n = (v_1, \ldots, v_{n+3})$, so we will glue the triangulations $T_n^{d,i}$ $(3 \leq d \leq n+2,\ 1 \leq i \leq I(n,d))$ of $\mathcal{V}'_n$ to v_0.

– For $4 \leq d \leq n+3$: pass from the triangulations $T_{n+1}^{d-1,i}$ $(1 \leq i \leq I(n+1, d-1))$ of degree $d-1$ of $\mathcal{V}_{n+1}$ to those of degree d by means of flips.

2.3. Details of the iteration step

To clarify the iteration step of our algortihm, we assume that the following basic functions are known. In the same way, the vertex v_k of $\mathcal{V}_n$ will be denoted by k.

• **Function $f(T)$ where T is a triangulation of $\mathcal{V}_n$:**
transforms a triangulation $T = [[0, n+1, n+2], [a_1, b_1, c_1], \dots, [a_k, b_k, c_k]]$ of $\mathcal{V}_n$ into the triangulation $[[0, n+2, n+3]$, $[0, n+2, 1]$, $[0, 1, n+3]$, $[a_1+1, b_1+1, c_1+1]$, …, $[a_k+1, b_k+1, c_k+1]]$ of $\mathcal{V}_{n+1}$ of degree 3.

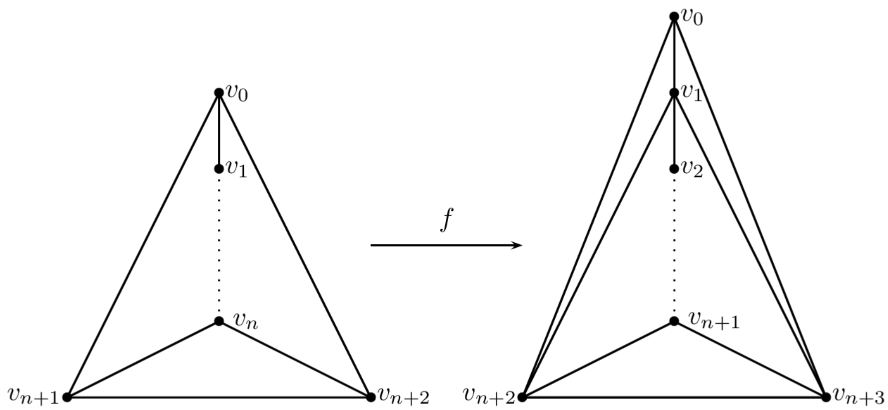

Figure 6: Function f

• **Function $g(C, n)$ where C is a combinatorics of $\mathcal{V}_n$:**
transforms a combinatorics $C = [d_0, \dots, d_n, d_{n+1}, d_{n+2}]$ of $\mathcal{V}_n$ into the combinatorics $[3, d_0+1, \dots, d_n, d_{n+1}+1, d_{n+2}+1]$ of $f(T)$.

• **Function $h(T, n, d)$ where T is a triangulation of $\mathcal{V}_n$ of degree d:**
returns the list $[i_0 = n+1, i_1, \dots, i_{d_2}, i_{d-1} = n+2]$ of neighbors of 0 for a triangulation T of degree d of $\mathcal{V}_n$.

• **Function $\mathit{flip}(T, a, b, c, d)$ where T is a triangulation:**
returns the new triangulation obtained after the flip of the two triangles $[a, b, c]$ and $[b, c, d]$ of T, that is:
$\mathit{flip}([[0, n+1, n+2], \dots, [a, b, c], [b, c, d], \dots], a, b, c, d) = [[0, n+1, n+2], \dots, [a, b, d], [a, d, c], \dots]$.

• **Function $\mathit{cflip}(C, a, b, c, d)$ where C is a combinatorics:**
computes the new combinatorics of C after the flip of the two triangles $[a, b, c]$ and $[b, c, d]$, that is: $\mathit{cflip}([\dots, \mathit{deg}(a), \dots, \mathit{deg}(b), \dots, \mathit{deg}(c), \dots, \mathit{deg}(d), \dots], a, b, c, d) = [\dots, \mathit{deg}(a)+1, \dots, \mathit{deg}(b)-1, \dots, \mathit{deg}(c)-1, \dots, \mathit{deg}(d)+1, \dots]$.

We are now ready for writing the iteration step (cf. 2.2) where the list $T[n]$ of triangulations of $\mathcal{V}_n$ is known, as well as the list $C[n]$ of corresponding combinatorics and the list $I[n]$ of indices ($I[n][0] = I[n][1] + \dots + I[n][n]$ and $I[n][d-2]$ is the number of triangulations of degree d).

For j=1 to $I[n][0]$: *#triangulations of degree* 3 *of* $\mathcal{V}_{n+1}$
$T[n+1][3][j] = f(T[n][3][j])$
$C[n+1][3][j] = g(C[n][3][j], n)$

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    I[n+1][1] = I[n+1][1] + 1
For d = 4 to n+3:                        #triangulations of degree 4 to n+3 of V_{n+1}
        For k = 1 to I[n+1][d-3]:        # from degree d-1 to degree d for V_{n+1}
            c = 0
            u = h(T[n+1][d-1][k], n+1, d-1)
            For l = 0 to d-3:
                For m = 1 to n+1:
                    If (m not in u) and ([u[l], u[l+1], m] ∈ T[n+1][d-1][k]):
                        c = c+1
                        T[n+1][d][c] = flip(T[n+1][d-1][k], 0, u[l], u[l+1], m)
                        C[n+1][d][c] = cflip(C[n+1][d-1][k], 0, u[l], u[l+1], m)
            I[n+1][d-2] = c
        I[n+1][0] = I[n+1][0] + I[n+1][d-2]
```

2.4. Numerical results and unicity

After translating the pseudo-code of the algorithm in the Python programming language (thanks to my colleague Jean Fromentin [4] for his contribution), we have obtained the following combinatorics ordered by the degree of v_0.

$\mathcal{V}_1$: $(3,3,3,3)$.

$\mathcal{V}_2$: $(3,4,3,4,4)$,
$(4,3,4,3,4)$, $(4,3,4,4,3)$.

$\mathcal{V}_3$: $(3,4,4,3,5,5)$, $(3,5,3,4,5,4)$, $(3,5,3,4,4,5)$,
$(4,3,5,3,4,5)$, $(4,3,5,3,5,4)$, $(4,4,4,4,4,4)$, $(4,4,3,5,5,3)$, $(4,4,3,5,3,5)$,
$(5,3,4,4,3,5)$, $(5,3,4,4,5,3)$, $(5,3,3,5,4,4)$, $(5,3,4,5,4,3)$, $(5,3,4,5,3,4)$.

The problem now is to recognize equivalent combinatorics. A necessary condition for two combinatorics $\mathcal{C}$ and $\mathcal{C}'$ to be equivalent is the existence of a permutation $\sigma \in \mathfrak{S}_{n+3}$ such that $\mathcal{C}' = \sigma(\mathcal{C})$. The converse is false, but can we expect that this is true for a permutation from the algorithm? That is:

Conjecture. *Let T and T' two triangulations of $\mathcal{V}_n$ such that there exists some flips $f_1, \dots, f_k$ verifying $T' = f_k \circ \dots \circ f_1(T)$, and $\sigma \in \mathfrak{S}_{n+3}$ (the symmetric group on $n+3$ letters) such that $\mathcal{C}' = \sigma(\mathcal{C})$. Then $\mathcal{C}'$ is equivalent to $\mathcal{C}$.*

The conjecture is true for $n = 1,2,3$. Indeed all the combinatorics of $\mathcal{V}_3$, except $(4,4,4,4,4,4)$, are equivalent because there are all coming from three glued tetrahedra. Another way to see that is to apply the relation $deg(v_0) + \dots + deg(v_5) = 2E = 24$.

This implies that all the degrees are equal to 4 or there exists at least one vertex of degree 3, and it remains to triangulate the set obtained by removing this point. So, the same object seen from different locations produces permuted combinatorics. But we don't know yet if the previous assumption remains true in the general case. Another problem is the number of triangulations. It's rising really rapidly; our program has given us several hundred of combinatorics just for $\mathcal{V}_5$. Therefore, it would be interesting to further develop theoretical tools for finding the different classes of equivalent combinatorics.

Finally, I would like to thank the referee for pointing out to me that, for each triangulated polyhedron, there is a polynomial with the polyhedron's squared volume as a root, and the polynomial only depends on the combinatorics of the polyhedron. In this sense, the problem of enumerative combinatorics helps to count how many polynomials of Sabitov type exist ([6]).

References

[1] A.D. Alexandrov: *Convex polyhedra*. Springer, 2010 (A.D. Alexandrov's original edition 1950 was in Russian. This English edition has been updated and extended by V.A. Zalgaller).

[2] K. Wagner: *Bemerkungen zum Vierfarbenproblem*. Jahresber. Dtsch. Math.-Ver. **46**, Abt. 1, 26–32 (1936).

[3] S.L. Devadoss, J. O'Rourke: *Discrete and computational geometry*. Princeton University Press, Princeton/NJ 2011.

[4] J. Fromentin: *Private communication*. Univ. Littoral Côte d'Opale LMPA – F-62228 Calais (France).

[5] S. Negami: *Diagonal flips of triangulations on surfaces, a survey*. Yokohama Math. J. **47** (1999).

[6] I.Kh. Sabitov: *The volume as a metric invariant of polyhedra*. Discrete Comput. Geom. **20**/4, 405–425 (1998).

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