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The Generalized Biquaternionic M-J Sets

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Abstract. The Mandelbrot-Julia sets (henceforth abbrev. M-J sets) and their properties have been extensively studied since their discovery. Many studies are focused on properties and dynamics of generalized M-J sets in complex and hypercomplex vector spaces, however there are still many variations of M-J sets which have not been studied yet. The following paper discusses one of such variations — the M-J sets in the biquaternionic vector space. Starting from theoretical fundamentals on an algebra of biquaternions and its closedness under addition and multiplication, which is required for constructing biquaternionic M-J sets, the author defines the generalized biquaternionic M-J sets and their relation both with complex M-J sets as well as with their 4-space analogues: quaternionic and bicomplex M-J sets. The connectedness and dynamics of J sets is also studied. Moreover, the analysis of 3D cross-sections of J sets allows validating the relationships with other hypercomplex fractal sets and evaluating a symmetry of resulting biquaternionic sets.

Key Words: biquaternionic Mandelbrot-Julia sets, algebra of biquaternions, fractals, generalized Mandelbrot-Julia sets

MSC 2010: 28A80

1. Introduction

Mandelbrot and Julia sets have been extensively studied in terms of their properties such as self-similarity, multifractality, symmetry, periodicity, stability and many others. Most of these studies are concerned with the classical Mandelbrot and Julia sets, given by the following recursive equation

$$z \leftarrow z^2 + c \tag{1}$$

in the complex number space $\mathbb{C}$ (see, e.g., [19, 27]) as well as almost infinite number of their variations with respect to powers of z [13, 14, 3], trigonometric and other functions applied in the recursive iteration of (1) [2, 26], polynomials (a great survey was presented in [22]) and others. Numerous researchers have studied the hypercomplex versions of M-J sets since

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NORTON introduced their generalization in quaternions $\mathbb{H}$ [24, 23]. The latter paper introduces intensive studies on variations of quaternionic M-J sets. The first published studies on the stability of the quaternionic version of (1) were presented in [10], where the authors performed the mathematical analysis of multi-cycle stability of the M-sets. Further, the generalizations of (1) with respect to a power α, we studied

$$z \leftarrow z^{\alpha} + c, \quad \alpha \in \mathbb{N}, \tag{2}$$

and provided the stability analysis of M-J sets.

A further extension of (1) to octonions $\mathbb{O}$ was proposed in [12] immediately after the extension of M-J sets to quaternions and in [11] the authors presented some preliminaries of stability analysis of octonionic J-sets. The next studies were performed by DIXON et al. [4], where the first trials of generalization of M-J sets in higher-dimensional vector spaces were presented.

Simultaneously, the tensor products of algebras were considered in order to construct M-J sets. The first study on generalized bicomplex $\mathbb{C} \otimes \mathbb{C}$ (or equivalently $\mathbb{C}_2$) M-J set was presented by ROCHON [29]. The bicomplex M-J set has unique properties, i.e., it is a four-dimensional (4D) vector space, similar to the quaternionic M-J sets. The deep studies of these properties were performed by the authors of [20, 21]. Then, ROCHON and his team proposed further generalization of M-J sets in a tricomplex $\mathbb{C}_3$ 8-space with extended analysis of 3D cross-sections of M-J sets and finally, the multicomplex $\mathbb{C}_n$ M-J sets with an analysis of dynamic behavior of such systems and a generalization of the Fatou-Julia theorem for multicomplex number spaces [6, 25].

The presented study is focused on the generalization of M-J sets in a biquaternionic $\mathbb{C} \otimes \mathbb{H}$ vector space. Only the few mentions in the available literature can be found in this area. GINTZ [7] presented preliminaries of biquaternionic fractal sets and their several 3D cross-sections. BOGUSH et al. [1] studied symmetry properties of biquaternionic J sets. However, the mathematical formalism of the generalized biquaternionic M-J sets has never been introduced to date. Following this, the appropriate mathematical description and analysis of biquaternionic M-J sets is necessary in order to introduce a new class of hypercomplex fractal sets.

2. The algebra of biquaternions

The algebra of biquaternions (known also as an algebra of complex quaternions) is a tensor product 4-algebra with a basis $1, i_1, i_2, i_3$. The symbolic representation of a biquaternion $\tilde{q}$ is as follows:

$$\mathbb{C} \otimes \mathbb{H} := \left\{ \tilde{q} = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_3 \,|\, a_n \in \mathbb{C} \right\}, \tag{3}$$

which can be also presented in the alternative form:

$$\mathbb{C} \otimes \mathbb{H} := \left\{ \tilde{q} = (g_1 + h_1 j) + (g_2 + h_2 j)\, i_1 + (g_3 + h_3 j)\, i_2 + (g_4 + h_4 j)\, i_3 \,|\, g_n, h_n \in \mathbb{R} \right\}, \tag{4}$$

where i_1, i_2, i_3 and j are imaginary units, and $j^2 = -1$, $i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1$. The biquaternions contain zero divisors, idempotents and nilpotents [30]. The idempotent representation of biquaternions is unique and is given by the following (see Theorem 2 in [30]):

$$e_{1,2} = \frac{1}{2} \pm \frac{1}{2} \xi j\,, \tag{5}$$

where $e_{1,2}$ is represented in the form $e_{1,2} = w_1 + \xi w_2$ with $w_1, w_2 \in \mathbb{C}$, $\xi \in \mathbb{C} \otimes \mathbb{H}$ and $\xi^2 = -1$. This is a very useful property which allows performing addition and multiplication of biquaternions element-wise. Based on this observation, the addition and multiplication of two biquaternions $\tilde{q}_1 = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_3$ and $\tilde{q}_2 = b_1 + b_2 i_1 + b_3 i_2 + b_4 i_3$ with $a_n, b_n \in \mathbb{C}$ can be defined as

$$\tilde{q}_1 + \tilde{q}_2 := (a_1 + b_1) + (a_2 + b_2)\, i_1 + (a_3 + b_3)\, i_2 + (a_4 + b_4)\, i_3, \tag{6}$$

$$\begin{aligned} \tilde{q}_1 \cdot \tilde{q}_2 := {} & (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4) + (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3)\, i_1 \\ & + (a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2)\, i_2 + (a_1 b_4 + a_2 b_3 - a_3 b_2 - a_4 b_1)\, i_3, \end{aligned} \tag{7}$$

or, in equivalent form, using the definition of $\tilde{q}$ given by (4).

3. The generalized M-J sets in biquaternionic vector space

Considering quadratic polynomial of type (1) defined in biquaternions, one can construct its generalized version in the form

$$z \leftarrow z^p + c \quad \text{for} \quad p \geq 2, \quad z, c \in \mathbb{C} \otimes \mathbb{H}. \tag{8}$$

The condition $p \geq 2$ leads to several interesting properties of M-J sets defined in $\mathbb{C} \otimes \mathbb{H}$. When this condition is fulfilled the point at infinity is a super-attracting fixed point, and taking into consideration that for every $z \neq \infty$ a sequence resulting from (8) has finite values, this fixed point is also the exceptional point. However, when the condition of $p \geq 2$ is not fulfilled the resulting sets are not M sets. Let us consider five possible cases for values of p when $p \ngeq 2$.

- When $0 \leq p < 1$, the critical point is not at zero, but at the point of infinity. This leads to a fact that c in (8) is not on the trajectory of infinity, thus M-J sets do not exist for such values of p.
- In the second case, when $p = 1$, the critical point does not exist; so the M-J sets do not exist, too (similar observations were obtained by the authors of [31] for hypercomplex M-J sets).
- Considering the third case, when $1 < p < 2$, the critical point is located at zero as for $p \geq 2$, the resulting M-J sets resemble integer-valued M-J sets and hold the axial symmetry, however the rotational symmetry is not fulfilled. Such sets hold their fractal properties for arbitrary $p > 1$, $p \in \mathbb{N}$ following the studies of the authors of [18].

  The next two cases consider the negative values of p.
- The case, when $-1 < p < 0$, is similar to the case of $0 \leq p < 1$, i.e., the M-J sets in this case do not exist.
- Finally, for $p < -1$, $p \in \mathbb{N}$, the dynamics of resulting sets reveals a completely different behavior with respect to $p > 2$. In this case the critical point is at infinity which means that the set of prisoners has an attractor at the point at infinity and covers the whole $\mathbb{C} \otimes \mathbb{H}$-space except some small regions near c. Therefore, the obtained structures in this case are not M-J sets.

Considering the properties of the generalized M-J sets, one can define the M set and the J set following the next definitions.

Definition 1. Let $f_c(z) = z^p + c$, where $z, c \in \mathbb{C} \otimes \mathbb{H}$ and $p > 2$ $(p \in \mathbb{R})$, be a mapping in the biquaternionic vector space (a $\mathbb{C} \otimes \mathbb{H}$-space). Then, the resulting M set is a *generalized biquaternionic* M *set*, for which the trajectory of c is limited

$$\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}} = \left\{c \in \mathbb{C} \otimes \mathbb{H} \,|\, f_c^{(s)}(0) \not\to \infty \text{ if } s \to \infty\right\}, \tag{9}$$

and thus $\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$ is bounded.

Definition 2. The *generalized "filled"* J *set*, obtained while iterating the recursive equation $f_c(z) = z^p + c$, where $z, c \in \mathbb{C} \otimes \mathbb{H}$ and $p > 2$, is mapped into the biquaternionic vector space with a limited trajectory of c, and

$$\mathrm{J}^p_{\mathbb{C}\otimes\mathbb{H}} = \left\{c \in \mathbb{C} \otimes \mathbb{H} \,|\, f_c^{(s)}(z) \not\to \infty \text{ if } s \to \infty\right\}, \tag{10}$$

and thus $\mathrm{J}^p_{\mathbb{C}\otimes\mathbb{H}}$ is bounded.

The Definitions 1 and 2 introduce the generalized biquaternionic M-J sets with dynamics similar to bicomplex and hypercomplex (Cliffordean) sets (see the extended description presented in [34, 31]). These definitions lead to a possibility of constructing $\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$-$\mathrm{J}^p_{\mathbb{C}\otimes\mathbb{H}}$ sets in a 4-space with a certain bailout value B (known also as the escaping-time limit).

Definition 3. $X \subseteq \mathbb{C} \otimes \mathbb{H}$ is a $\mathbb{C} \otimes \mathbb{H}$-*Cartesian product determined by* X_1 *and* X_2 if $X = X_1 \times_e X_2 := \{z_1 + z_2 j \in \mathbb{C} \otimes \mathbb{H} : \; z_1 + z_2 j = w_1 e_1 + w_2 e_2 \mid \; w_1, w_2 \in X_1 \times X_2\}$.

Definition 3 leads to the formulation of a theorem of connectedness of the $\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$-$\mathrm{J}^p_{\mathbb{C}\otimes\mathbb{H}}$ sets. Knowing that the $\mathrm{M}^p_{\mathbb{C}}$ sets are connected (see [5] for a recursive equation of type (1) and [34] for its generalized version of type (2)), and $\mathrm{M}^p_{\mathbb{H}}$ sets are connected [35], it is possible to show that $\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$ sets are also connected.

Theorem 1. $\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}} = \mathrm{M}^p_{\mathbb{C}} \times_e \mathrm{M}^p_{\mathbb{H}}$.

Proof. Let $c \in \mathbb{C} \otimes \mathbb{H}$. For $f_c(z) = z^p + c$, $p > 2$, $p \in \mathbb{N}$, $z, c \in \mathbb{C} \otimes \mathbb{H}$ and $f_c^{(s)}(z) := \left(f_c^{(s-1)} \circ f_c\right)(z)$, according to Definition 1, $f_c^{(s)}(0)$ has a bounded orbit $\forall s \in \mathbb{N}$. Additionally, when $p > 2$ one obtains

$$f_c(z) = z^p + c = \left[(z_1 - z_2 j)^p + (c_1 - c_2 j)\right] e_1 + \left[(\tilde{z}_1 + \tilde{z}_2 j)^p + (\tilde{c}_1 + \tilde{c}_2 j)\right] e_2, \tag{11}$$

where $z = (z_1 - z_2 j)\, e_1 + (\tilde{z}_1 + \tilde{z}_2 j)\, e_2$, and $c = (c_1 - c_2 j)\, e_1 + (\tilde{c}_1 + \tilde{c}_2 j)\, e_2$, thus

$$f_c^{(s)}(z) = f^{(s)}_{c_1 - c_2 j}\,(z_1 - z_2 j)\, e_1 + f^{(s)}_{\tilde{c}_1 + \tilde{c}_2 j}\,(\tilde{z}_1 + \tilde{z}_2 j)\, e_2. \tag{12}$$

Considering that $f_c^{(s)}(0) = f^{(s)}_{c_1 - c_2 j}(0) e_1 + f^{(s)}_{\tilde{c}_1 + \tilde{c}_2 j}(0) e_2$ is bounded when $s \to \infty$, $f^{(s)}_{c_1 - c_2 j}(0) e_1$ and $f^{(s)}_{\tilde{c}_1 + \tilde{c}_2 j}(0) e_2$ are also bounded when $s \to \infty$. Then $c_1 - c_2 j \in \mathrm{M}^p_{\mathbb{C}}$, $\tilde{c}_1 + \tilde{c}_2 j \in \mathrm{M}^p_{\mathbb{H}}$, and $c = (c_1 - c_2 j)\, e_1 + (\tilde{c}_1 + \tilde{c}_2 j)\, e_2 \in \mathrm{M}^p_{\mathbb{C}} \times_e \mathrm{M}^p_{\mathbb{H}}$, and $\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}} \subset \mathrm{M}^p_{\mathbb{C}} \times_e \mathrm{M}^p_{\mathbb{H}}$. $\square$

Theorem 2. *The* $\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$ *sets are connected.*

Proof. Defining the mapping e as

$$\mathbb{C} \otimes \mathbb{H} = \mathbb{C} \times \mathbb{H} \xrightarrow{e} \mathbb{C} \times_e \mathbb{H} = \mathbb{C} \otimes \mathbb{H}\,(z_1, z_2) \;\mapsto\; z_1 e_1 + z_2 e_2, \tag{13}$$

one can see that the mapping e is a homeomorphism. Thus, if $X_1 \subset \mathbb{C}$ and $X_2 \subset \mathbb{H}$ are connected, then $e\,(X_1 \times X_2) = X_1 \times_e X_2$ is also connected. Following this and considering Theorem 1, $\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$ is also connected. $\square$

Considering Definition 2, it is possible to investigate the connectedness of "filled" $\mathrm{J}^p_{\mathbb{C}\otimes\mathbb{H}}$ sets and relationships between $\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$ and $\mathrm{J}^p_{\mathbb{C}\otimes\mathbb{H}}$ sets. Having in mind the theorem on structural dichotomy of M-J sets on a $\mathbb{C}$-plane proposed by Douady and Hubbard [5], one can formulate the theorem which can support the proof of a similar theorem for $\mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$-$\mathrm{J}^p_{\mathbb{C}\otimes\mathbb{H}}$ sets.

Theorem 3. $\mathrm{J}^p_{c,\mathbb{C}\otimes\mathbb{H}} = \mathrm{J}^p_{(c_1-c_2j)e_1+(\tilde{c}_1+\tilde{c}_2j)e_2,\mathbb{C}\otimes\mathbb{H}} = \mathrm{J}^p_{c_1-c_2j,\mathbb{C}} \times_e \mathrm{J}^p_{\tilde{c}_1+\tilde{c}_2j,\mathbb{H}}$.

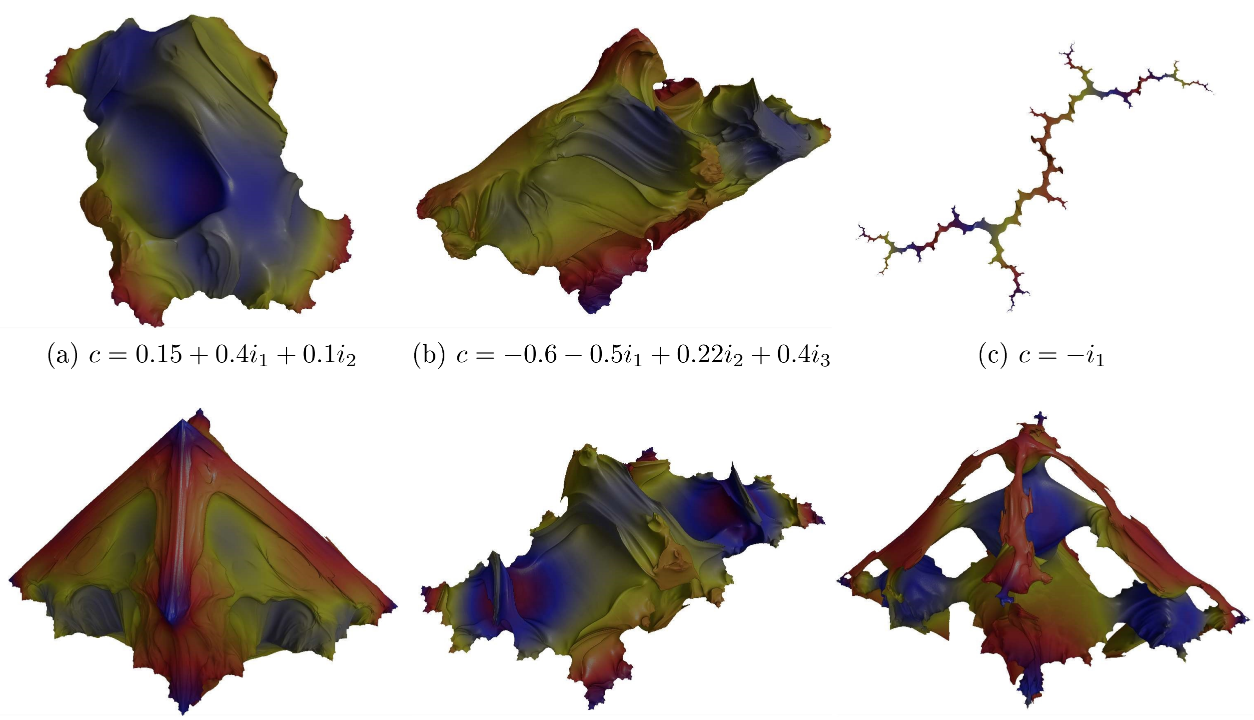
The proof of Theorem 3 proceeds along the same lines as that for Theorem 1.

Theorem 4. $c \in \mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}} \Leftrightarrow \mathrm{J}^p_{c,\mathbb{C}\otimes\mathbb{H}}$ *are connected.*

Proof. Theorem 3 indicates that $\mathrm{J}^p_{c,\mathbb{C}\otimes\mathbb{H}} = \mathrm{J}^p_{c_1-c_2j,\mathbb{C}} \times_e \mathrm{J}^p_{\tilde{c}_1+\tilde{c}_2j,\mathbb{H}}$. Considering the homeomorphism of the mapping e in the proof of Theorem 1, $\mathrm{J}^p_{c_1-c_2j,\mathbb{C}} \times \mathrm{J}^p_{\tilde{c}_1+\tilde{c}_2j,\mathbb{H}}$ are connected iff $\mathrm{J}^p_{c_1-c_2j,\mathbb{C}} \times_e \mathrm{J}^p_{\tilde{c}_1+\tilde{c}_2j,\mathbb{H}}$ are connected. Then $\mathrm{J}^p_{c_1-c_2j,\mathbb{C}} \times_e \mathrm{J}^p_{\tilde{c}_1+\tilde{c}_2j,\mathbb{H}}$ are connected iff $\mathrm{J}^p_{c_1-c_2j,\mathbb{C}}$ and $\mathrm{J}^p_{\tilde{c}_1+\tilde{c}_2j,\mathbb{H}}$ are connected. Hence, generalizing the theorem on structural dichotomy of M-J sets on a $\mathbb{C}$-plane [5], $\mathrm{J}^p_{c,\mathbb{C}\otimes\mathbb{H}}$ is connected iff $c = (c_1 - c_2j)\,e_1 + (\tilde{c}_1 + \tilde{c}_2j)\,e_2 \in \mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$. □

4. Analysis of 3D cross-sections of generalized biquaternionic M-J sets

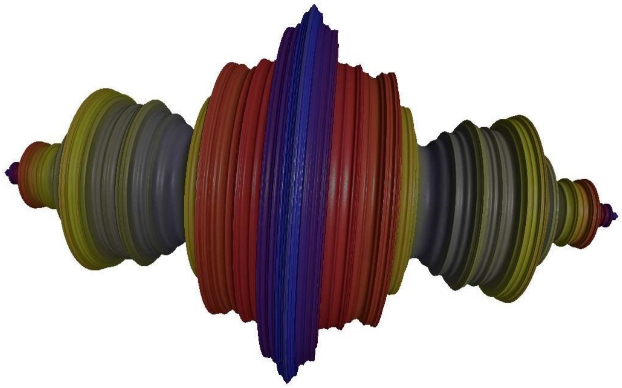
In order to investigate a geometric structure of generalized biquaternionic M-J sets and their specific properties as well as to show the character of their evolution, while changing the degree p of an iterated polynomial, and finally, to prove additionally the above theorems, experimentally a series of numerical simulations was performed. Since the biquaternionic vector space is four-dimensional, the biquaternionic M-J sets can be visualized in the form of 3D projections in $\mathbb{R}^3$. For this purpose, the last element in a biquaternion presented in the



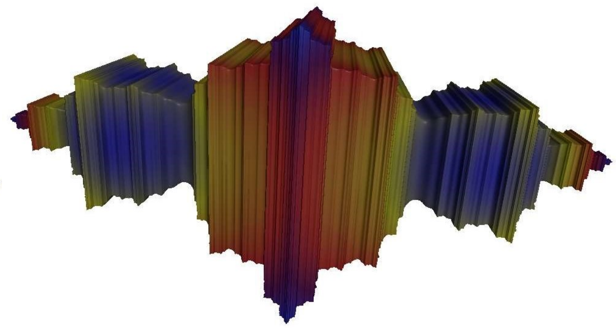
(a) $c = 0.15 + 0.4i_1 + 0.1i_2$ (b) $c = -0.6 - 0.5i_1 + 0.22i_2 + 0.4i_3$ (c) $c = -i_1$

(d) $c = -0.75$ (e) $c = -0.390541 - 0.586788i_1$ (f) $c = -1 - 0.1i_1$

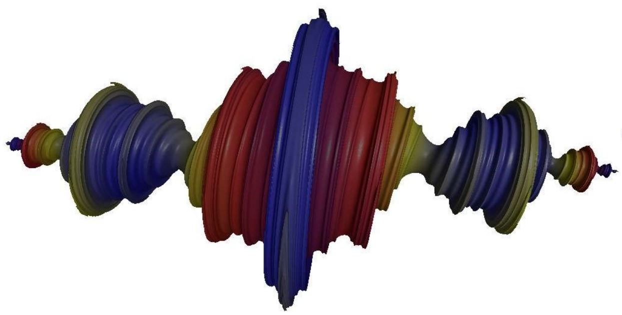
Figure 1: Examples of biquaternionic J sets.



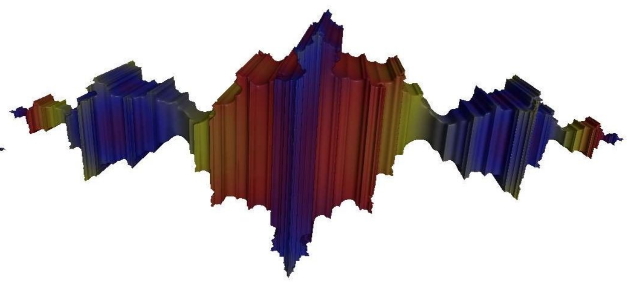
(a) $\mathrm{J}^2_{\mathbb{H}}, c = -0.75$



(b) $\mathrm{J}^2_{\mathbb{C}_2}, c = -0.75$



(c) $\mathrm{J}^2_{\mathbb{H}}, c = -1 - 0.1i$



(d) $\mathrm{J}^2_{\mathbb{C}_2}, c = -1 - 0.1i$

Figure 2: Analogues of the degenerate biquaternionic J sets.

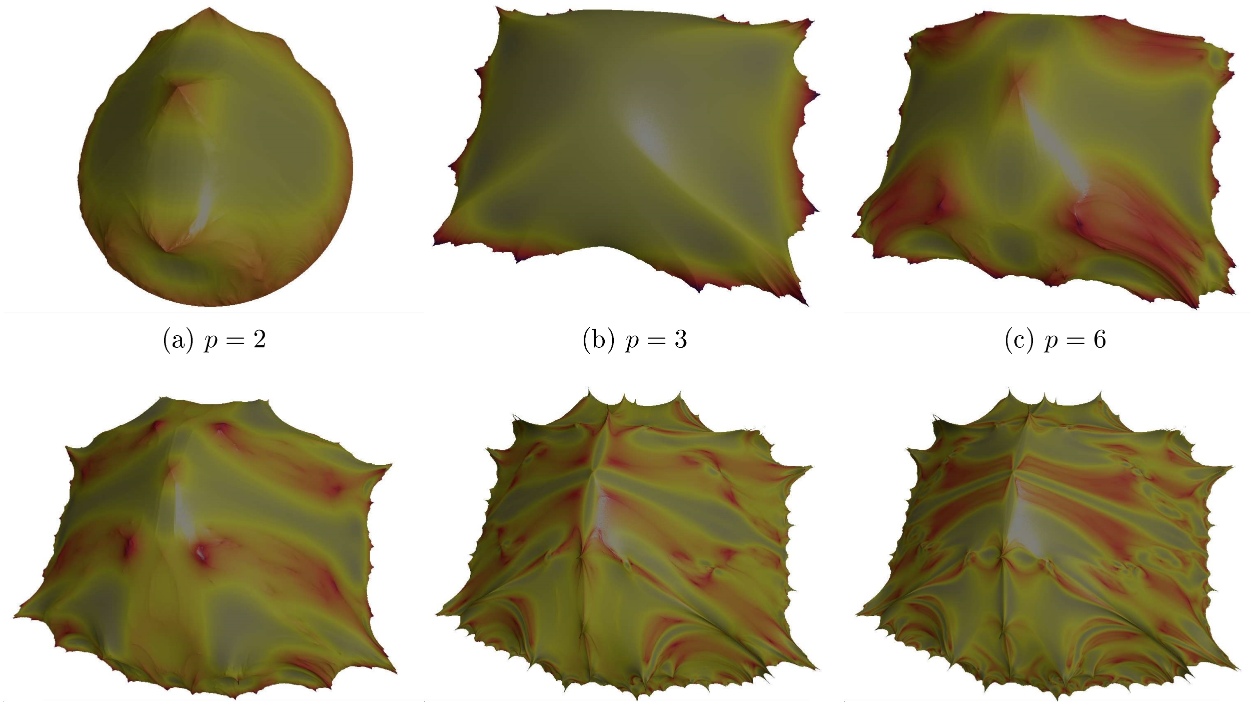
form (3) remains fixed. This approach is widely applied during the visualization of 4D fractal sets [29, 35, 34, 21]. Further, a characteristic perspective view is selected to prepare a 2D image. Several examples of views of 3D cross-sections of biquaternionic J sets for $p = 2$ with various c values are presented in Figure 1.

From Figure 1 a variety and complexity of biquaternionic J sets can be observed, but it is also possible to perceive several similarities of biquaternionic J sets with their complex analogues. By cutting the resulting 3D cross-sections of fractals presented in the Figures 1(c) – (f) by a plane along the axis of reals, one can obtain the well known J sets on a $\mathbb{C}$-plane. That is, the biquaternionic analogues of the *Dendrite, San Marco fractal* and *Siegel Disk* are presented in the Figures 1(c) – (e), respectively. Based on this observation, one can introduce the following remark.

Remark. Let c be represented by two non-zero coefficients a_1 and a_2 only, following the symbolic representation of a biquaternion given by (3). Then, the J set with $a_3 = 0$ and $a_4 = 0$ is a degenerate biquaternionic J set.

Moreover, when analyzing the shapes of 3D cross-sections of biquaternionic J sets one can observe that their symmetry properties are broken with respect to quaternionic analogues of these J sets. This is very well visible in the Figures 1(d),(f) which are neither rotationally symmetric as their quaternionic analogues (see Figures 2(a),(b)) nor quadrilaterally symmetric as their bicomplex analogues (see Figures 2(c),(d)), which results from the complexification of a quaternion.

What is interesting, the behavior of a biquaternionic quadratic map of type (1) with $c = 0$



(a) $p = 2$ (b) $p = 3$ (c) $p = 6$

(d) $p = 10$ (e) $p = 50$ (f) $p = 100$

Figure 3: Examples of biquaternionic J sets for various values of p and with $c = 0$.

also does not reveal symmetry typical for complex, quaternionic and other higher-dimensional hypercomplex generalizations of J sets. As it was shown in [15], for $c = 0$ the J sets in the mentioned vector spaces are $(d-1)$-spheres, where d is the dimension of the given vector space. This is a direct consequence when iterating (1) for $c = 0$. However, in the case of biquaternionic quadratic map the resulting set (see Figure 3(a)) is not a $(d-1)$-sphere, due to the tensor product of dimensionally unequal algebras. When the order of an iterated polynomial increases the resulting J sets resemble a shape of a pillow with multiple horn-like protruding geometric structures on the boundaries, therefore we can call these sets the *Devil's pillows*. The Devil's pillows are not rotationally symmetric, however they retain several symmetries, e.g., they have two symmetry planes. Moreover, in contrast to the mentioned complex and hypercomplex J sets, the geometric structure of biquaternionic J sets with $c = 0$ remains fractal.

(a) $p = 2, c = 0.45 + 0.45i_1$ (b) $p = 3, c = 0.8i_1$ (c) $p = 3, c = 0.54$

Figure 4: Examples of disconnected but not totally disconnected biquaternionic J sets.

Considering the results of exploration of biquaternionic J sets, one observes that some of these sets are connected, some of them are totally disconnected, and the other ones are disconnected but not totally (see examples in Figure 4). This classification is similar to the results of studies on the connectedness of multicomplex J sets described in [28]. Following this, one can formulate the following conjecture.

Conjecture. *The connectedness of biquaternionic J sets can be fully described by the following three cases:*

1. *for* $c \in \mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$ *the* $\mathrm{J}^p_{\mathbb{C}\otimes\mathbb{H}}$ *sets are connected;*
2. *for* $c \notin \mathrm{M}^p_{\mathbb{C}\otimes\mathbb{H}}$ *the* $\mathrm{J}^p_{\mathbb{C}\otimes\mathbb{H}}$ *sets are totally disconnected (e.g., homeomorphic to the Cantor dust);*
3. *there exist other cases of* $\mathrm{J}^p_{\mathbb{C}\otimes\mathbb{H}}$ *sets that are disconnected, but not totally.*

5. Conclusions

This paper introduces a new class of hypercomplex fractal sets — the generalized biquaternionic M-J sets with appropriate mathematical description and analysis. The generalized M-J sets in the biquaternionic vector space are defined and analyzed. In particular it was proven that the bailout value (escaping-time limit) equal to 2 is the best possible value, same as for other complex and hypercomplex analogues of M-J sets. It was proven that the biquaternionic M set and the corresponding "filled" J sets are connected. The analysis of 3D cross-sections of generalized biquaternionic J sets shows that, due to the specificity of construction of these sets in a vector space resulting from a tensor product of two algebras ($\mathbb{C}\otimes\mathbb{H}$), the symmetry of these sets is broken, which was confirmed by appropriate examples. The graphical analysis also allows analyzing relations with complex J sets as well as its hypercomplex analogues defined in the bicomplex and quaternionic vector spaces. Finally, the conjecture of connectedness of J sets was formulated which is a topic of further studies related to biquaternionic M-J sets.

References

[1] A.A. Bogush, A.Z. Gazizov, Y.A. Kurochkin, V.T. Stosui: *Symmetry properties of quaternionic and biquaterionic analogs of Julia sets.* Ukrainian Journal of Physics **48**/4, 295–299 (2003).

[2] Y.S. Chauhan, R. Rana, A. Negi: *A computational study of logarithmic funtion for Mann and Ishikawa iterates.* World Journal of Science and Technology **1**, 92–101 (2011).

[3] S.V. Dhurandhar, V.C. Bhavsar, U.G. Gujar: *Analysis of z-plane fractal images from* $z \leftarrow z^\alpha + c$ *for* $\alpha < 0$. Computers & Graphics **17**/1, 89–94 (1993).

[4] S.L. Dixon, K.L. Steele, R.P. Burton: *Generation and graphical analysis of Mandelbrot and Julia sets in more than four dimensions.* Computers & Graphics **20**/3, 451–456 (1996).

[5] A. Douady, J.H. Hubbard: *Itération des polynômes quadratiques complexes.* Comptes Rendus de l'Académie des Sciences – Series I – Mathematics **294**, 123–126 (1982).

[6] V. Garant-Pelletier, D. Rochon: *On a generalized Fatou-Julia theorem in multicomplex spaces.* Fractals **17**/3, 241–255 (2009).

[7] T.W. Gintz: *Artist's statement CQUATS – a non-distributive quad algebra for 3D renderings of Mandelbrot and Julia sets*. Computers & Graphics **26**/2, 367–370 (2002).

[8] A.K. Golmankhaneh, D. Baleanu: *Non-local Integrals and Derivatives on Fractal Sets with Applications*. Open Physics **14**/1, 542–548 (2016).

[9] A.K. Golmankhaneh, D. Baleanu: *Diffraction from fractal grating Cantor sets*. Journal of Modern Optics **63**/14, 1364–1369 (2016).

[10] J. Gomatham, J. Doyle, B. Steves, I. McFarlane: *Generalization of the Mandelbrot set: Quaternionic quadratic maps*. Chaos Solitons Fractals **5**/6, 971–986 (1995).

[11] C.J. Griffin, G.C. Joshi: *Transition points in octonionic Julia sets*. Chaos Solitons Fractals **3**/1, 67–88 (1993).

[12] C.J. Griffin, G.C. Joshi: *Octonionic Julia sets*. Chaos Solitons Fractals **2**/1, 11–24 (1992).

[13] U.G. Gujar, V.C. Bhavsar: *Fractals from $z \leftarrow z^\alpha + c$ in the complex c-plane*. Computers & Graphics **15**/3, 441–449 (1991).

[14] U.G. Gujar, V.C. Bhavsar, N. Vangala: *Fractals from $z \leftarrow z^\alpha + c$ in the complex z-plane*. Computers & Graphics **16**/1, 45–49 (1992).

[15] A. Katunin, K. Fedio: *On a visualization of the convergence of the boundary of generalized Mandelbrot set to $(n-1)$-sphere*. J. Appl. Math. Comput. Mech. **14**/1, 63–69 (2015).

[16] T. Kim: *Quaternion Julia Set Shape Optimization*. Computer Graphics Forum **34**/5, 167–176 (2015).

[17] K.A. Lindsey: *Shapes of polynomial Julia sets*. Ergodic Theory and Dynamical Systems **35**/6, 1913–1924 (2015).

[18] S. Liu, X. Cheng, C. Lan, W. Fu, J. Zhou, Q. Li, G. Gao: *Fractal property of generalized M-set with rational number exponent*. Appl. Math. Comput. **220**, 668–675 (2013).

[19] B.B. Mandelbrot: *The Fractal Geometry of Nature*. W.H. Freeman and Co, New York 1982.

[20] É. Martineau, D. Rochon: *On a bicomplex distance estimation for the Tetrabrot*. International Journal of Bifurcation and Chaos **15**/9, 3039–3050 (2005).

[21] C. Matteau, D. Rochon: *The inverse iteration method for Julia sets in the 3-dimensional space*. Chaos Solitons Fractals **75**, 272–280 (2015).

[22] A. Negi, M. Rami, R. Chugh: *Julia sets and Mandelbrot sets in Noor orbit*. Appl. Math. Comput. **228**, 615–631 (2014).

[23] A.V. Norton: *Julia sets in the quaternions*. Computers & Graphics **13**/2, 267–278 (1989).

[24] A.V. Norton: *Generation and display of geometric fractals in 3-D*. Computer Graphics **16**/3, 61–67 (1982).

[25] P.-O. Parisé, D. Rochon: *A study on dynamics of the tricomplex polynomial $\eta^p + c$*. Nonlinear Dynam. **82**/1, 157–171 (2015).

[26] R. Rana, Y.S. Chauhan: *Generation of new Julia sets and Mandelbrot sets for inverse sine function*. International Journal of Innovative Research in Computer and Communication Engineering **1**, 101–111 (2013).

[27] P.H. Richter: *The Beauty of Fractals: Images of Complex Dynamical Systems.* Springer, Berlin 1986.

[28] D. Rochon: *On a generalized Fatou-Julia theorem.* Fractals **11**/3, 213–219 (2003).

[29] D. Rochon: *A generalized Mandelbrot set for bicomplex numbers.* Fractals **8**/4, 355–368 (2000).

[30] S.J. Sangwine, D. Alfsmann: *Determination of the biquaternion divisors of zero, including the idempotents and nilpotents.* Adv. Appl. Clifford Algebr. **20**/2, 401–410 (2010).

[31] X. Wang, T. Jin: *Hyperdimensional generalized M–J sets in hypercomplex number space.* Nonlinear Dynam. **73**/1, 843–852 (2013).

[32] X.Y. Wang, X.D. Liu, W.Y. Zhu: *Researches on generalized Mandelbrot sets from complex mapping* $z \leftarrow z^\alpha + c(\alpha > 0)$. Acta Mathematica Scientia **19**/1, 73–79 (1999).

[33] X.Y. Wang, X.D. Liu, W.Y. Zhu, S.S. Gu: *Analysis of c-plane fractal images from* $z \leftarrow z^\alpha + c$ *for* $\alpha > 0$. Fractals **8**, 307–314 (2000).

[34] X.-Y. Wang, W.J. Song: *The generalized M-J sets for bicomplex numbers.* Nonlinear Dynam. **72**/1, 17–26 (2013).

[35] X.-Y. Wang, Y.-Y. Sun: *The general quaternionic M–J sets on the mapping* $z \leftarrow z^\alpha + c$ $(\alpha \in \mathbf{N})$. Comput. Math. Appl. **53**/11, 1718–1732 (2007).

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