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Dual Tiling Origami

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Abstract. This paper proposes a class of origami, dual tiling origami, which is an extension of LANG's *"Octet Truss"*. Dual tiling origami lies between two parallel planes and tessellates both sides with polygonal tiles. The tiling patterns on both surfaces are dual, and the structure comprises cones of these tiling polygons on both sides with tetrahedra between them. Three conditions for the existence of dual tiling origami are shown. If the folded shape is flat, dual tiling origami can tessellate a family of parallelogram lattices. For the thick case, the relation between parameters of the parallelograms and the height between two planes is discussed. These dual tiling origami can converge to *Miura-Ori* under a certain condition.

Key Words: Origami tessellation, tiling, core panel

MSC 2010: 51M20, 53A05, 68U05

1. Introduction

LANG's origami *"Octet Truss"* tessellates two parallel planes with square lattices, as shown in Figure 1 [4]. This origami consists of square pyramids having bases on both sides and regular tetrahedra composed of folded squares. The bottom and top lattices are dual. Each square pyramid is formed with a square face and its corresponding vertex in the dual graph. Similarly, each tetrahedron is formed by the corresponding edges of the dual graphs.

In this paper, we propose a class of origami called *dual tiling origami*, which is a generalization of *Octet Truss* from square lattices to arbitrary dual graphs drawn on two parallel planes. We define dual tiling origami and summarize the necessary and sufficient condition for the graphs to form dual tiling origami in Section 2. In Section 3, we discuss a case when two planes are coplanar (flat case), wherein the design problem turns out to be an overconstrained system. Thus, we explore some symmetric cases that can give a family of solutions consisting of parallelogram lattices. In Section 4, we explore parallelogram dual tiling origami with some thickness that has parallelogram lattices and clarify the relationship between the parameters of the parallelogram and the thickness. We also discuss the relationship between parallelogram dual tiling origami and *Miura-Ori*.

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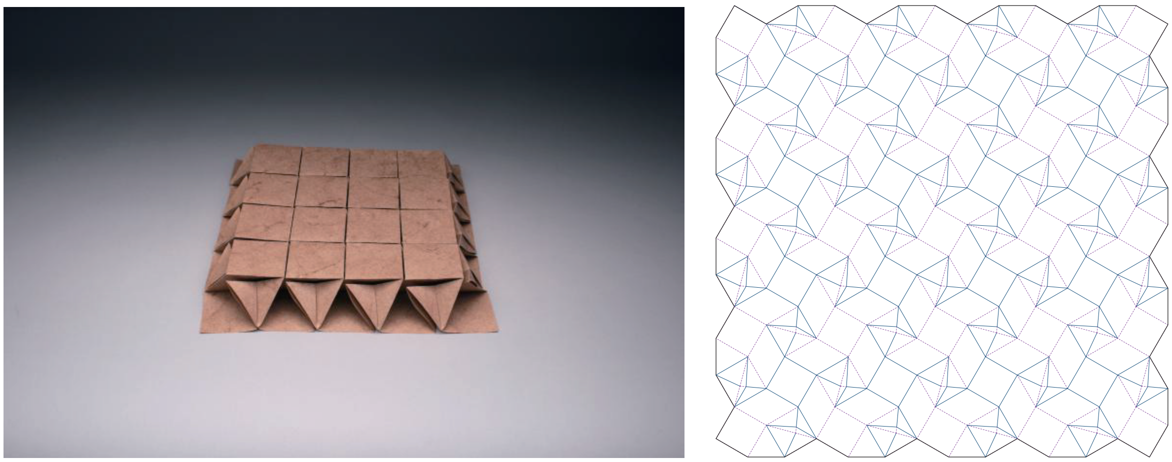

Figure 1: Folded state (left) and crease pattern (right) of the *"Octet Truss"* designed by *Lang*. (Reprinted from *Artwork: Octet Truss, opus 652* in *Robert J. Lang Origami*, R.J. Lang, 2014, Retrieved from `http://www/langoigami.com/artwork/octet-truss-opus-652`, and *Crease Pattern: Octet Truss, opus 652* in *Robert J. Lang Origami*, R.J. Lang, 2014, Retrieved from `http://www/langoigami.com/creasepattern/octet-truss-opus-652`. Copyright 2004–2018 by Robert J. Lang. Reprinted with permission)

Concerning the application of dual tiling origami, we expect it to be used as sandwich core material. Corrugated fibreboard is a popular sandwich core material. *Zeta-core*, which consists of *Miura-Ori* between two planes, has high stiffness [6]. Saito and Nojima [7] proposed a method to make arbitrary cross-section honeycomb cores from one sheet by folding and cutting. The bonding surface between the existing sandwich core materials and their liners are edges in their primitive forms. When liners are laminated on parallelogram dual tiling origami, as shown in the left side of Figure 2, the origami is bonded on the entire liner surfaces. Therefore, the resulting structure becomes rigid and is expected to be as stiff as *Octet Truss* as shown in the right side of Figure 2.

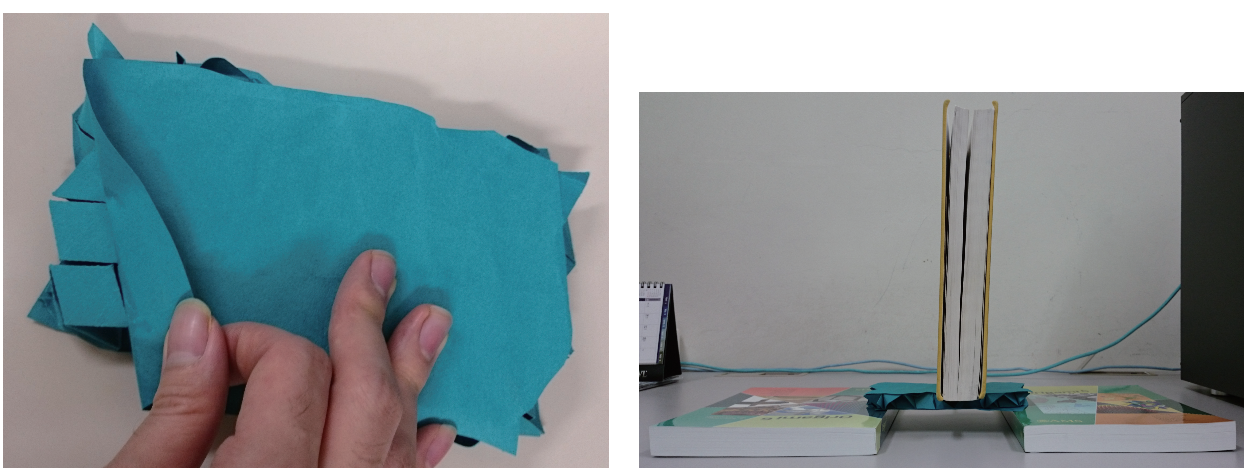

Figure 2: Parallelogram dual tiling origmi with liners. It is folded from sheets of cartridge paper. It is stiff as it can even support a heavy book.

2. Definition and the existence condition

2.1. Concept of the dual tiling origami

Octet Truss has a truss structure that fills up the space with regular octahedra and regular tetrahedra. A single layer of *Octet Truss* is shown in Figure 3, where the octahedra are sliced into their halves, that is, into square pyramids. LANG's work *Octet Truss* folds a paper into this single layer of *Octet Truss*. There are square lattices on the top and bottom surfaces of the layer. The lattices have duality where the apices and bases of the square pyramids are corresponding vertices and faces in the dual graphs. The regular tetrahedra are similarly formed with the corresponding dual edges.

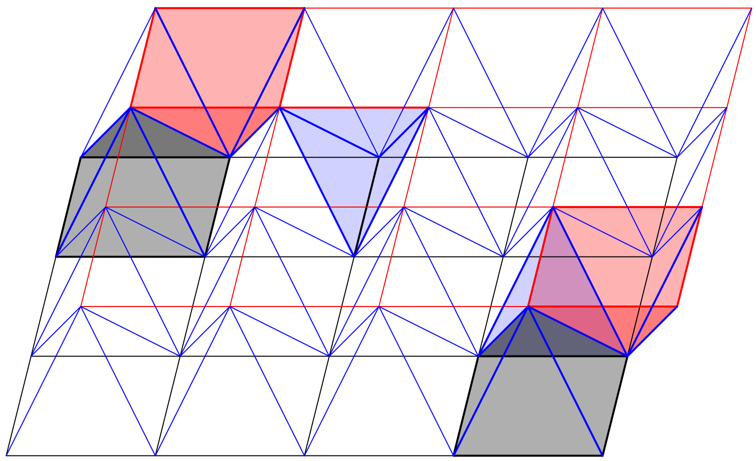

Figure 3: *Octet Truss*

The rest of this section describes how such an *Octet Truss* can be developed into a sheet of paper, that is, the inverse operation of folding the *Octet Truss*. More precisely, we give dual graphs G_1 and G_2 on two planes P_1 and P_2, respectively, and consider a structure to be folded, namely, *dual tiling truss*. This structure consists of pyramids connecting each vertex of G_1 (or G_2), the corresponding face of G_2 (or G_1), tetrahedra connecting each edge on G_1, and its corresponding dual edge in G_2 on the other plane.

Each k-gonal pyramid is developed by base edges with cutting along the lateral edges to form a k-gonal star shape (refer to Figure 4). Here, we have the base point O, which is the orthogonal projection of the apex to the base polygon. Each split apex corresponding to a tip of the star shape exists on the line passing through O and is perpendicular to the base edge of the lateral face. The top and bottom pyramids sharing a lateral edge in the folded state also share an edge in their developed star shapes (Figure 5). There are two ways of connectivity, and the choice results in the chirality of the pattern. As we can always obtain the other connectivity by mirroring the whole shape without the loss of generality, we choose one connectivity.

If the developed pyramids can tessellate a plane while sharing edges and do not overlap, the uncovered regions are quadrangles. We call the star shapes *pyramid regions* and the uncovered regions *tetrahedron regions*. Each tetrahedron region is, with some crease pattern, folded in a tetrahedron between folded pyramids. For example, tetrahedron regions on the crease pattern of *Octet Truss* are squares, which are folded in regular tetrahedra.

If the above construction works and results in a proper folding, we call it the *dual tiling origami* of G_1 and G_2.

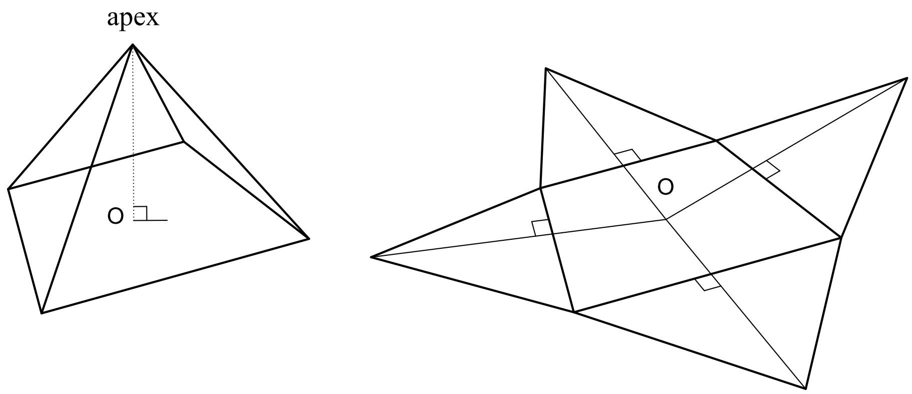

Figure 4: Quadrangular pyramid and its developed star shape.

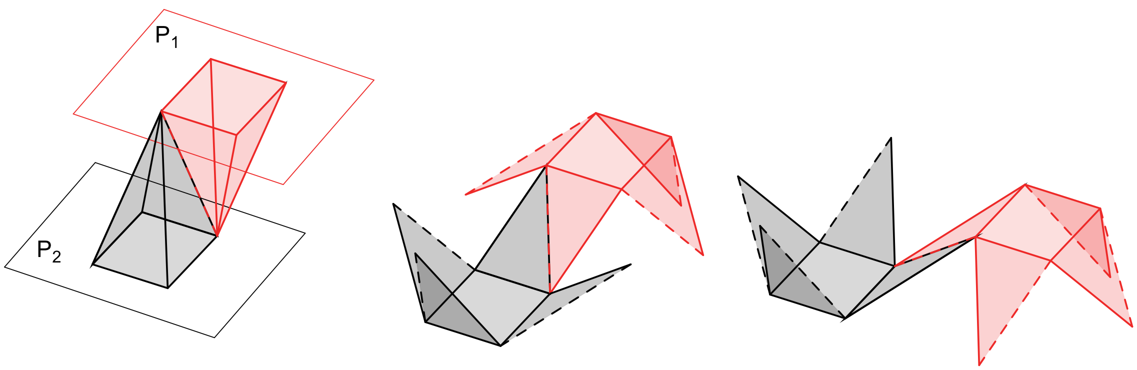

Figure 5: Pyramids sharing an edge

2.2. Condition for the existence

Now, this conversion from a dual tiling truss to a crease pattern does not always work, specifically, when

(1) a dual tiling truss self-intersects (Figure 6);

(2) developed pyramids, that is, star shapes, cannot lie on a plane while keeping the sharing edges connected without overlap (Figure 7); or

(3) tetrahedron regions do not have enough material to compose the desired tetrahedron.

The necessary and sufficient condition for the existence of a dual tiling origami that tiles G_1 and G_2 is the intersection of the following three conditions:

1. Pyramids made from the dual graphs do not overlap each other except at the sharing edges (*non-intersecting condition*).
2. Developed pyramids are connected along the sharing edges on a plane without overlap (*pyramid condition*).
3. Each tetrahedron region can be folded in the corresponding tetrahedron (*tetrahedron condition*).

The non-intersecting condition is satisfied if there exists a parallel projection from G_1 to P_2 by which every vertex of G_1 is projected to the interior of the corresponding dual face of G_2. A necessary condition for the tetrahedron condition is that the diagonal lines of each tetrahedron region are longer than or equal to the corresponding dual edges on G_1 and G_2, forming a tetrahedron [1]. For any quadrangular paper, the distance between the opposite vertices cannot get longer in any folded state, because paper cannot stretch; however, this can

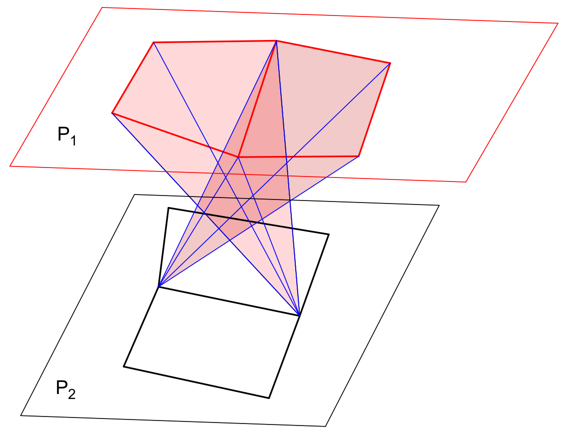

Figure 6: An example of pyramid intersection

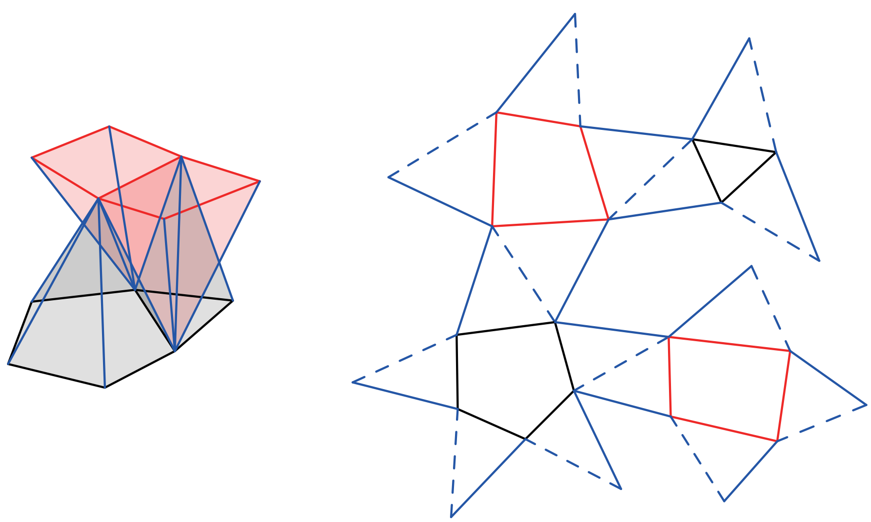

Figure 7: Disconnected pyramid regions

get arbitrarily shorter using a pattern of crimp fold called *Tsubasa-Ori* by Ito and Nara [2] (Figure 8). *Tsubasa-Ori* is used in *Octet Truss* to shorten the distance between the corners to $\frac{1}{\sqrt{2}}$. If the distance between the opposite vertices in the folded state has the same length as the diagonal line, then the tetrahedron region is folded only along that diagonal line (Figure 9).

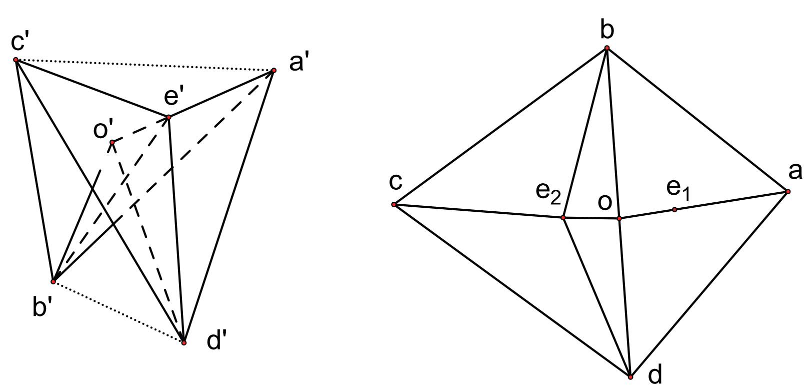

Figure 8: *Tsubasa-Ori*. The developed state (left) and the folded state (right).

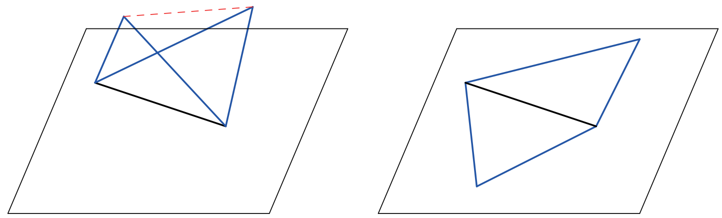

Figure 9: A tetrahedron region of the diagonal line keeping the original length

3. Flat case

We consider a special case wherein planes P_1 and P_2 coincide. The resulting origami is flat folding; that is, the folded image lies on a plane. The existence condition is simplified in this case. We call such a folding *flat dual tiling origami*.

We assume that the apex of each pyramid lies interior to its base. In this case, the base is surrounded by creases that are actually folded. This means that the flat dual tiling origami with the assumption is a special case of flagstone tessellation [5]. Therefore, we name this assumption *flagstone assumption*. Pyramids made from dual graphs do not intersect on the flagstone assumption (non-intersecting condition).

In addition, each tetrahedron degenerates into a flat quadrangle and thus must be congruent to a tetrahedron region (tetrahedron condition). The boundaries of the degenerated tetrahedra, that is, lateral edges of the pyramids, draw a planer graph. We call it the *ridge graph*.

The faces of each category of flat dual tiling origami lie on and tessellate a plane. The bases of the pyramids form the dual graphs. The lateral faces of a pyramid tile its base. Tetrahedron regions form the ridge graph. Therefore, a flat dual tiling origami has five layers at arbitrary points, except for the points on creases.

3.1. Existence condition

Each tetrahedron is congruent to a tetrahedron region, which means that tetrahedron regions are not folded. The lateral edges shared by two pyramids are not folded either. Then, the crease pattern consists of base edges of pyramids and boundaries of tetrahedron regions. All

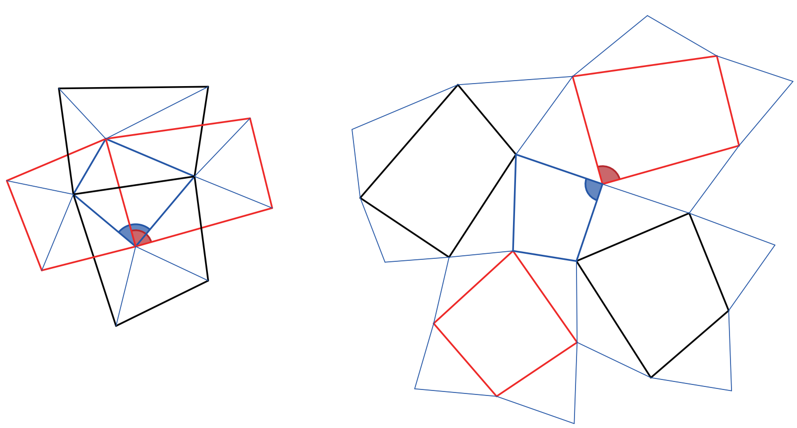

Figure 10: Local views of flat dual tiling origami folded state (left) and in the developed state (right). Actual creases are indicated by solid lines.

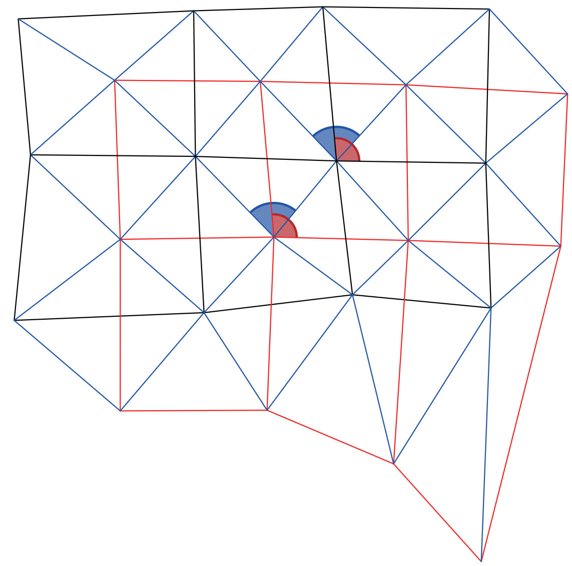

Figure 11: Angle condition. Ridge angles are shown in blue, and dual angles are shown in yellow.

the vertices of the crease pattern are vertices of the bases. The degree of the vertices of the crease pattern is four, as shown in Figure 10.

Kawasaki's theorem represents the local foldability of flat origami [3]. It states that the sum of odd angles around any point is π. This means that the degree of each vertex is even, and the sum of angles around the vertex is 2π. By applying this theorem to the flat dual tiling origami, it is concluded that the sum of the ridge angle specified in blue, that is, the angle of a face of the ridge graph, and the dual angle specified in red, that is, the angle of a face of the dual graphs, in Figure 11 is π. We call it the *angle condition*.

The existence condition consists of the non-intersecting condition, pyramid condition, and tetrahedron condition. As the non-intersecting condition is satisfied under the flagstone condition, the remaining two conditions must be fulfilled for the existence of dual tiling origami. Both conditions are fulfilled at the same time if the pyramid regions and tetrahedron regions that are degenerated tetrahedra in the flat case tessellate a plane; this means that Gaussian curvature at any point is zero in the developed state. This is satisfied if and only if all the vertices of dual graphs satisfy Kawasaki's theorem, namely, the angle condition. Therefore, the flat dual tiling origami can exist if all vertices satisfy the angle condition under the flagstone assumption.

The number of angle conditions for each vertex, n, is the same as that of the faces around the vertex of each graph, that is, one of the dual graphs and ridge graph. The sum of these n pairs of angles, namely, dual angles and ridge angles, is equal to $n\pi$. It is the same as the sum of the angles around the shared vertex of both graphs, which is 4π. Therefore, the number of faces adjacent to the vertex must be four. That is, all the faces of the graphs are quadrangles.

Let us consider the relationship between the ridge graph and dual graphs. The diagonal lines of the ridge graph are the edges of the dual graphs. Both the dual graphs can be reconstructed from the ridge graph. Thus, we will discuss the existence condition of the ridge graph instead of that of the dual graphs. The degree of any vertex of the ridge graph is four, because all the faces of the dual graphs are quadrangles. Each vertex of a ridge graph has two degrees of freedom and has three constraints given as angle conditions. Three angle conditions around a vertex of degree four satisfy the fourth angle condition automatically. When the ridge graph is infinite, its vertices are overconstrained by angle conditions. We will see two different assumptions on regularity in the next subsection.

3.2. Parallelogram case

Firstly, assume that any two adjacent faces of the ridge graph are point symmetrical to the middle point of the shared edge, where all of the faces of the ridge graph are congruent. Secondly, assume that the faces of the ridge graph are parallelograms. On both assumptions, the ridge graph satisfies the angle conditions if and only if the faces of the ridge graph are congruent parallelograms and the diagonal lines are $\sqrt{2}$ times longer than the corresponding edges, as shown in Figure 12. These types of parallelograms have one degree of freedom except for their scaling factor. They are parameterized by the smaller interior angle. The faces of the dual graphs are similar to the parallelograms of the ridge graph, which are $\sqrt{2}$ times larger and reversed. Figure 13 shows an example of the flat dual tiling origami. This type of flat dual tiling origami is an instance of iso-area folding. When both interior angles are the same, the unit parallelogram becomes a square, as shown in Figure 14.

4. Thick case

In this section, we discuss a class of dual tiling origami whose planes are parallel, and the dual graphs on them are congruent parallelogram lattices that construct right pyramids. We name this class of dual tiling origami *parallelogram dual tiling origami*. Figure 15 illustrates a sample of the parallelogram dual tiling origami with some thickness.

4.1. Parallelogram dual tiling origami

The pyramids are right, so the projection of each apex on its base by the perpendicular line coincides with the centroid of the base. Therefore, dual tiling origami satisfies the non-intersecting condition.

The base of the right pyramid is a parallelogram. The developed pyramids are congruent to each other and have rotational symmetry of order two. This means that all the pyramid

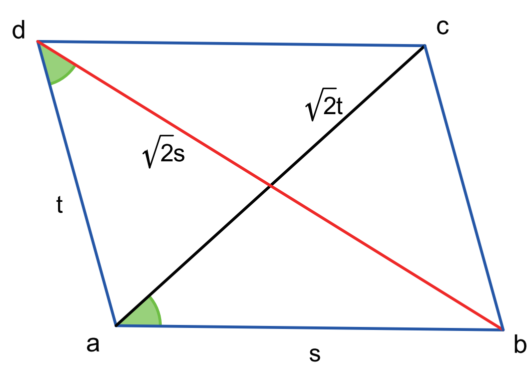

Figure 12: A face of the ridge graph of the parallelogram flat dual tiling origami satisfying the angle condition. In this case, the angles indicated in green are the same, which is equivalent to the angle condition.

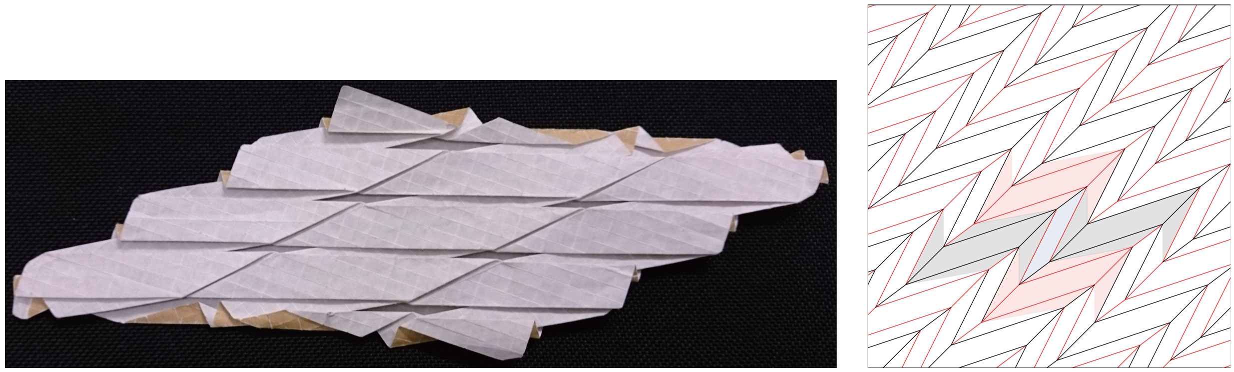

Figure 13: A sample of the flat dual tiling origami

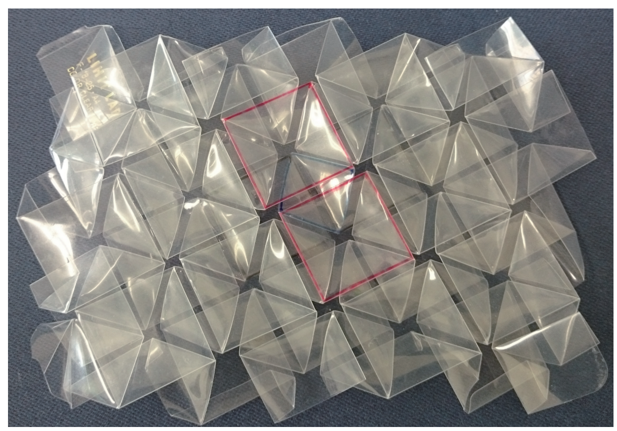

Figure 14: The square lattice in a flat dual tiling origami. This is known as Ron Resch's *square pattern with twisted flaps*.

regions on a developed plane are made with parallel translation, as specified with green and orange arrows in Figure 16. Therefore, the pyramid condition is that the development of the

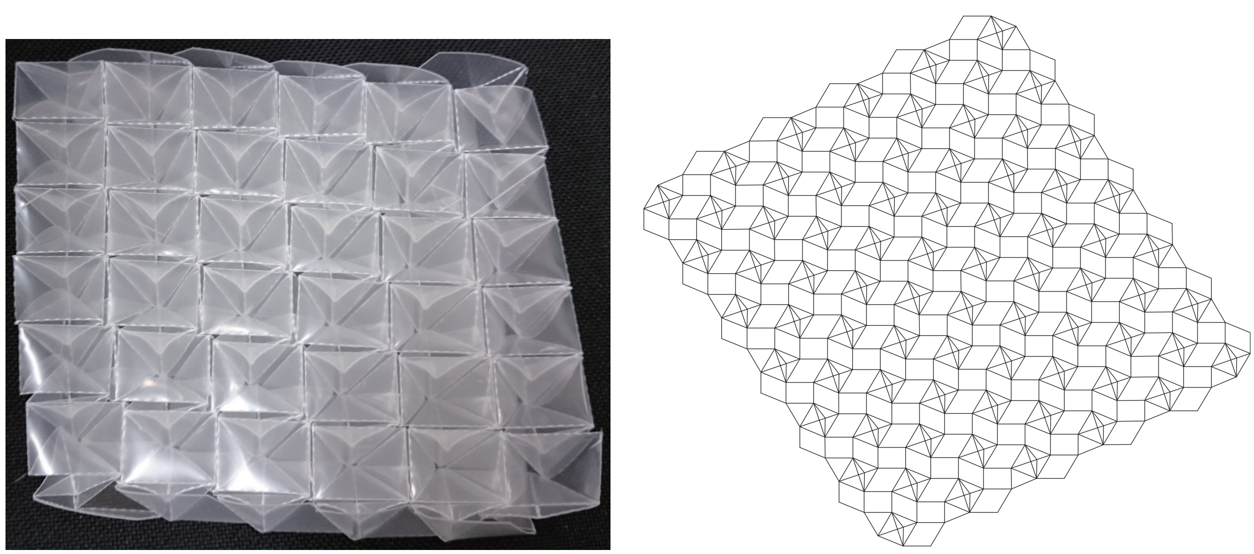

Figure 15: A sample of the parallelogram dual tiling origami

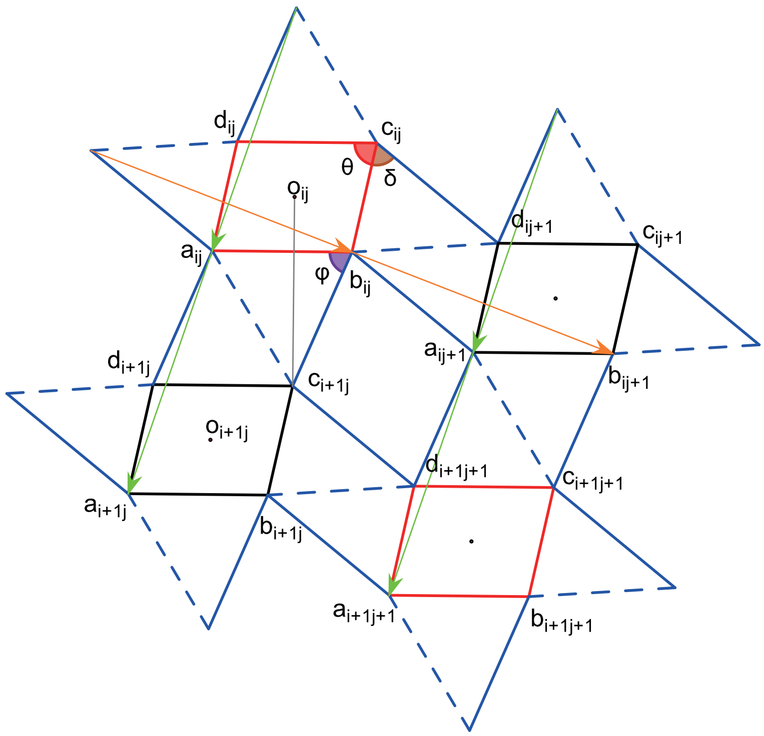

Figure 16: Pyramid regions of a parallelogram dual tiling origami

pyramids does not intersect, which means that the tetrahedron region actually exists. The pyramid condition turns out to be the condition that every interior angle of the tetrahedron region is larger than zero. It is given by

$$\varphi < \theta + \delta < \varphi + \pi. \tag{1}$$

Here, the symbols are shown in Figure 16. The tetrahedron condition is given by

$$\|b_{i,j}d_{i+1,j+1}\| \geq \|b_{i,j}a_{i,j}\|, \tag{2}$$

$$\|c_{i+1,j}a_{i,j+1}\| \geq \|c_{i+1,j}b_{i+1,j}\|. \tag{3}$$

The left-hand sides of the inequalities are the lengths of the diagonal lines of the tetrahedron region. The right-hand sides of the inequalities are the lengths of the base edges, where the second-term vertices of both hand sides of the inequalities coincide with each other in the folded state. *Tsubasa-Ori* folds the parallelogram tetrahedron region satisfying the inequalities into the tetrahedron. Let k and l be the edge lengths of parallelogram, $\|b_{i,j}a_{i,j}\|$ and $\|b_{i,j}c_{i,j}\|$. Let h be the thickness of the parallelogram dual tiling origami in the folded state. The length of the diagonal line $\|b_{i,j}d_{i+1,j+1}\|$ is given by

$$\|b_{i,j}d_{i+1,j+1}\| = f(k,l,\theta,h), \tag{4}$$

where $f(k,l,\theta,h)$ satisfies

$$(f(k,l,\theta,h))^2 = \left(-k + \left(\frac{k}{2}\sin\theta + \sqrt{h^2 + \left(\frac{k}{2}\sin\theta\right)^2}\right)\sin\theta\right)^2$$
$$+ \left(\frac{1}{2}m\sin\theta + \sqrt{h^2 + \left(\frac{1}{2}\sin\theta\right)^2} + \left(\frac{k}{2}\sin\theta + \sqrt{h^2 + \left(\frac{k}{2}\sin\theta\right)^2}\right)\cos\theta\right)^2. \tag{5}$$

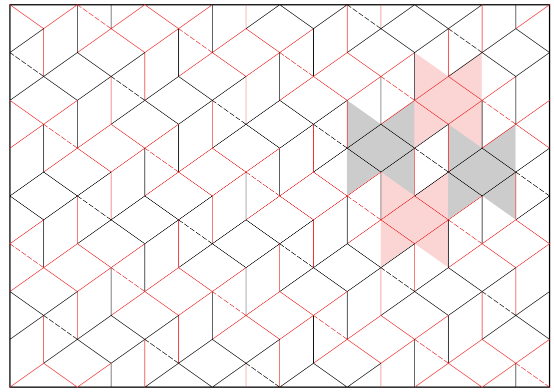

Figure 17: A crease pattern of the diagonal-fold parallelogram dual tiling origami with rhombus lattice whose diagonal lines have a ratio of $1 : \sqrt{2}$.

The length of the other diagonal line $\|c_{i+1,j}a_{i,j+1}\|$ is given by

$$\|c_{i+1,j}a_{i,j+1}\| = f(l, k, \pi - \theta, h). \tag{6}$$

By substituting all the lengths in equalities (2) and (3), the following conditions are obtained:

$$f(k, 1, \theta, h) \geq k, \tag{7}$$

$$f(1, k, \pi - \theta, h) \geq 1. \tag{8}$$

When h approaches infinity, both φ and δ approach $\frac{\pi}{2}$, and f diverges to infinity. This indicates that the existence condition is satisfied if the folded structure is thick enough. In the previous section, the special case is discussed wherein even zero thickness, that is, $h \geq 0$, satisfies the existence condition when the diagonal lines of the base parallelograms are $\sqrt{2}$ times longer than the corresponding edges.

In particular, when either inequality (2) or (3) is satisfied as an equation, tetrahedron regions are folded along only the diagonal line corresponding to the left-hand side of the equation. We name this simple type of origami *diagonal-fold parallelogram dual tiling origami*. The diagonal-fold parallelogram dual tiling origami is an instance of iso-area folding. It is a generarized pattern of the flat dual tiling origami discussed in Section 3.2. Furthermore, the crease pattern becomes simple if the unit parallelogram of the lattices is a rhombus and the ratio of its diagonal lines is $1 : \sqrt{2}$. All the vertices of the crease pattern are on the lattice points of a rhombus lattice whose unit rhombus is congruent to the unit of the dual graphs (Figure 17).

4.2. Convergence to *Miura-Ori*

Let us consider a variation of the diagonal-fold parallelogram dual tiling origami whose interior angle θ and length of the diagonal fold line, namely, the corresponding edge of the parallelogram, are constant. When the length of the other edge of the unit parallelogram is approaching zero, the dual tiling origami converges to a certain type of *Miura-Ori* [6]. *Miura-Ori* has a crease pattern with congruent parallelograms, as shown in Figure 18. This crease pattern can be seen as a combination of straight lines in one direction, that is, vertical

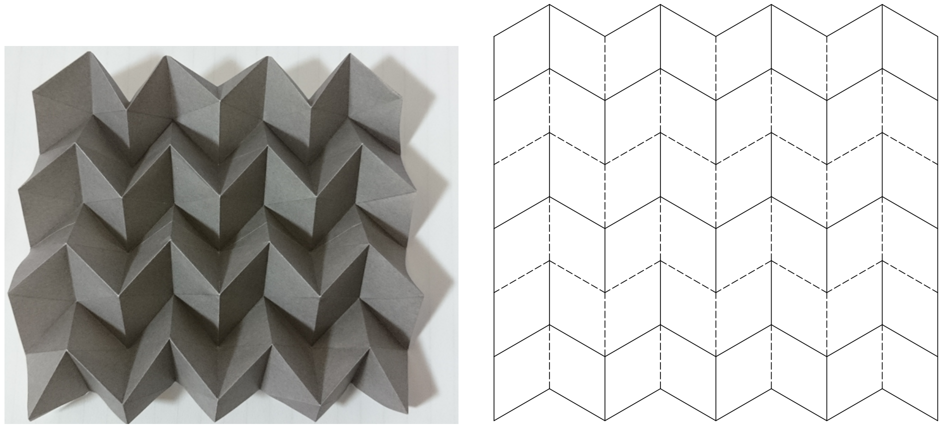

Figure 18: Folded state (left) and crease pattern (right) of *Miura-Ori*

direction in Figure 18, and zigzag lines in the other direction, that is, horizontal direction in Figure 18. The unit parallelogram of *Miura-Ori* as a limit of the diagonal-fold parallelogram dual tiling origami has a nature so that two vertices have perpendicular feet to the edges of the zigzag lines at their mid points. The folded and opened states of a diagonal-fold parallelogram dual tiling origami almost converged to *Miura-Ori* are shown in Figs. 19 and 20, respectively.

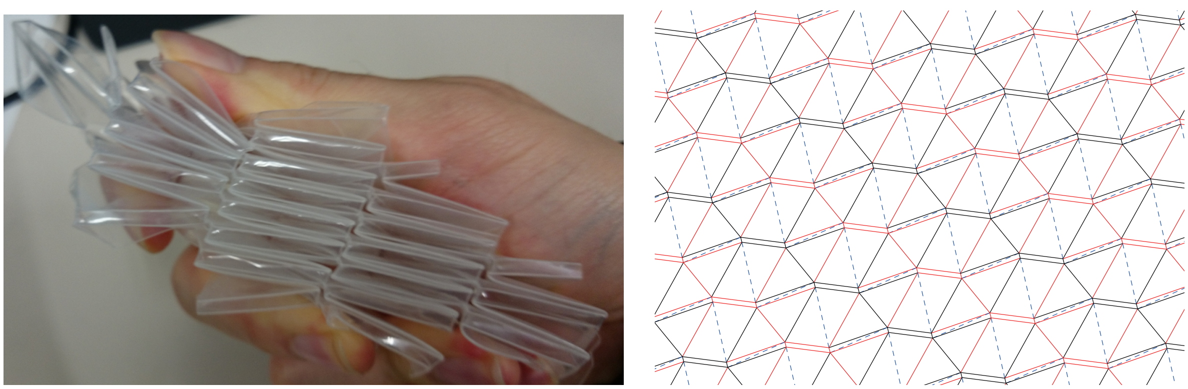

Figure 19: Diagonal-fold parallelogram dual tiling origami almost converged to *Miura-Ori*

Figure 20: The opened state of Figure 19

5. Conclusion and discussion

We discussed dual tiling origami, which is an extension of LANG's work *Octet Truss*. First, we specified the necessary and sufficient condition for the existence of dual tiling origami. Then, we considered this condition in two cases, flat dual tiling origami and parallelogram dual tiling origami. The following results are achieved:

1. The necessary and sufficient condition for the existence of flat dual tiling origami under the flagstone assumption is the angle condition that the sum of the ridge and dual angles shown in Figure 8 is π.
2. A flat dual tiling origami exists if its dual graphs or ridge graph indicate(s) a parallelogram lattice where the diagonal lines of the unit face are $\sqrt{2}$ times longer than the corresponding edges of the face.
3. For any parallelogram, a parallelogram dual tiling origami exists if its thickness is sufficiently large.
4. In case of parallelogram dual tiling origami, a tetrahedron region can be folded only with one of its diagonals under a certain condition, which is called diagonal-fold parallelogram dual tiling origami.
5. The limit of the diagonal-fold parallelogram dual tiling origami can be *Miura-Ori* as the shorter edge of the unit parallelogram approaches zero length.

Acknowledgements

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