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The Mean Width and Integral Geometric Properties of the Oloid

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Abstract. The oloid is the convex hull of two circles with equal radius in perpendicular planes so that the center of each circle lies on the other circle. We calculate the mean width of the oloid in two ways, first via the integral of mean curvature, and then directly. Using this result, the surface area and the volume of the parallel body are obtained. Furthermore, we derive the expectations of the mean width, the surface area and the volume of the intersections of a fixed oloid and a moving ball, as well as of a fixed and a moving oloid.

Key Words: Oloid, convex hull, integral of mean curvature, mean width, Steiner formula, parallel body, intrinsic volumes, principal kinematic formula

MSC: 53A05, 52A22, 52A15, 60D05

1. Introduction

The oloid Ω_r is the convex hull of two circles k_A, k_B with equal radius r in perpendicular planes so that the center of each circle lies on the other circle (see Figures 1 and 2). Dirnböck abd Stachel [4, p. 117] calculated the surface area and the volume of the oloid (see also [14], [15], and Equations (5), (6), (7), and (8) of the present paper). The surface $\partial\Omega_r$ is part of a developable surface [4], [2], [14], [15].

Finch [5] calculated surface areas, volumes and mean widths of the convex hulls of three different configurations of two orthogonal disks with equal radius. The mean width $\bar{b}$ of every convex hull is determined twice:
1) using the integral M of the mean curvature and the relation $\bar{b} = M/(2\pi)$,
2) calculating $\bar{b}$ directly.

According to [4, pp. 105–106], the circles with $r = 1$ can be defined by

$$\begin{aligned} k_A &:= \{(x,y,z) \in \mathbb{R}^3 \colon x^2 + (y+1/2)^2 = 1 \wedge z = 0\}, \\ k_B &:= \{(x,y,z) \in \mathbb{R}^3 \colon (y-1/2)^2 + z^2 = 1 \wedge x = 0\} \end{aligned} \tag{1}$$

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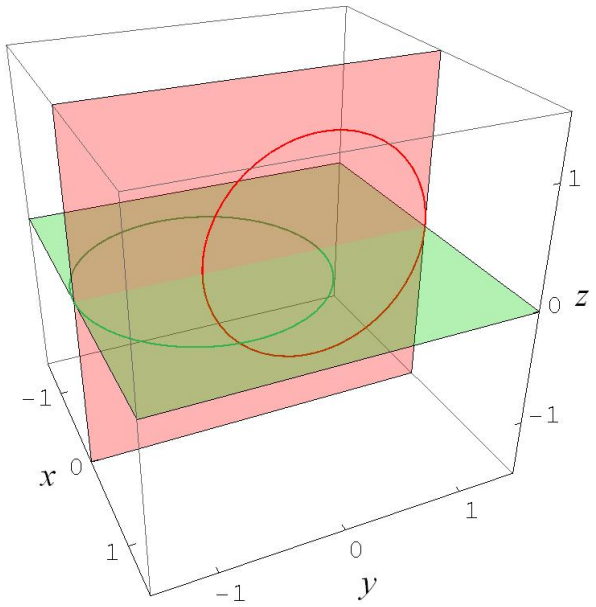

Figure 1: The circles k_A and k_B with $r = 1$

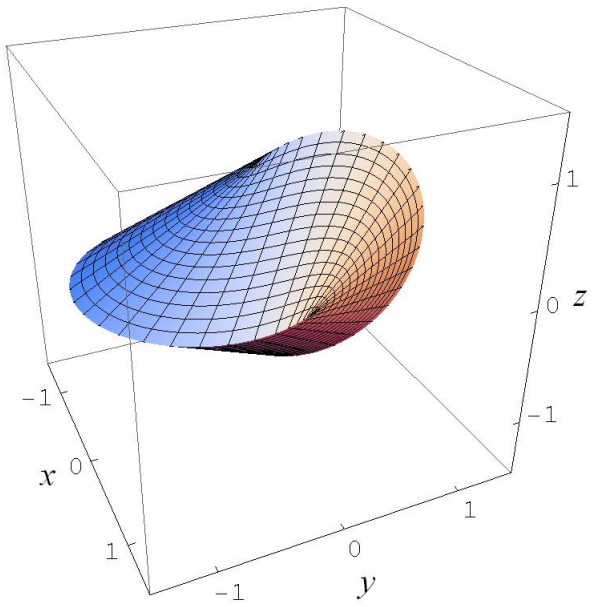

Figure 2: The oloid Ω_1

(see Figure 1). A parametrization of the surface $\partial\Omega_1$ [2, p. 165, Eq. (2)] is

$$(x, y, z)^\top = \vec{\omega}(m, t) = \big(\omega_1(m, t),\, \omega_2(m, t),\, \pm\omega_3(m, t)\big)^\top ,\quad 0 \le m \le 1 ,\quad -\frac{2\pi}{3} \le t \le \frac{2\pi}{3} , \tag{2}$$

with

$$\left.\begin{aligned} \omega_1(m, t) &= (1 - m)\sin t\,, \\ \omega_2(m, t) &= \frac{2(m-1)\cos^2 t + (2m-3)\cos t + 2m - 1}{2(1 + \cos t)}\,, \\ \omega_3(m, t) &= \frac{m\sqrt{1 + 2\cos t}}{1 + \cos t}\,. \end{aligned}\right\} \tag{3}$$

To the author's knowledge, the mean width of the oloid is not yet known. In Section 3 we calculate the mean width of Ω_r using the integral of mean curvature, and in Section 4 we calculate it directly. With the help of this result we derive the volume, the surface area and the mean width of the parallel body of Ω_r in Section 5. Using the principal kinematic formula of integral geometry, the expectations of the mean width, the surface area and the volume of the intersections of a fixed oloid and a moving ball, as well as of a fixed and a moving oloid are calculated in Section 6.

2. Preliminaries

In the following, we work in the real vector space $\mathbb{R}^3$ with its standard scalar product $\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot \vec{b}$ and its vector product $\vec{a} \times \vec{b}$ for two vectors $\vec{a} = (a_1, a_2, a_3)^\top$, $\vec{b} = (b_1, b_2, b_3)^\top$. We denote the partial derivatives $\dfrac{\partial\vec{\omega}}{\partial m}$ and $\dfrac{\partial\vec{\omega}}{\partial t}$ of $\vec{\omega} = \vec{\omega}(m, t)$ (see (2)) by $\vec{\omega}_m$ and $\vec{\omega}_t$, and so on.

Using (3), we find for the coefficients $g_{11} = E$, $g_{12} = F = g_{21}$, $g_{22} = G$ of the first

fundamental form (see e. g. [6, pp. 87–88], translation: p. 68)

$$\left.\begin{aligned} g_{11} &= \langle \vec{\omega}_m, \vec{\omega}_m \rangle = 3\,, \qquad g_{12} = \langle \vec{\omega}_m, \vec{\omega}_t \rangle = \tan(t/2)\,, \\ g_{22} &= \langle \vec{\omega}_t, \vec{\omega}_t \rangle = \frac{2(3m^2-4m+1)\cos^2 t - (4m-3)\cos t + 1}{(1+\cos t)(1+2\cos t)}\,, \\ g &= \det(g_{jk}) = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11}\,g_{22} - g_{12}^2 = \frac{2[(3m-2)\cos t - 1]^2}{(1+\cos t)(1+2\cos t)}\,. \end{aligned}\right\} \tag{4}$$

Now, we able to calculate the surface area of the oloid Ω_1:

$$\begin{aligned} S(\Omega_1) &= \int_{\partial\Omega_1} \mathrm{d}S = \int_{\partial\Omega_1} \mathrm{d}S(m,t) = 2\int_{t=-2\pi/3}^{2\pi/3}\int_{m=0}^{1} \sqrt{g(m,t)}\,\mathrm{d}m\,\mathrm{d}t \\ &= 4\int_{t=0}^{2\pi/3}\int_{m=0}^{1} \sqrt{g(m,t)}\,\mathrm{d}m\,\mathrm{d}t = 4\int_{t=0}^{2\pi/3}\int_{m=0}^{1} \frac{\sqrt{2}\,[(2-3m)\cos t + 1]}{\sqrt{(1+\cos t)(1+2\cos t)}}\,\mathrm{d}m\,\mathrm{d}t \\ &= 2\sqrt{2}\int_{0}^{2\pi/3} \frac{2+\cos t}{\sqrt{(1+\cos t)(1+2\cos t)}}\,\mathrm{d}t\,. \end{aligned} \tag{5}$$

Mathematica evaluates this integral to

$$S(\Omega_1) = 2\sqrt{2}\cdot\sqrt{2}\,\pi = 4\pi\,. \tag{6}$$

Now, we calculate the volume of Ω_1, and start with

$$\begin{aligned} V(\Omega_1) &= 2\iint z\,\mathrm{d}x\,\mathrm{d}y = 2\int_{t=-2\pi/3}^{2\pi/3}\int_{m=0}^{1} \omega_3(m,t)\left|\frac{\partial(\omega_1(m,t),\,\omega_2(m,t))}{\partial(m,t)}\right|\mathrm{d}m\,\mathrm{d}t \\ &= 4\int_{t=0}^{2\pi/3}\int_{m=0}^{1} \omega_3(m,t)\left|\frac{\partial(\omega_1(m,t),\,\omega_2(m,t))}{\partial(m,t)}\right|\mathrm{d}m\,\mathrm{d}t\,. \end{aligned}$$

From (3) it follows that

$$\frac{\partial\,(\omega_1(m,t),\,\omega_2(m,t))}{\partial(m,t)} = \begin{vmatrix} \dfrac{\partial\omega_1(m,t)}{\partial m} & \dfrac{\partial\omega_1(m,t)}{\partial t} \\ \dfrac{\partial\omega_2(m,t)}{\partial m} & \dfrac{\partial\omega_2(m,t)}{\partial t} \end{vmatrix} = -\frac{1+(2-3m)\cos t}{1+\cos t}\,.$$

So we have

$$\begin{aligned} V(\Omega_1) &= 4\int_{t=0}^{2\pi/3}\int_{m=0}^{1} \frac{m\sqrt{1+2\cos t}}{1+\cos t}\,\frac{1+(2-3m)\cos t}{1+\cos t}\,\mathrm{d}m\,\mathrm{d}t \\ &= 2\int_{0}^{2\pi/3} \frac{\sqrt{1+2\cos t}}{(1+\cos t)^2}\,\mathrm{d}t\,. \end{aligned} \tag{7}$$

Mathematica finds

$$V(\Omega_1) = \frac{2}{3}\left[K\big(\sqrt{3}/2\big) + 2E\big(\sqrt{3}/2\big)\right], \tag{8}$$

where

$$K(k) = F(\pi/2, k) = \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1-k^2\sin^2 x}} \tag{9}$$

is the complete elliptic integral of the first kind, and

$$E(k) = E(\pi/2, k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 x}\, \mathrm{d}x \tag{10}$$

is the complete elliptic integral of the second kind. A numerical integration of (7) and evaluation of (8) with *Mathematica* yields the decimal expansion

$$V(\Omega_1) \approx 3.052418468424374856697200531 93$$

(see also [1]).

3. The integral of mean curvature

The surface $\partial\Omega_1$ of the oloid Ω_1 is piecewise continuously differentiable. We denote by H the mean curvature in one point of $\partial\Omega_1$. The circles k_A and k_B (see (1)) produce two edges ε_1 and ε_2, respectively, which are smooth curves. Let $\alpha = \alpha(t)$ denote the angle between the two unit normal vectors at each point of ε_1. Applying the general formula for the integral M of the mean curvature (see [12, pp. 76–77, Eqs. (3.5), (3.7)]; cp. the formula for the mean width in [5, p. 3]) to Ω_1 gives

$$M(\Omega_1) = \int_{\partial\Omega_1} H\, \mathrm{d}S + \frac{1}{2} \sum_{j=1}^{2} \int_{\varepsilon_j} \alpha\, \mathrm{d}s = \int_{\partial\Omega_1} H(m,t)\, \mathrm{d}S(m,t) + \int_{\varepsilon_1} \alpha(t)\, \mathrm{d}t \tag{11}$$

For the unit normal vector one finds

$$\vec{n} = (n_1, n_2, n_3)^\top = \frac{\vec{\omega}_m \times \vec{\omega}_t}{|\vec{\omega}_m \times \vec{\omega}_t|} = \left(\sin(t/2)\,,\; -\frac{\cos t}{2\cos(t/2)}\,,\; \frac{\sqrt{1+2\cos t}}{2\cos(t/2)} \right)^\top . \tag{12}$$

Since $\partial\Omega_1$ is part of a developable surface and the line segments $t = \text{const}$, $0 \le m \le 1$ are part of the generators of $\partial\Omega_1$ (see [4], [2]), it is not surprising that $\vec{n}$ does not depend on m. The mean curvature in a point of a surface is defined by

$$H = \frac{1}{2}\,(\kappa_1 + \kappa_2) = \frac{1}{2g}\,(g_{11} b_{22} - 2 g_{12} b_{12} + g_{22} b_{11})\,,$$

where κ_1, κ_2 are the principal curvatures, b_{ik} are the coefficients of the second fundamental form (see e. g. [6, p. 99], translation: p. 79), and g_{ik}, g are given by (4). In our case we have

$$H(m,t)\, \mathrm{d}S(m,t) = H(m,t)\, \sqrt{g(m,t)}\, \mathrm{d}m\, \mathrm{d}t = \frac{1}{2\sqrt{g}}\,(g_{11} b_{22} - 2 g_{12} b_{12} + g_{22} b_{11})\, \mathrm{d}m\, \mathrm{d}t$$

and

$$\begin{aligned} b_{11} &= L = \langle \vec{\omega}_{mm}, \vec{n} \rangle = 0\,, \qquad b_{12} = M = \langle \vec{\omega}_{mt}, \vec{n} \rangle = 0\,, \\ b_{22} &= N = \langle \vec{\omega}_{tt}, \vec{n} \rangle = \frac{(3m-2)\cos t - 1}{\sqrt{2}\,(1+2\cos t)\,\sqrt{1+\cos t}}\,. \end{aligned}$$

It follows that

$$\begin{aligned} H(m,t)\, \mathrm{d}S(m,t) &= \frac{g_{11}(m,t) b_{22}(m,t)}{2\sqrt{g(m,t)}}\, \mathrm{d}m\, \mathrm{d}t \\ &= \frac{3}{2} \cdot \frac{(3m-2)\cos t - 1}{\sqrt{2}\,(1+2\cos t)\,\sqrt{1+\cos t}} \cdot \frac{\sqrt{1+\cos t}\,\sqrt{1+2\cos t}}{\sqrt{2}\,[(3m-2)\cos t - 1]}\, \mathrm{d}m\, \mathrm{d}t \\ &= \frac{3}{4\sqrt{1+2\cos t}}\, \mathrm{d}m\, \mathrm{d}t \end{aligned}$$

and

$$\int_{\partial\Omega_1} H\,\mathrm{d}S = 2\int_{m=0}^{1}\int_{t=-2\pi/3}^{2\pi/3} H(m,t)\,\mathrm{d}S(m,t) = \frac{3}{2}\int_0^1 \mathrm{d}m \int_{-2\pi/3}^{2\pi/3} \frac{1}{\sqrt{1+2\cos t}}\,\mathrm{d}t$$
$$= \frac{3}{2}\int_{-2\pi/3}^{2\pi/3} \frac{1}{\sqrt{1+2\cos t}}\,\mathrm{d}t = 3\int_0^{2\pi/3} \frac{1}{\sqrt{1+2\cos t}}\,\mathrm{d}t\,.$$

Mathematica finds

$$\int_0^{2\pi/3} \frac{\mathrm{d}t}{\sqrt{1+2\cos t}}\,\mathrm{d}t = K\big(\sqrt{3}/2\big)\,, \tag{13}$$

where K is the complete elliptic integral of the first kind (9), hence

$$\int_{\partial\Omega_1} H\,\mathrm{d}S = 3K\big(\sqrt{3}/2\big)\,. \tag{14}$$

A handmade proof for the relation in (13) may be found in Section 7.

Now we calculate the integral of mean curvature for the edges ε_1, ε_2 (see (11)). Therefore, we consider ε_1. The first unit normal vector $\vec{n} = \vec{n}(t)$ at a point $t \in [-2\pi/3, 2\pi/3]$, $m = 0$ is given by (12), the second is $\vec{n}^* = \vec{n}^*(t) = (n_1, n_2, -n_3)^\top$. So one gets

$$\alpha(t) = \arccos\langle\vec{n}(t), \vec{n}^*(t)\rangle = \arccos\left(-\frac{\cos t}{1+\cos t}\right),$$

hence

$$\int_{\varepsilon_1} \alpha\,\mathrm{d}t = \int_{-2\pi/3}^{2\pi/3} \alpha(t)\,\mathrm{d}t = 2\int_0^{2\pi/3} \alpha(t)\,\mathrm{d}t = 2\int_0^{2\pi/3} \arccos\left(-\frac{\cos t}{1+\cos t}\right)\mathrm{d}t\,.$$

We observe that the inverse function of the integrand is equal to the integrand, and hence the graph of the integrand symmetrical with respect to the line $f(t) = t$. As solution of

$$f(t) = t = \arccos\left(-\frac{\cos t}{1+\cos t}\right)$$

we find $t = \pi/2$, hence

$$\int_{\varepsilon_1} \alpha\,\mathrm{d}t = 4\int_0^{\pi/2}\left[\arccos\left(-\frac{\cos t}{1+\cos t}\right) - t\right]\mathrm{d}t = 4\int_0^{\pi/2} \arccos\left(-\frac{\cos t}{1+\cos t}\right)\mathrm{d}t - 4\int_0^{\pi/2} t\,\mathrm{d}t$$
$$= 4\int_0^{\pi/2} \arccos\left(-\frac{\cos t}{1+\cos t}\right)\mathrm{d}t - \frac{\pi^2}{2} = 4\int_0^{\pi/2}\left[\pi - \arccos\frac{\cos t}{1+\cos t}\right]\mathrm{d}t - \frac{\pi^2}{2}$$
$$= 4\pi\int_0^{\pi/2}\mathrm{d}t - 4\int_0^{\pi/2} \arccos\frac{\cos t}{1+\cos t}\,\mathrm{d}t - \frac{\pi^2}{2}$$
$$= \frac{3\pi^2}{2} - 4\int_0^{\pi/2} \arccos\frac{\cos t}{1+\cos t}\,\mathrm{d}t\,. \tag{15}$$

Mathematica and we, too, are not able to solve the last integral. It looks similar to Coxeter's integral in [7, pp. 194–201]. The `NIntegrate`-function of *Mathematica* provides

$$I := \int_0^{\pi/2} \arccos\frac{\cos t}{1+\cos t}\,\mathrm{d}t \approx 1.87738105428247449505835371657\,, \tag{16}$$

hence

$$\int_{\varepsilon_1} \alpha \, \mathrm{d}t \approx 7.294882384504139948018321633 53 \, .$$

From (11), (14), (15), with (9) and (16), it follows that the integral M of the mean curvature of Ω_r is given by

$$M(\Omega_r) = \left(3K\big(\sqrt{3}/2\big) + \frac{3\pi^2}{2} - 4I \right) r \approx 13.764429327003069654334346629 9 \, r \, .$$

For a convex body K, the mean width $\bar{b}$ is given by the relation $\bar{b}(K) = M(K)/2\pi$ (see [12, p. 78, Eq. (3.9)]). So we proved the following theorem.

Theorem 1. *The mean width of the oloid Ω_r is*

$$\bar{b}(\Omega_r) = \left(\frac{3}{2\pi} K\big(\sqrt{3}/2\big) + \frac{3\pi}{4} - \frac{2}{\pi} I \right) r \approx 2.190676966231588766332630494 36 \, r,$$

where K is the complete elliptic integral of the first kind (9)*, and*

$$I = \int_0^{\pi/2} \arccos \frac{\cos x}{1 + \cos x} \, \mathrm{d}x \, .$$

4. Direct calculation of the mean width

Let

$$P = \{(x, y, z) \in \mathbb{R}^3 \colon \ ax + by + cz = d\}$$

be a supporting plane of Ω_1, given in the Hesse normal form. So $\vec{N} = (a, b, c)^\top$ with $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$ is the normal unit vector of P and $|d|$ is the distance of P from the origin. P intersects the plane $z = 0$ in the line

$$L_{xy} = \{(x, y) \in \mathbb{R}^2 \colon \ ax + by = d\} \, ,$$

and the plane $x = 0$ in the line

$$L_{yz} = \{(y, z) \in \mathbb{R}^2 \colon \ by + cz = d\} \, .$$

The equation of L_{xy} in Hesse normal form is

$$\frac{ax}{\sqrt{a^2 + b^2}} + \frac{by}{\sqrt{a^2 + b^2}} = \frac{d}{\sqrt{a^2 + b^2}} \, ,$$

therefore, the distance d_1 of L_{xy} from the center $(0, -1/2, 0)$ of k_A is

$$d_1 = \left| \frac{a}{\sqrt{a^2 + b^2}} \cdot 0 + \frac{b}{\sqrt{a^2 + b^2}} \cdot \left(-\frac{1}{2} \right) - \frac{d}{\sqrt{a^2 + b^2}} \right| = \frac{b/2 + d}{\sqrt{a^2 + b^2}}$$

(see e. g. [3, p. 172]). Since L_{xy} is tangent to k_A, we have

$$\frac{b/2 + d}{\sqrt{a^2 + b^2}} = 1 \quad \Longrightarrow \quad d = \sqrt{a^2 + b^2} - \frac{b}{2} \, .$$

Analogously one finds that the distance d_2 of L_{yz} from the center $(0, 1/2, 0)$ of k_B is

$$d_2 = \frac{b/2 - d}{\sqrt{a^2 + b^2}} = 1\,,$$

hence

$$d = \sqrt{a^2 + b^2} + \frac{b}{2}\,.$$

It follows that the distance p between the support plane P and the origin is

$$p = \max\left\{\sqrt{a^2 + b^2} - \frac{b}{2},\ \sqrt{a^2 + b^2} + \frac{b}{2}\right\}.$$

Now we use spherical coordinates $0 \le \varphi \le \pi/2$ and $0 \le \vartheta \le \pi/2$ as coordinates of the unit normal vector $\vec{N}$:

$$a = \cos\varphi \sin\vartheta\,, \quad b = \sin\varphi\sin\vartheta\,, \quad c = \cos\vartheta\,.$$

So we have

$$\begin{aligned}
\sqrt{a^2 + b^2} - \frac{b}{2} &= \left(1 - \frac{1}{2}\,\sin\varphi\right)\sin\vartheta\,,\\
\sqrt{a^2 + b^2} + \frac{b}{2} &= \frac{1}{2}\,\sin\varphi\sin\vartheta + \sqrt{\sin^2\varphi\sin^2\vartheta + \cos^2\vartheta}\,,
\end{aligned}$$

and can write p as

$$p(\varphi,\vartheta) = \max\left\{\left(1 - \frac{1}{2}\sin\varphi\right)\sin\vartheta,\ \frac{1}{2}\sin\varphi\sin\vartheta + \sqrt{\sin^2\varphi\sin^2\vartheta + \cos^2\vartheta}\right\}.$$

Clearly, $p = p(\varphi, \vartheta)$ is the support function of Ω_1 in the direction (φ, ϑ). Hence, the width b of Ω_r in this direction is given by

$$b(\varphi,\vartheta) = p(\varphi,\vartheta) + p(\pi + \varphi, \pi - \vartheta)\,.$$

In order to calculate the mean width $\bar{b}$ of Ω_1 we have to integrate over all directions, hence over the unit hemisphere. Let $\mathrm{d}S = \mathrm{d}S(\varphi,\vartheta) = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$ denote the surface element of the unit sphere, we have

$$\begin{aligned}
\bar{b}(\Omega_1) &= \frac{\int_0^\pi\int_0^\pi b(\varphi,\vartheta)\,\mathrm{d}S(\varphi,\vartheta)}{\int_0^\pi\int_0^\pi \mathrm{d}S(\varphi,\vartheta)} = \frac{\int_0^\pi\int_0^\pi [p(\varphi,\vartheta) + p(\pi+\varphi,\pi-\vartheta)]\,\mathrm{d}S(\varphi,\vartheta)}{\int_0^\pi\int_0^\pi \mathrm{d}S(\varphi,\vartheta)}\\
&= \frac{\int_0^\pi\int_0^\pi p(\varphi,\vartheta)\,\mathrm{d}S(\varphi,\vartheta) + \int_0^\pi\int_0^\pi p(\pi+\varphi,\pi-\vartheta)\,\mathrm{d}S(\varphi,\vartheta)}{\int_0^\pi\int_0^\pi \mathrm{d}S(\varphi,\vartheta)} = \frac{2\int_0^\pi\int_0^\pi p(\varphi,\vartheta)\,\mathrm{d}S(\varphi,\vartheta)}{\int_0^\pi\int_0^\pi \mathrm{d}S(\varphi,\vartheta)}
\end{aligned}$$

(cp. [12, p. 78, Eq. (3.9)]), where the last equal sign follows from the fact that there are two congruent portions of Ω_1 in the half spaces $y \le 0$ and $y \ge 0$. Due to the symmetry of Ω_1 with respect to the planes $z = 0$ and $x = 0$ we can restrict the spherical coordiates to the intervals

$0 \le \vartheta \le \pi/2$ and $0 \le \varphi \le \pi/2$, respectively, hence

$$\begin{aligned}
\bar{b}(\Omega_1) &= \frac{2\int_0^{\pi/2}\int_0^{\pi/2} p(\varphi,\vartheta)\,\mathrm{d}S(\varphi,\vartheta)}{\int_0^{\pi/2}\int_0^{\pi/2}\mathrm{d}S(\varphi,\vartheta)} = \frac{2\int_0^{\pi/2}\int_0^{\pi/2} p(\varphi,\vartheta)\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi}{\int_0^{\pi/2}\int_0^{\pi/2}\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi} \\
&= \frac{4}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} p(\varphi,\vartheta)\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \\
&= \frac{4}{\pi}\left[\int_{\varphi=0}^{\pi/6}\int_{\vartheta=0}^{\xi(\varphi)}\left(\frac{1}{2}\sin\varphi\sin\vartheta + \sqrt{\sin^2\varphi\sin^2\vartheta+\cos^2\vartheta}\right)\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\right. \\
&\quad + \int_{\varphi=0}^{\pi/6}\int_{\vartheta=\xi(\varphi)}^{\pi/2}\left(1-\frac{1}{2}\sin\varphi\right)\sin^2\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \qquad (17) \\
&\quad \left. + \int_{\varphi=\pi/6}^{\pi/2}\int_{\vartheta=0}^{\pi/2}\left(\frac{1}{2}\sin\varphi\sin\vartheta + \sqrt{\sin^2\varphi\sin^2\vartheta+\cos^2\vartheta}\right)\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\right],
\end{aligned}$$

where

$$\xi(\varphi) = \arccos\frac{\sqrt{-1+2\sin\varphi}}{\sqrt{-2+2\sin\varphi}}$$

is the solution of the equation

$$\left(1-\frac{1}{2}\sin\varphi\right)\sin\vartheta = \frac{1}{2}\sin\varphi\sin\vartheta + \sqrt{\sin^2\varphi\sin^2\vartheta+\cos^2\vartheta}$$

for $\vartheta = \xi(\varphi)$. Numerical integration of (17) with *Mathematica* gives

$$\bar{b}(\Omega_1) \approx 2.19067696623\,.$$

5. The parallel body

For a convex body $K \subset \mathbb{R}^n$ and $\varrho > 0$, the set (Minkowski sum)

$$K + B_\varrho^n = \{x \in \mathbb{R}^n\colon\ d(x,K) \le \varrho\}$$

is the *parallel body* of K at distance ϱ, where B_ϱ^n is the n-ball of radius ϱ,

$$B_\varrho^n = \{x \in \mathbb{R}^n\colon\ \|x\| \le \varrho\}\,,$$

and $d(x,K)$ is the distance between the point x and K. The volume of the parallel body is given by the Steiner formula

$$V_n(K + B_\varrho^n) = \sum_{j=0}^{n} \varrho^{n-j}\kappa_{n-j}V_j(K)\,, \qquad (18)$$

where

$$\kappa_k = \frac{\pi^{k/2}}{\Gamma(1+k/2)} \qquad (19)$$

is the volume of the k-dimensional unit ball B_1^k, and $V_0,\ldots,V_{n-1}$ are the intrinsic volumes of K [11, p. 2, pp. 12–13, p. 600]

Using the relations in [10, p. 301], where χ denotes the Euler characteristic, the intrinsic volumes of Ω_r are

$$\left.\begin{aligned} V_0(\Omega_r) = \chi = 1\,, \quad V_1(\Omega_r) = 2\bar{b}(\Omega_r) = \frac{M(\Omega_r)}{\pi} = \left[\frac{3}{\pi}\,K\big(\sqrt{3}/2\big) + \frac{3\pi}{2} - \frac{4}{\pi}\,I\right] r\,, \\ V_2(\Omega_r) = \frac{1}{2}\,S(\Omega_r) = 2\pi r^2\,, \quad V_3(\Omega_r) = V(\Omega_r) = \frac{2}{3}\left[2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right] r^3 \end{aligned}\right\} \tag{20}$$

with E (see (10)), K (see (9)), and I (see (16)). Let $\Omega_{r,\,\varrho}$ denote the parallel body of Ω_r at distance ϱ. Due to (18), its volume is

$$\begin{aligned} V(\Omega_{r,\,\varrho}) = V_3(\Omega_{r,\,\varrho}) &= \kappa_0 V_3(\Omega_r) + \kappa_1 V_2(\Omega_r)\varrho + \kappa_2 V_1(\Omega_r)\varrho^2 + \kappa_3 V_0(\Omega_r)\varrho^3 \\ &= V_3(\Omega_r) + 2V_2(\Omega_r)\varrho + \pi V_1(\Omega_r)\varrho^2 + \frac{4\pi}{3} V_0(\Omega_r)\varrho^3 \\ &= V(\Omega_r) + S(\Omega_r)\varrho + M(\Omega_r)\varrho^2 + \frac{4\pi}{3}\varrho^3 \\ &= V(\Omega_1)r^3 + S(\Omega_1)r^2\varrho + M(\Omega_1)r\varrho^2 + \frac{4\pi}{3}\varrho^3\,. \end{aligned}$$

Applying [12, p. 82, (3.17)] allows to calculate the surface area S of $\Omega_{r,\,\varrho}$:

$$S(\Omega_{r,\,\varrho}) = S(\Omega_r) + 2M(\Omega_r)\varrho + 4\pi\varrho^2 = S(\Omega_1)r^2 + 2M(\Omega_1)r\varrho + 4\pi\varrho^2\,.$$

Clearly, the mean width of $\Omega_{r,\,\varrho}$ is equal to $\bar{b}(\Omega_r) + 2\varrho$, hence

$$M(\Omega_{r,\,\varrho}) = 2\pi\left[\bar{b}(\Omega_r) + 2\varrho\right] = 2\pi\bar{b}(\Omega_r) + 4\pi\varrho = M(\Omega_r) + 4\pi\varrho = M(\Omega_1)r + 4\pi\varrho$$

(see also [12, p. 82, (3.17)]). The results of the following theorem follow immediately.

Theorem 2. *The integral M of mean curvature, the surface area S and the volume V of the parallel body $\Omega_{r,\,\varrho}$ are given by*

$$\begin{aligned} M(\Omega_{r,\,\varrho}) &= M(\Omega_1)\,r + 4\pi\varrho\,, \qquad S(\Omega_{r,\,\varrho}) = 4\pi r^2 + 2M(\Omega_1)\,r\varrho + 4\pi\varrho^2\,, \\ V(\Omega_{r,\,\varrho}) &= \frac{2}{3}\left[2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right] r^3 + 4\pi r^2\varrho + M(\Omega_1)\,r\varrho^2 + \frac{4\pi}{3}\,\varrho^3 \end{aligned}$$

with

$$M(\Omega_1) = 3K\big(\sqrt{3}/2\big) + \frac{3\pi^2}{2} - 4\int_0^{\pi/2} \arccos\frac{\cos x}{1+\cos x}\,\mathrm{d}x\,.$$

6. Intersections with an oloid

Now, we apply our results and the principal kinematic formula to derive some expectations for the intersections of the oloid Ω_r and the three-dimensional ball $B_r := B_r^3$ of radius r, and of two oloids Ω_r.

The principal kinematic formula (see [10, p. 301]) for a fixed convex body K and a moving convex body M is for $j \in \{0, \ldots, n\}$ given by

$$I_j(K, M) := \int_{SO_n}\int_{\mathbb{R}^n} V_j(K \cap (\vartheta M + \vec{x}))\,\mathrm{d}\lambda(\vec{x})\,\mathrm{d}\nu(\vartheta) = \sum_{k=j}^{n} \alpha_{njk} V_k(K) V_{n+j-k}(M) \tag{21}$$

with the notation

SO_n group of proper (orientation-preserving) rotations [11, p. 13],
ϑ proper rotation, $\vartheta \in SO_n$,
$\vec{x}$ translation vector,
λ Lebesgue measure on $\mathbb{R}^n$,
ν unique Haar measure on SO_n with $\nu(SO_n) = 1$ [11, p. 584],

and

$$\alpha_{njk} = \frac{k!\,\kappa_k (n+j-k)!\,\kappa_{n+j-k}}{j!\,\kappa_j\, n!\,\kappa_n}\,, \quad \alpha_{njk} = \alpha_{nj(n+j-k)}\,, \quad \alpha_{njj} = \alpha_{njn} = 1\,,$$

where κ_k is the volume of the unit k-ball (see (19)). Since the intersection of two convex sets is a convex set, we have

$$\begin{aligned} I_0(K,M) &= \int_{SO_n} \int_{\mathbb{R}^n} \chi(K \cap (\vartheta M + \vec{x}))\, \mathrm{d}\lambda(\vec{x})\, \mathrm{d}\nu(\vartheta) \\ &= \int_{SO_n} \int_{\mathbb{R}^n} \mathbb{1}_{K \cap (\vartheta M + \vec{x}) \neq \emptyset}\, \mathrm{d}\lambda(\vec{x})\, \mathrm{d}\nu(\vartheta)\,, \end{aligned} \tag{22}$$

where $\mathbb{1}_B$ is the indicator function of the event B. So we see that $I_0(K,M)$ is the measure of the set of rigid motions bringing M into a hitting position with K (see [11, p. 175], [8, p. 262, p. 267]).

For $n = 3$, (21) gives

$$\left.\begin{aligned} I_0(K,M) &= V_0(K)\,V_3(M) + \frac{1}{2}\,V_1(K)\,V_2(M) + \frac{1}{2}\,V_2(K)\,V_1(M) + V_3(K)\,V_0(M)\,, \\ I_1(K,M) &= V_1(K)\,V_3(M) + \frac{\pi}{4}\,V_2(K)\,V_2(M) + V_3(K)\,V_1(M)\,, \\ I_2(K,M) &= V_2(K)\,V_3(M) + V_3(K)\,V_2(M)\,, \\ I_3(K,M) &= V_3(K)\,V_3(M)\,. \end{aligned}\right\} \tag{23}$$

From (22) it follows that

$$\mathbb{E}\,[V(K \cap M)] = \mathbb{E}\,[V_3(K \cap M)] = \frac{I_3(K,M)}{I_0(K,M)} \tag{24}$$

is the expected volume of $K \cap M$. Analogously, we get the expected mean width and the expected surface area:

$$\mathbb{E}\big[\bar{b}\big(K \cap M\big)\big] = \frac{1}{2}\,\mathbb{E}\big[V_1(K \cap M)\big] = \frac{I_1(K,M)}{2I_0(K,M)}\,, \tag{25}$$

$$\mathbb{E}\big[S\big(K \cap M\big)\big] = 2\,\mathbb{E}\big[V_2(K \cap M)\big] = \frac{2I_2(K,M)}{I_0(K,M)}\,. \tag{26}$$

Clearly, it is possible to reverse the roles of the fixed body and the moving body.

Example 1. As an example, we calculate the expected values (25), (26) and (24) for $K = \Omega_r$ and $M = B_r$, or, equivalently, for $K = B_r$ and $M = \Omega_r$. For the ball B_r one easily gets

$$\left.\begin{aligned} &V_0(B_r) = \chi(B_r) = 1\,, \quad V_1(B_r) = 2\bar{b}(B_r) = 4r\,, \\ &V_2(B_r) = \frac{1}{2}\,S(B_r) = 2\pi r^2\,, \quad V_3(B_r) = V(B_r) = \frac{4\pi r^3}{3}\,. \end{aligned}\right\} \tag{27}$$

Note that these terms also follow from the general formula

$$V_k(B_r) = V_k\big(B_r^3\big) = V_k(B_1^3)\, r^k = \binom{3}{k} \frac{\kappa_3}{\kappa_{3-k}}\, r^k$$

[10, p. 300], where κ_k is the volume of the unit k-ball (see (19)). Plugging (20) and (27) in (23) gives

$$\begin{aligned}
I_0(\Omega_r, B_r) &= \frac{r^3}{6}\left[9\pi^2 + 32\pi + 8E\big(\sqrt{3}/2\big) + 22K\big(\sqrt{3}/2\big) - 24I\right],\\
I_1(\Omega_r, B_r) &= \frac{r^4}{3}\left[3\pi^3 + 6\pi^2 + 16E\big(\sqrt{3}/2\big) + 20K\big(\sqrt{3}/2\big) - 16I\right],\\
I_2(\Omega_r, B_r) &= \frac{4\pi r^5}{3}\left[2\pi + 2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right],\\
I_3(\Omega_r, B_r) &= \frac{8\pi r^6}{9}\left[2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\big[\bar{b}(\Omega_r \cap B_r)\big] &= \frac{I_1(\Omega_r, B_r)}{2I_0(\Omega_r, B_r)} = \frac{3\pi^3 + 6\pi^2 + 16E\big(\sqrt{3}/2\big) + 20K\big(\sqrt{3}/2\big) - 16I}{9\pi^2 + 32\pi + 8E\big(\sqrt{3}/2\big) + 22K\big(\sqrt{3}/2\big) - 24I}\, r,\\
\mathbb{E}\big[S(\Omega_r \cap B_r)\big] &= \frac{2I_2(\Omega_r, B_r)}{I_0(\Omega_r, B_r)} = \frac{16\pi\left[2\pi + 2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right]}{9\pi^2 + 32\pi + 8E\big(\sqrt{3}/2\big) + 22K\big(\sqrt{3}/2\big) - 24I}\, r^2,\\
\mathbb{E}\big[V(\Omega_r \cap B_r)\big] &= \frac{I_3(\Omega_r, B_r)}{I_0(\Omega_r, B_r)} = \frac{16\pi\left[2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right]}{3\left[9\pi^2 + 32\pi + 8E\big(\sqrt{3}/2\big) + 22K\big(\sqrt{3}/2\big) - 24I\right]}\, r^3.
\end{aligned}$$

Example 2. In the case $K = \Omega_r = M$, we have

$$\begin{aligned}
I_0(\Omega_r, \Omega_r) &= \frac{r^3}{3}\left[9\pi^2 + 8E\big(\sqrt{3}/2\big) + 22K\big(\sqrt{3}/2\big) - 24I\right],\\
I_1(\Omega_r, \Omega_r) &= \frac{r^4}{3\pi}\left[3\pi^4 + 2\left(2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right)\left(3\pi^2 + 6K\big(\sqrt{3}/2\big) - 8I\right)\right],\\
I_2(\Omega_r, \Omega_r) &= \frac{8\pi r^5}{3}\left[2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right],\\
I_3(\Omega_r, \Omega_r) &= \frac{4r^6}{9}\left[2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right]^2,
\end{aligned}$$

hence

$$\begin{aligned}
\mathbb{E}\big[\bar{b}(\Omega_r \cap \Omega_r)\big] &= \frac{I_1(\Omega_r, \Omega_r)}{2I_0(\Omega_r, \Omega_r)} = \frac{3\pi^4 + 2\left[2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right]\left[3\pi^2 + 6K\big(\sqrt{3}/2\big) - 8I\right]}{2\pi\left[9\pi^2 + 8E\big(\sqrt{3}/2\big) + 22K\big(\sqrt{3}/2\big) - 24I\right]}\, r,\\
\mathbb{E}\big[S(\Omega_r \cap \Omega_r)\big] &= \frac{2I_2(\Omega_r, \Omega_r)}{I_0(\Omega_r, \Omega_r)} = \frac{16\pi\left[2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right]}{9\pi^2 + 8E\big(\sqrt{3}/2\big) + 22K\big(\sqrt{3}/2\big) - 24I}\, r^2,\\
\mathbb{E}\big[V(\Omega_r \cap \Omega_r)\big] &= \frac{I_3(\Omega_r, \Omega_r)}{I_0(\Omega_r, \Omega_r)} = \frac{4\left[2E\big(\sqrt{3}/2\big) + K\big(\sqrt{3}/2\big)\right]^2}{3\left[9\pi^2 + 8E\big(\sqrt{3}/2\big) + 22K\big(\sqrt{3}/2\big) - 24I\right]}\, r^3.
\end{aligned}$$

The following table shows numerical approximations for the expectations of the intersections.

K	M	$\mathbb{E}\big[\bar{b}(K\cap M)\big]/r$	$\mathbb{E}\big[S(K\cap M)\big]/r^2$	$\mathbb{E}\big[V(K\cap M)\big]/r^3$
B_r	B_r	0.9626377063	3.141592654	0.5235987756
Ω_r	B_r	0.9169621588	2.710463736	0.3808512243
Ω_r	Ω_r	0.8585694641	2.280916270	0.2770215506

7. Appendix

Now we are going to show that the integral

$$J := \int_0^{2\pi/3} \frac{\mathrm{d}t}{\sqrt{1+2\cos t}}$$

is equal to $K\big(\sqrt{3}/2\big)$ (see (13)) without the use of *Mathematica*. With

$$1+2\cos t = 1+2\left(1-2\sin^2\frac{t}{2}\right) = 3-4\sin^2\frac{t}{2} = 3\left[1-\left(\frac{2}{\sqrt{3}}\right)^2\sin^2\frac{t}{2}\right]$$

we have

$$J = \int_0^{2\pi/3} \frac{\mathrm{d}t}{\sqrt{3}\,\sqrt{1-\big(2/\sqrt{3}\,\big)^2\sin^2(t/2)}}\,.$$

Now, following the argumentation in [13], we put

$$\csc\frac{t_0}{2} = \frac{2}{\sqrt{3}} \qquad \Longrightarrow \qquad t_0 = \frac{2\pi}{3}\,.$$

This gives

$$J = \frac{1}{2}\int_0^{t_0} \frac{1}{\sin(t_0/2)\,\sqrt{1-\csc^2(t_0/2)\sin^2(t/2)}}\,\mathrm{d}t\,.$$

Now let

$$\sin(t/2) = \sin(t_0/2)\,\sin\varphi\,,$$

so the angle t is transformed to

$$\varphi = \arcsin\frac{\sin(t/2)}{\sin(t_0/2)}\,,$$

hence

$$t=0 \implies \varphi = 0\,, \qquad t=t_0 \implies \varphi = \frac{\pi}{2}\,.$$

Taking the differentials gives

$$\frac{1}{2}\cos\left(\frac{t}{2}\right)\mathrm{d}t = \sin\left(\frac{t_0}{2}\right)\cos\varphi\,\mathrm{d}\varphi\,, \quad \text{or} \quad \frac{1}{2}\sqrt{1-\sin^2\left(\frac{t}{2}\right)}\,\mathrm{d}t = \sin\left(\frac{t_0}{2}\right)\cos\varphi\,\mathrm{d}\varphi\,,$$

hence

$$\frac{1}{2}\sqrt{1-\sin^2\left(\frac{t_0}{2}\right)\sin^2\varphi}\;\mathrm{d}t = \sin\left(\frac{t_0}{2}\right)\cos\varphi\,\mathrm{d}\varphi\,.$$

Plugging this in gives

$$\begin{aligned} J &= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sin(t_0/2)\sqrt{1-\sin^2\varphi}} \frac{\sin(t_0/2)\cos\varphi\,\mathrm{d}\varphi}{(1/2)\sqrt{1-\sin^2(t_0/2)\sin^2\varphi}} \\ &= \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-\sin^2(t_0/2)\sin^2\varphi}} = \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-\sin^2(\pi/3)\sin^2\varphi}} \\ &= \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-\left(\sqrt{3}/2\right)^2\sin^2\varphi}} = K\left(\sqrt{3}/2\right). \end{aligned}$$

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