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PH Curves with Non-Primitive Hodographs

Petroula Dospra

Dept. of Mathematics, Aristotle University of Thessaloniki
54124 Thessaloniki, Greece
emails: petdospra@math.auth.gr, petroula.dospra@gmail.com

Abstract

In this paper we present a necessary and sufficient condition under which a PH curve generated by a primitive quaternion polynomial has a non-primitive hodograph. Such curves are regular PH curves, and we give a characterization of these curves in terms of their associated quaternion polynomial. This work leads to the problem of the production of RRMF curves by others of lower degree. Furthermore, we present some geometrical properties of RRMF curves with non-primitive hodographs of degrees 5 and 7.

Key Words: Pythagorean-hodograph curves, rotation-minimizing frame, curves with rational rotation-minimizing frame, non-primitive hodograph

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1. Introduction

A *polynomial* space curve defined by $x(t), y(t), z(t) \in \mathbb{R}[t]$ is the set

$$C = \{(x(t), y(t), z(t)) \in \mathbb{R}^3 \,|\, t \in \mathbb{R}\}.$$

We denote by $\mathbf{r}(t)$ the parametrization of C, i.e. the map defined by the correspondence $t \mapsto (x(t), y(t), z(t))$. In the following we shall refer to the polynomial space curve C by giving its parametrization $\mathbf{r}(t)$.

A moving frame along a curve describes the orientation of a rigid body when it moves along its trajectory. An *adapted* frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ on a space curve $\mathbf{r}(t)$ is an orthonormal basis defined at each curve point, where $\mathbf{f}_1$ coincides with the curve tangent $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ and $\mathbf{f}_2, \mathbf{f}_3$ span the normal plane, such that $\mathbf{f}_1 \times \mathbf{f}_2 = \mathbf{f}_3$. The variation of such a frame may be specified by its angular velocity $\boldsymbol{\omega} = \omega_1\mathbf{f}_1 + \omega_2\mathbf{f}_2 + \omega_3\mathbf{f}_3$ through the differential relations

$$\mathbf{f}_1' = \sigma\boldsymbol{\omega} \times \mathbf{f}_1\,, \qquad \mathbf{f}_2' = \sigma\boldsymbol{\omega} \times \mathbf{f}_2\,, \qquad \mathbf{f}_3' = \sigma\boldsymbol{\omega} \times \mathbf{f}_3\,,$$

where $\sigma(t) = |\mathbf{r}'(t)|$ is the parametric speed of $\mathbf{r}(t)$. A familiar adapted frame is the Frenet frame but it is often not suitable for applications since its normal $\mathbf{h}$ and binormal $\mathbf{b}$ vectors

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may appear to execute a rotation about the tangent vector $\mathbf{t}$ which is not desirable for the study of space motions. The most appropriate adapted frame for applications is the *rotation-minimizing* frame (RMF) which is characterized by the property that its angular velocity satisfies $\boldsymbol{\omega}\cdot\mathbf{f}_1 \equiv 0$, i.e., $\boldsymbol{\omega}$ has no component along $\mathbf{f}_1$, which is equivalent to $\mathbf{f}_3\cdot\mathbf{f}_2' = \mathbf{f}_3'\cdot\mathbf{f}_2 = 0$. This means that $\mathbf{f}_2, \mathbf{f}_3$ rotate as little as possible around $\mathbf{f}_1$ and thus RMFs minimize the amount of rotation along the curve. Thus RMF is very useful in computer graphics, swept surface constructions, motion design and other similar applications [9, 10, 15, 14, 17, 18, 19].

Since it is very important for computer aided design applications to have a frame rationally dependent on the curve parameter, recently there has been great interest in constructing polynomial curves which have *rational* rotation-minimizing frames (RRMF curves). The search of such curves is restricted to the particular class of polynomial curves with a special structure, the *Pythagorean-hodograph curves* (PH curves) [8], since only PH curves have rational unit tangents. Due to this fact, PH curves admit an exceptional kind of frame, the Euler-Rodrigues frame (ERF) which is rational by its construction [3] and is defined only on PH curves.

For a given space curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ the *hodograph* is its parametric derivative $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ regarded as a curve in its own right. The curve $\mathbf{r}(t)$ is said to have a *Pythagorean-hodograph* [8] if there exists a real polynomial $\sigma(t)$ such that

$$x'^2(t) + y'^2(t) + z'^2(t) \;=\; \sigma^2(t). \tag{1}$$

By [4] and [6], the equality (1) is satisfied if and only if there are $u(t), v(t), p(t), q(t) \in \mathbb{R}[t]$ such that we have:

$$\begin{aligned} x'(t) &= \; u^2(t) + v^2(t) - p^2(t) - q^2(t), \\ y'(t) &= 2\,[\,u(t)q(t) + v(t)p(t)\,], \\ z'(t) &= 2\,[\,v(t)q(t) - u(t)p(t)\,]. \end{aligned} \tag{2}$$

The polynomial

$$\sigma(t) \;=\; u^2(t) + v^2(t) + p^2(t) + q^2(t) \tag{3}$$

defines the *parametric speed* of the curve $\mathbf{r}(t)$, i.e., the rate of change $\mathrm{d}s/\mathrm{d}t$ of its arc length s with respect to the curve parameter t.

We shall say that the curve $\mathbf{r}(t)$ is called *regular* if $|\mathbf{r}'(t)| \neq 0$, for all t. If $k = \max\{\deg u(t), \deg v(t), \deg p(t), \deg q(t)\}$, then the PH curves obtained by integrating the hodograph $\mathbf{r}'(t)$ are said to be of *degree* $n = 2k+1$. If $k = 1$ and $k = 2$, then we shall call them *cubics* and *quintics*, respectively.

A *primitive* hodograph $\mathbf{r}'(t)$ is characterized by the fact that $\gcd((x'(t),\, y'(t),\, z'(t)) = 1$. Otherwise it is called *non-primitive.* Primitive hodographs are desirable in practice since at a common real root of $x'(t), y'(t), z'(t)$ may incur a cusp or inflection point. This is why we consider polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$ having $\gcd(u(t), v(t), p(t), q(t)) = 1$, since common real roots of these polynomials incur cusps on the curve. However, we can see in [8] that $\gcd(u, v, p, q) = 1$ does not ensure that the hodograph is primitive. The hodograph components may have common quadratic factors with complex conjugate roots even if $\gcd(u, v, p, q) = 1$. In this case the hodograph $\mathbf{r}'(t)$ is non-primitive but the PH curve is regular i.e., $|\mathbf{r}'(t)| \neq 0$, for all real t.

In [4] Choi et al. introduced two equivalent characterization of solutions to condition (1) using the algebra of *quaternions* and the *Hopf map* which are greatly useful in the research of spatial PH curves. Recall that a quaternion number is of the form $\mathcal{Q} = x_0 + \mathbf{i}\,x_1 + \mathbf{j}\,x_2 + \mathbf{k}\,x_3$, where $x_1, x_2, x_3, x_4 \in \mathbb{R}$, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}, \quad \mathbf{j}\,\mathbf{k} = -\mathbf{k}\,\mathbf{j} = \mathbf{i}, \quad \mathbf{k}\,\mathbf{i} = -\mathbf{i}\,\mathbf{k} = \mathbf{j}.$$

The *conjugate* of $\mathcal{Q}$ is defined as $\mathcal{Q}^* = x_0 - \mathbf{i}\,x_1 - \mathbf{j}\,x_2 - \mathbf{k}\,x_3$. The *real* and the *imaginary part* of $\mathcal{Q}$ are $\mathrm{Re}\mathcal{Q} = x_0$ and $\mathrm{Im}\mathcal{Q} = \mathbf{i}\,x_1 + \mathbf{j}\,x_2 + \mathbf{k}\,x_3$, respectively. The *norm* $|\mathcal{Q}|$ of $\mathcal{Q}$ is defined to be the quantity

$$|\mathcal{Q}| = \sqrt{\mathcal{Q}\mathcal{Q}^*} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

We denote by $\mathbb{H}$ the skew field of real quaternions. Let now $\mathbb{H}[t]$ be the polynomial ring in the variable t over $\mathbb{H}$. Every polynomial $\mathcal{A}(t) \in \mathbb{H}[t]$ is written as $\mathcal{A}_0 t^n + \mathcal{A}_1 t^{n-l} + \cdots + \mathcal{A}_n$ where n is an integer ≥ 0 and $\mathcal{A}_0, \ldots, \mathcal{A}_n \in \mathbb{H}$ with $\mathcal{A}_0 \neq 0$. The addition and the multiplication of polynomials are defined in the same way as in the commutative case, where the variable t is assumed to commute with quaternion coefficients [16, Chapter 5, Section 16].

The *quaternion* form generates a PH curve $\mathbf{r}(t)$ in $\mathbb{R}^3$ from a polynomial

$$\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t) \tag{4}$$

through the product

$$\begin{aligned} \mathbf{r}'(t) \;=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) \;=\; & [\,u^2(t) + v^2(t) - p^2(t) - q^2(t)\,]\,\mathbf{i} \\ & + 2\,[\,u(t)q(t) + v(t)p(t)\,]\,\mathbf{j} + 2\,[\,v(t)q(t) - u(t)p(t)\,]\,\mathbf{k}, \end{aligned} \tag{5}$$

where $\mathcal{A}^*(t) = u(t) - \mathbf{i}\,v(t) - \mathbf{j}\,p(t) - \mathbf{k}\,q(t)$ is the conjugate of $\mathcal{A}(t)$.

The *Hopf map* form generates a Pythagorean hodograph from polynomials $\alpha(t) = u(t) + \mathbf{i}\,v(t)$ and $\beta(t) = q(t) + \mathbf{i}\,p(t)$ through the expression

$$\mathbf{r}'(t) \;=\; \big(|\alpha(t)|^2 - |\beta(t)|^2,\; 2\,\mathrm{Re}\,(\alpha(t)\bar{\beta}(t)),\; 2\,\mathrm{Im}\,(\alpha(t)\bar{\beta}(t))\big) \tag{6}$$

The equivalence of (5) and (6) can be taken by

$$\mathcal{A}(t) = \alpha(t) + \mathbf{k}\,\beta(t). \tag{7}$$

The polynomial $\mathcal{A}(t)$ is called *primitive*, if $\gcd(u(t), v(t), p(t), q(t)) = 1$, and *non-primitive*, otherwise. Although a primitive quaternion polynomial generates a regular PH curve $\mathbf{r}(t)$, the hodograph $\mathbf{r}'(t)$ may be primitive or non-primitive. Indeed, $\mathbf{r}'(t)$ may have the form $\mathbf{r}'(t) = \phi(t)\mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t)$, where $\phi(t)$ is a real polynomial of positive even degree and $\mathcal{B}(t)$ is primitive quaternion polynomial with lower degree than the degree of $\mathcal{A}(t)$. In this case the polynomial $\phi(t)$ does not have real roots, the hodograph is non-primitive and the curve $\mathbf{r}(t)$ is regular [8, §23.3]. The question that arises is, when the hodograph generated by a primitive quaternion polynomial is primitive and when it is non-primitive.

The main goal of this work is to obtain necessary and sufficient conditions for the quaternion polynomial $\mathcal{A}(t)$ such that a regular PH curve generated by $\mathcal{A}(t)$ has a non-primitive hodograph. We prove that regular PH curves with non-primitive hodographs are those whose associated quaternion polynomials have right complex factors. Through the study of non-primitive hodographs, we see that there are RRMF curves which are produced by others of lower degree. In particular, the RRMF curves of degree 7 with non-primitive hodograph for which the ERF is an RMF are generated by a specific subset of quintic RRMF curves.

The organization of this paper is as follows. In Section 2, the Euler-Rodrigues frame, which is the key to identify RRMF curves, is introduced and the condition which characterizes the RRMF curves is also described. Since quaternion polynomials are used to represent PH curves, in Section 3 we recall an useful criterion for such a polynomial to have a root in $\mathbb{C}$. Necessary and sufficient conditions for a regular PH curve to have a non-primitive hodograph in terms of its associated quaternion polynomial, are presented in Section 4. Working on regular curves with non-primitive hodographs, we deal with the problem of generating PH curves by others of lower degree and we study their geometrical properties in Section 5. Finally, in Section 6 we focus in our study on these quintics and RRMF curves of degree 7 with non-primitive hodographs which have the ERF as an RMF and we characterize these special sets since this property is of interest in applications. Moreover, a number of examples is given as well.

2. Euler-Rodrigues frame and RRMF curves

In this section we recall some basic facts about a special frame, the Euler-Rodrigues frame, and the condition which characterizes curves with rational rotation-minimizing frames. In [3] Choi and Han introduced a special adapted frame, the *Euler-Rodrigues frame* (ERF) which is defined on any spatial PH curve and was an important step for identifying the RRMF curves as a subset of PH curves. As we mentioned, the ERF is rational by its construction and additionally has non-singular behavior at inflection points. The ERF is not always an RMF. The first true spatial RRMF curves for which the ERF is itself rotation minimizing (ERF = RMF) are PH curves of degree 7. The conditions under which the ERF of a PH curve can be an RMF were also investigated in [3].

The ERF on the PH curve specified by (4) – (5) is the set of orthonormal vectors defined by

$$
\begin{aligned}
\mathbf{e}_1 &= \frac{(u^2+v^2-p^2-q^2)\,\mathbf{i} \,+\, 2(uq+vp)\,\mathbf{j} \,+\, 2(vq-up)\,\mathbf{k}}{u^2+v^2+p^2+q^2}, \\
\mathbf{e}_2 &= \frac{2(vp-uq)\,\mathbf{i} \,+\, (u^2-v^2+p^2-q^2)\,\mathbf{j} \,+\, 2(uv+pq)\,\mathbf{k}}{u^2+v^2+p^2+q^2}, \\
\mathbf{e}_3 &= \frac{2(up+vq)\,\mathbf{i} \,+\, 2(pq-uv)\,\mathbf{j} \,+\, (u^2-v^2-p^2+q^2)\,\mathbf{k}}{u^2+v^2+p^2+q^2},
\end{aligned}
\tag{8}
$$

where $\mathbf{e}_1$ is the curve tangent, while $\mathbf{e}_2$ and $\mathbf{e}_3$ span the normal plane of the curve.

The ERF is a useful frame since it can be used as a reference frame to identify rational rotation minimizing frames [3]. By [13], if the PH curve defined by (4)–(5) admits a rational RMF $(\mathbf{f}_1(t),\ \mathbf{f}_2(t), \mathbf{f}_3(t))$, then $\mathbf{e}_1 = \mathbf{f}_1$ is the curve tangent, and the normal-plane vectors $\mathbf{f}_2(t), \mathbf{f}_3(t)$ must be obtainable from the ERF normal-plane vectors $\mathbf{e}_2(t), \mathbf{e}_3(t)$ by a rational rotation — i.e., for relatively prime polynomials $a(t), b(t)$ we must have

$$
\begin{bmatrix} \mathbf{f}_2(t) \\ \mathbf{f}_3(t) \end{bmatrix} = \frac{1}{a^2(t)+b^2(t)} \begin{bmatrix} a^2(t)-b^2(t) & -2\,a(t)b(t) \\ 2\,a(t)b(t) & a^2(t)-b^2(t) \end{bmatrix} \begin{bmatrix} \mathbf{e}_2(t) \\ \mathbf{e}_3(t) \end{bmatrix}.
$$

In [13], Han proved that a PH curve, defined by (4) – (5), has a rational RMF if and only if relatively prime polynomials $a(t)$, $b(t)$ exist, such that

$$
\frac{u(t)v'(t)-u'(t)v(t)-p(t)q'(t)+p'(t)q(t)}{u(t)^2+v(t)^2+p(t)^2+q(t)^2} = \frac{a(t)b'(t)-a'(t)b(t)}{a(t)^2+b(t)^2}. \tag{9}
$$

A PH curve which satisfies the above condition, is called a *Rational Rotation-Minimizing Frame curve* (RRMF curve). Using the Hopf map form (6), condition (9) can be phrased by requiring the existence of a complex polynomial $w(t) = a(t) + \mathrm{i}\, b(t)$, with $\gcd(a(t), b(t)) = 1$, such that

$$\frac{\operatorname{Im}\left(\bar{\alpha}(t)\alpha'(t) + \bar{\beta}(t)\beta'(t)\right)}{|\alpha(t)|^2 + |\beta(t)|^2} = \frac{\operatorname{Im}\left(\bar{w}(t)w'(t)\right)}{|w(t)|^2}. \tag{10}$$

When $w(t)$ is either a real polynomial or a constant, the angle $\theta(t)$ between the ERF and RMF is constant. This is equivalent to

$$\operatorname{Im}\left(\bar{\alpha}(t)\alpha'(t) + \bar{\beta}(t)\beta'(t)\right) = 0. \tag{11}$$

Since in the computation of the RMF appears an integration constant, we may consider (11) as the condition identifying coincidence of the RMF and ERF (for short, we shall write ERF = RMF). Further analysis of this condition was presented in [3]. Finally, if $\mathcal{A}(t) = u(t) + \mathbf{i}\, v(t) + \mathbf{j}\, p(t) + \mathbf{k}\, q(t)$, then (11) is equivalent to

$$u(t)v'(t) - u'(t)v(t) - p(t)q'(t) + p'(t)q(t) = 0. \tag{12}$$

3. Complex roots of quaternion polynomials

In 1971, Barnett determined the degree of the greatest common divisor of several univariate polynomials with coefficients in an integral domain by means of the rank of several matrices involving theirs coefficients [1, 2]. In [5], a formulation of Barnett's theorem is given by using the hybrid Bézout matrices and, as it is noticed, these matrices have the best computational behaviour. In [7], we used the results of [5] in order to study the existence of complex roots of a quaternion polynomial (i.e., roots which are in $\mathbb{C}$).

Let $\mathbb{F}$ be a field of characteristic zero and $\mathbb{F}[t]$ the ring of polynomials with coefficients in $\mathbb{F}$. Consider polynomials

$$P(t) = p_0 t^n + p_1 t^{n-1} + \cdots + p_n \quad \text{and} \quad Q(t) = q_0 t^m + q_1 t^{m-1} + \cdots + q_m,$$

in $\mathbb{F}[t]$ with $n \geq m$. The hybrid Bézout matrix, denoted by $\operatorname{Hbez}(P, Q)$, is a square matrix of size n whose entries are defined by:

- for $1 \leq i \leq m$, $1 \leq j \leq n$, the (i, j)-entry is the coefficient of t^{n-j} in the polynomial

$$\begin{aligned} K_{m-i+1} = {} & (p_0 t^{m-i} + \cdots + p_{m-i})(q_{m-i+1} t^{n-m+i-1} + \cdots + q_m t^{n-m}) \\ & - (p_{m-i+1} t^{n-m+i-1} + \cdots + p_n)(q_0 t^{m-i} + \cdots + q_{m-i}); \end{aligned}$$

- for $m + 1 \leq i \leq n$, $1 \leq j \leq n$, the (i, j)-entry is the coefficient of t^{n-j} in the polynomial $t^{n-i} Q(t)$.

Let $R(P, Q)$ be the Sylvester resultant of $P(t)$ and $Q(t)$. By [5, Corollary 5.2], we have $\det(\operatorname{Hbez}(P, Q)) = R(P, Q)$.

In the next section, we shall use the following results:

Lemma 1. *Let $D(t) = \gcd(P(t), Q(t))$ and $r = \operatorname{rank} \operatorname{Hbez}(P, Q)$. Then*

$$\deg D(t) = n - r.$$

Proof. See [5, Theorem 5.1]. □

Lemma 2. *Let $\mathcal{Q}(t) \in \mathbb{H}[t] \setminus \mathbb{C}[t]$ be a monic polynomial with $\deg \mathcal{Q} = n \geq 1$ and $f(t), g(t) \in \mathbb{C}[t]$ with $f(t)g(t) \neq 0$ such that $\mathcal{Q}(t) = f(t) + \mathbf{k}g(t)$. Set $E(t) = \gcd(f(t), g(t))$. Then the roots of $E(t)$ are precisely the complex roots of $\mathcal{Q}(t)$. Furthermore, the following are equivalent:*

(a) *$\mathcal{Q}(t)$ has a complex root.*

(b) $\deg E(t) > 0$.

(c) $\det(\mathrm{Hbez}(f, g)) = 0$.

(d) $R(f, g) = 0$.

Proof. See [7, Theorem 1]. □

Lemma 3. *Let $\mathcal{Q}(t) = t^2 + \mathcal{Q}_1 t + \mathcal{Q}_0$ be a quadratic polynomial of $\mathbb{H}[t] \setminus \mathbb{C}[t]$ with no real factor. Set $\mathcal{Q}_1 = b_1 + \mathbf{k}\, c_1$ and $\mathcal{Q}_0 = b_0 + \mathbf{k}\, c_0$, where $b_0, b_1, c_0, c_1 \in \mathbb{C}$. Then $\mathcal{Q}(t)$ has a complex root if and only if*

$$c_0^2 - c_0 b_1 c_1 + b_0 c_1^2 = 0.$$

Proof. See [7, Theorem 2]. □

4. Characterization of non-primitive hodographs

Let $\mathcal{A}(t)$ be a monic primitive quaternion polynomial of degree m. We write $\mathcal{A}(t) = f(t) + \mathbf{k}g(t)$, where $f(t), g(t) \in \mathbb{C}[t]$. Let $\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$ be the hodograph generated by $\mathcal{A}(t)$. Recall that it is possible for a PH curve to be regular even when its hodograph $\mathbf{r}'(t)$ is non-primitive.

We say that the quaternion polynomial $\mathcal{B}(t)$ is a right (left) factor of $\mathcal{A}(t)$ if there is a quaternion polynomial $\mathcal{C}(t)$ such that $\mathcal{A}(t) = \mathcal{C}(t)\,\mathcal{B}(t)$ $(\mathcal{A}(t) = \mathcal{B}(t)\mathcal{C}(t))$.

The next theorem characterizes non-primitive hodographs.

Theorem 1. *The following statements are equivalent:*

(a) *The hodograph defined by $\mathbf{r}'(t)$ is non-primitive.*

(b) *The polynomial $\mathcal{A}(t)$, has a right factor which is a polynomial of $\mathbb{C}[t] \setminus \mathbb{C}$.*

(c) *We have $\mathbf{r}'(t) = h(t)\mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t)$, where $h(t)$ is a real monic polynomial with no real roots and $\mathcal{B}(t)$ is a left factor of $\mathcal{A}(t)$.*

(d) *We have $R(f(t), g(t)) = 0$.*

Proof. Write $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$, where $u(t), v(t), p(t), q(t)$ are real polynomials. Suppose that the hodograph $\mathbf{r}'(t)$ is not primitive. Then, there is a real monic irreducible polynomial $\wp(t)$ which divides the polynomials

$$u^2(t) + v^2(t) - p^2(t) - q^2(t), \quad u(t)q(t) + v(t)p(t), \quad v(t)q(t) - u(t)p(t).$$

The relations $\wp(t) \mid u(t)q(t) + v(t)p(t)$ and $\wp(t) \mid v(t)q(t) - u(t)p(t)$ imply

$$\wp(t) \mid u^2(t)q(t) + u(t)v(t)p(t) \quad \text{and} \quad \wp(t) \mid v^2(t)q(t) - u(t)v(t)p(t)$$

and whence we get $\wp(t) \mid (u^2(t) + v^2(t))q(t)$.

Suppose first that $\wp(t) \nmid q(t)$. Then, we have $\wp(t) \,|\, u^2(t) + v^2(t)$, and so, the relation $\wp(t) \,|\, u^2(t)+v^2(t)-p^2(t)-q^2(t)$ yields $\wp(t) \,|\, p^2(t)+q^2(t)$. If $\wp(t) = t-a$, then the real number a is a root of $p^2(t)+q^2(t)$, and so a common root of $p(t)$ and $q(t)$. Similarly, a is a common root of $u(t)$ and $v(t)$. Thus, we have $\gcd(u(t), v(t), p(t), q(t)) > 1$ which is a contradiction, and so, $\deg \wp(t) \geq 2$. Since the irreducible polynomials of $\mathbb{R}[t]$ are linear or quadratic, we deduce that $\deg \wp(t) = 2$.

Next, suppose that $\wp(t) \,|\, q(t)$. Thus, we have

$$\wp \,|\, u^2(t) + v^2(t) - p^2(t), \quad \wp(t) \,|\, v(t)p(t), \quad \wp(t) \,|\, u(t)p(t).$$

If $\wp(t) \nmid p(t)$, then $\wp(t) \,|\, v(t)$, $\wp(t) \,|\, u(t)$ and the relation $\wp(t) \,|\, u^2(t)+v^2(t)-p^2(t)$ implies that $\wp(t) \,|\, p(t)$, which is a contradiction. Therefore $\wp(t) \,|\, p(t)$ and so, we have $\wp(t) \,|\, p^2(t)+q^2(t)$, whence follows that $\wp(t) \,|\, u^2(t)+v^2(t)$. If $\wp(t) = t-a$, a is a common root of $u(t)$ and $v(t)$, and since $t-a|p(t)$, $t-a \,|\, q(t)$ we have $\gcd(u(t), v(t), p(t), q(t)) > 1$, which is a contradiction. Hence, we have $\deg \wp(t) = 2$. Therefore, in both cases, we have that $\wp(t) \,|\, u^2(t)+v^2(t)$, $\wp(t) \,|\, p^2(t)+q^2(t)$ and $\wp(t) = (t-r)\,(t-\bar{r})$, where $r \in \mathbb{C} \setminus \mathbb{R}$ and $\bar{r}$ is the complex conjugate of r.

We remark that the divisibility relations

$$\wp(t) \,|\, u(t)q(t) + v(t)p(t) \quad \text{and} \quad \wp(t) \,|\, v(t)q(t) - u(t)p(t)$$

can be equivalently presented by the relation

$$(t-r)(t-\bar{r}) \,|\, (u(t) + v(t)\,\mathbf{i})\,(q(t) - p(t)\,\mathbf{i}).$$

We also have

$$(t-r)(t-\bar{r}) \,|\, (u(t)+v(t)\,\mathbf{i})\,(u(t)-v(t)\,\mathbf{i}), \quad (t-r)(t-\bar{r}) \,|\, (q(t)+p(t)\,\mathbf{i})\,(q(t)-p(t)\,\mathbf{i}).$$

Suppose that $t-r \,|\, u(t)+v(t)\,\mathbf{i}$. If $t-r \nmid q(t)+p(t)\,\mathbf{i}$, then $t-r \,|\, q(t)-p(t)\,\mathbf{i}$ and so, $t-\bar{r} \,|\, q(t)+p(t)\,\mathbf{i}$. On the other hand, we have that the relation $t-r \nmid q(t)+p(t)\,\mathbf{i}$ implies $t-\bar{r} \nmid q(t)-p(t)\,\mathbf{i}$. Thus, the relation

$$(t-r)(t-\bar{r}) \,|\, (u(t)+v(t)\,\mathbf{i})\,(q(t)-p(t)\,\mathbf{i})$$

implies that $t-\bar{r} \,|\, u(t)+v(t)\,\mathbf{i}$. Thus, we deduce that the polynomials $u(t)+v(t)\,\mathbf{i}$ and $q(t)+p(t)\,\mathbf{i}$ have a common complex root. If $t-r \,|\, q(t)+p(t)\,\mathbf{i}$, then we also have that $u(t)+v(t)\,\mathbf{i}$ and $q(t)+p(t)\,\mathbf{i}$ have a common complex root. Since $\mathcal{A}(t) = u(t)+v(t)\,\mathbf{i} + \mathbf{k}\,(q(t)+p(t)\,\mathbf{i})$, it follows that $\mathcal{A}(t)$ has a complex root.

Suppose that $\mathcal{A}(t) = \mathcal{B}(t)\,\mathcal{C}(t)$, where $\mathcal{B}(t) \in \mathbb{H}[t]$ and $\mathcal{C}(t)$ is a monic polynomial of $\mathbb{C}[t] \setminus \mathbb{R}[t]$ with $\deg C(t) > 0$. Then we have

$$\mathbf{r}'(t) = f(t)\,\mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t),$$

where $f(t) = \mathcal{C}(t)\,\mathcal{C}^*(t)$ is a real monic polynomial with non real root. It follows that the hodograph $\mathbf{r}'(t)$ is non-primitive.

Thus, we have established the equivalence of propositions (a), (b) and (c). Finally, Lemma 2 provides the equivalence of (b) and (d). $\square$

Corollary 1. *The hodograph* $\mathbf{r}'(t)$ *is primitive if and only if the quaternion polynomial* $\mathcal{A}(t)$ *has no complex roots.*

Lemma 3 and Corollary 1 give immediately the following result.

Corollary 2. *Suppose that the hodograph* $\mathbf{r}'(t)$ *is generated by the polynomial* $\mathcal{A}(t) = t^2+\mathcal{B}t+\mathcal{C}$. *Set* $\mathcal{B} = b_1 + \mathbf{k}\, c_1$ *and* $\mathcal{C} = b_0 + \mathbf{k}\, c_0$, *where* $b_0, b_1, c_0, c_1 \in \mathbb{C}$. *Then,* $\mathbf{r}'(t)$ *is non-primitive if and only if*

$$c_0^2 - c_0 b_1 c_1 + b_0 c_1^2 = 0.$$

Remark 1. From the proof of Theorem 1 we also conclude that if the hodograph $\mathbf{r}'(t)$ is non-primitive then $\gcd(x'(t), y'(t), z'(t))$ has no real roots.

We call the maximum of degrees of complex polynomials $C(t) \in \mathbb{C}[t]$ with the property, that there exists a quaternion polynomial $\mathcal{B}(t) \in \mathbb{H}[t]$ such that $\mathcal{A}(t) = \mathcal{B}(t)\, C(t)$, the *level of non-primitivity* of the hodograph $\mathbf{r}'(t)$ generated by the quaternion polynomial $\mathcal{A}(t)$, and we denote it by $\ell(\mathbf{r}'(t))$. Note that the level of non-primitivity of a primitive hodograph is zero.

Combining Lemma 1 and Theorem 1, we obtain the following result:

Theorem 2. *The level of non-primitivity of the hodograph* $\mathbf{r}'(t)$ *generated by the polynomial* $\mathcal{A}(t)$ *is*

$$\ell(\mathbf{r}'(t)) = \deg \mathcal{A} - \operatorname{rank} \operatorname{Hbez}(f, g).$$

Since the polynomial $\mathcal{B}(t)$ generates the hodograph $\hat{\mathbf{r}}'(t) = \mathcal{B}(t)\, \mathbf{i}\, \mathcal{B}^*(t)$ we can give the following definition. We say that a polynomial curve $\mathbf{r}(t)$ is *generated* by another polynomial curve $\hat{\mathbf{r}}(t)$, and we write $\mathbf{r}(t) \succeq \hat{\mathbf{r}}(t)$, if $\mathbf{r}'(t) = h(t)\, \hat{\mathbf{r}}'(t)$ for some monic real polynomial $h(t)$ with non real roots. We shall also say that the curve $\hat{\mathbf{r}}(t)$ *generates* the curve $\mathbf{r}(t)$.

Clearly, a PH curve with a non-primitive hodograph is generated by another PH curve, of lower degree. Such curves are defined by quaternion polynomials $\mathcal{A}(t)$ that admit factorizations of the form $\mathcal{A}(t) = \mathcal{B}(t)\, C(t)$, where $C(t)$ is a non-constant complex polynomial with no real roots. Thus, a PH curve $\mathbf{r}(t)$ is generated by another PH curve of lower degree if and only if the level of non-primitivity of its hodograph $\mathbf{r}'(t)$ is > 0.

Proposition 1. *The relation* $\succeq$ *is a partial ordering in the set of polynomial curves* $\mathcal{P}$.

Proof. For every $\mathbf{r}(t) \in \mathcal{P}$ we clearly have $\mathbf{r}(t) \succeq \mathbf{r}(t)$. Suppose that $\mathbf{r}(t) \succeq \hat{\mathbf{r}}(t)$ and $\hat{\mathbf{r}}(t) \succeq \mathbf{r}(t)$. Then there are real monic polynomials $f(t)$ and $g(t)$ with non-real roots such that $\mathbf{r}'(t) = f(t)\, \hat{\mathbf{r}}'(t)$ and $\hat{\mathbf{r}}'(t) = g(t)\mathbf{r}'(t)$. Thus, we get $\mathbf{r}'(t) = f(t)\, g(t)\mathbf{r}'(t)$, whence we obtain $f(t) = g(t) = 1$. Hence $\mathbf{r}(t) = \hat{\mathbf{r}}(t)$. Finally, suppose that $\mathbf{r}_1(t) \succeq \mathbf{r}_2(t)$ and $\mathbf{r}_2(t) \succeq \mathbf{r}_3(t)$. It follows that there are real monic polynomials $f_1(t)$ and $f_2(t)$ with non-real roots such that $\mathbf{r}_1'(t) = f_1(t)\, \mathbf{r}_2'(t)$ and $\mathbf{r}_2'(t) = f_2(t)\mathbf{r}_3'(t)$. Thus, we have $\mathbf{r}_1'(t) = f_1(t)\, f_2(t)\mathbf{r}_3'(t)$, whence we get $\mathbf{r}_1(t) \succeq \mathbf{r}_3(t)$. Hence, the relation $\succeq$ is reflexive, antisymmetric and transitive and so is a partial ordering in $\mathcal{P}$. □

Remark 2. The polynomial curves having primitive hodograph are the minimal elements of this ordering.

5. Geometrical properties

Here we discuss the geometrical interpretation of a PH curve which is generated by another PH curve of lower degree. More precisely, we are interested in finding the relation between these curves in the space and if the geometrical properties of the one are transferred to the other.

Let $\mathbf{r}(t)$ and $\hat{\mathbf{r}}(t)$ be PH curves defined by monic quaternion polynomials

$$\mathcal{A}(t) = u(t) + \mathbf{i}\, v(t) + \mathbf{j}\, p(t) + \mathbf{k}\, q(t), \quad \mathcal{B}(t) = \hat{u}(t) + \mathbf{i}\, \hat{v}(t) + \mathbf{j}\, \hat{p}(t) + \mathbf{k}\, \hat{q}(t)$$

satisfying $\mathbf{r}'(t) = f(t)\, \mathcal{B}(t)\, \mathbf{i}\, \mathcal{B}^*(t)$, for some monic real polynomial $f(t)$ with no real roots. Let $(\mathbf{t}, \mathbf{h}, \mathbf{b})$, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, $\kappa_1(t)$, $\tau_1(t)$, $\sigma_1(t)$ and $(\hat{\mathbf{t}}, \hat{\mathbf{h}}, \hat{\mathbf{b}})$, $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$, $\kappa_2(t)$, $\tau_2(t)$, $\sigma_2(t)$ be the Frenet frame, Euler-Rodrigues frame, curvature, torsion and parametric speed of $\mathbf{r}(t)$, $\hat{\mathbf{r}}(t)$ at each t, respectively.

If $\mathbf{r}(t) \succeq \hat{\mathbf{r}}(t)$, then at each t, the curves $\mathbf{r}(t)$ and $\hat{\mathbf{r}}(t)$ have the same Frenet and Euler-Rodrigues frames. Further, the parametric speed of $\mathbf{r}(t)$ is equal to that of $\hat{\mathbf{r}}(t)$ multiplied by $|f(t)|$, while the curvature and torsion of $\mathbf{r}(t)$ equal those of $\hat{\mathbf{r}}(t)$ divided by $|f(t)|$ and $f(t)$, respectively.

Indeed, if we substitute $\mathbf{r}' = f\, \hat{\mathbf{r}}'$ and its derivatives into the definitions of the tangent, principal normal, and binormal,

$$\mathbf{t} \;=\; \frac{\mathbf{r}'}{|\mathbf{r}'|}\,, \quad \mathbf{h} \;=\; \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \times \mathbf{t}\,, \quad \mathbf{b} \;=\; \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|}\,,$$

and the parametric speed, curvature, and torsion,

$$\sigma \;=\; |\mathbf{r}'|\,, \quad \kappa \;=\; \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}\,, \quad \tau \;=\; \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}\,.$$

We obtain

$$\mathbf{t} = \hat{\mathbf{t}}, \quad \mathbf{h} = \hat{\mathbf{h}}, \quad \mathbf{b} = \hat{\mathbf{b}}$$

and

$$\sigma_1(t) = |f(t)|\, \sigma_2(t), \qquad \frac{\kappa_1}{\kappa_2} = \frac{1}{|f(t)|}, \text{ and } \frac{\tau_1}{\tau_2} = \frac{1}{f(t)} \text{ for each } t.$$

By Theorem 1, we have $f(t) > 0$ and so, we get $\kappa_1(t)\tau_2(t) = \kappa_2(t)\tau_1(t)$, for the corresponding points of $\mathbf{r}(t)$ and $\hat{\mathbf{r}}(t)$.

Now, we shall prove that $\mathbf{e}_1 = \hat{\mathbf{e}}_1$, $\mathbf{e}_2 = \hat{\mathbf{e}}_2$, $\mathbf{e}_3 = \hat{\mathbf{e}}_3$, at each t. Indeed, since $\mathbf{r}'(t) = f(t)\hat{\mathbf{r}}'(t)$, we have that

$$\mathcal{A}(t)\, \mathbf{i}\, \mathcal{A}^*(t) = f(t)\mathcal{B}(t)\, \mathbf{i}\, \mathcal{B}^*(t), \tag{13}$$

which is equivalent to

$$\begin{aligned} u^2(t) + v^2(t) - p^2(t) - q^2(t) \;&=\; f(t)\, [\hat{u}^2(t) + \hat{v}^2(t) - \hat{p}^2(t) - \hat{q}^2(t)], \\ u(t)q(t) + v(t)p(t) \;&=\; f(t)\, [\hat{u}(t)\hat{q}(t) + \hat{v}(t)\hat{p}(t)], \\ v(t)q(t) - u(t)p(t) \;&=\; f(t)\, [\hat{v}(t)\hat{q}(t) - \hat{u}(t)\hat{p}(t)]. \end{aligned} \tag{14}$$

By multiplying (13) by $\mathcal{B}(t)$ and $\mathcal{A}^*(t)$ from the right and the left, respectively, we obtain

$$|\mathcal{A}(t)|^2\, \mathbf{i}\, \mathcal{A}^*(t)\, \mathcal{B}(t) = f(t)\, |\mathcal{B}(t)|^2\, \mathcal{A}^*(t)\, \mathcal{B}(t)\, \mathbf{i}$$

which implies that

$$|\mathcal{A}(t)|^2\, |\mathbf{i}\, \mathcal{A}^*(t)\, \mathcal{B}(t)| = |f(t)|\, |\mathcal{B}(t)|^2\, |\mathcal{A}^*(t)\, \mathcal{B}(t)\, \mathbf{i}|. \tag{15}$$

Since $f(t) > 0$ and $|\mathbf{i}\, \mathcal{A}^*(t)\, \mathcal{B}(t)| = |\mathcal{A}^*(t)\, \mathcal{B}(t)\, \mathbf{i}|$, (15) gives

$$|\mathcal{A}(t)|^2 = f(t)\, |\mathcal{B}(t)|^2. \tag{16}$$

Substituting (14) and (16) into (8) and simplifying, we deduce

$$\mathbf{e}_1 = \hat{\mathbf{e}}_1, \quad \mathbf{e}_2 = \hat{\mathbf{e}}_2 \;\text{ and }\; \mathbf{e}_3 = \hat{\mathbf{e}}_3 \text{ at each } t.$$

Next, we shall prove that if the $\mathbf{r}(t)$ is a planar curve (straight line) then $\hat{\mathbf{r}}(t)$ is a planar curve (straight line) and conversely. Write $\mathcal{A}(t) = \alpha(t) + \mathbf{k}\,\beta(t)$ and $\mathcal{B}(t) = \gamma(t) + \mathbf{k}\,\delta(t)$, where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ are polynomials of $\mathbb{C}[t]$ and $\mathcal{A}(t) = \mathcal{B}(t)C(t)$, where $C(t) \in \mathbb{C}[t] \backslash \mathbb{C}$. Then, we have $\alpha(t) = \gamma(t)C(t)$ and $\beta(t) = \delta(t)C(t)$. So, we deduce that $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$, where

$$x'(t) = |\alpha(t)|^2 - |\beta(t)|^2, \quad y'(t) = 2\,\mathrm{Re}\,(\alpha(t)\bar{\beta}(t)), \quad z'(t) = 2\,\mathrm{Im}\,(\alpha(t)\bar{\beta}(t)).$$

The curve $\mathbf{r}(t)$ is planar if and only if $x'(t)$, $y'(t)$ and $z'(t)$ are linearly dependent. Since $\mathcal{A}(t)$ is monic, the degree of $x'(t)$ is bigger than the degree of $y'(t)$ and $z'(t)$. Thus, $x'(t)$, $y'(t)$ and $z'(t)$ are linearly dependent if and only if $y'(t)$ and $z'(t)$ are linearly dependent. Thus, $\mathbf{r}(t)$ is a planar curve if and only if the polynomials $\mathrm{Re}(\alpha(t)\bar{\beta}(t))$ and $\mathrm{Im}(\alpha(t)\bar{\beta}(t))$ are linearly dependent. Similarly, $\hat{\mathbf{r}}(t)$ is planar if and only if $\mathrm{Re}(\gamma(t)\bar{\delta}(t))$ and $\mathrm{Im}(\gamma(t)\bar{\delta}(t))$ are linearly dependent. Since

$$\big(\mathrm{Re}\,(\alpha(t)\bar{\beta}(t)),\ \mathrm{Im}\,(\alpha(t)\bar{\beta}(t)\big) = |C(t)|^2 \left(\mathrm{Re}\,(\gamma(t)\bar{\delta}(t))\,,\ \mathrm{Im}\,\left(\gamma(t)\bar{\delta}(t)\right)\right),$$

we deduce that $\mathbf{r}(t)$ is a planar curve if and only if $\hat{\mathbf{r}}(t)$ is a planar curve.

The curve $\mathbf{r}(t)$ is a straight line if and only if $x'(t)$, $y'(t)$ and $x'(t)$, $z'(t)$ are linearly dependent. The degree of $x'(t)$ is bigger than the degrees of $y'(t)$, $z'(t)$ and so, we have that $\mathbf{r}(t)$ is a straight line if and only if $y'(t) = z'(t) = 0$. Thus $\mathbf{r}(t)$ is a straight line if and only if $\mathrm{Re}\,(\alpha(t)\bar{\beta}(t)) = \mathrm{Im}\,(\alpha(t)\bar{\beta}(t)) = 0$ which is equivalent to $\alpha(t)\bar{\beta}(t) = 0$. Since $\mathcal{A}(t)$ is monic, we have $\alpha(t) \neq 0$, and so $\mathbf{r}(t)$ is a straight line if and only if $\beta(t) = 0$ which is equivalent to $\mathcal{A}(t) \in \mathbb{C}[t]$. Similarly, $\hat{\mathbf{r}}(t)$ is a straight line if and only if and only if $\mathcal{B}(t) \in \mathbb{C}[t]$. Since $\mathcal{A}(t) = \mathcal{B}(t)C(t)$, where $C(t) \in \mathbb{C}[t]$, we have that $\mathcal{A}(t) \in \mathbb{C}[t]$ if and only if $\mathcal{B}(t) \in \mathbb{C}[t]$. Hence $\mathbf{r}(t)$ is a straight line if and only if $\hat{\mathbf{r}}(t)$ is a straight line.

Up to now, we have proved that $\mathbf{r}(t)$ is a planar curve if and only if $\hat{\mathbf{r}}(t)$ is a planar curve and $\mathbf{r}(t)$ is a straight line if and only if $\hat{r}(t)$ is a straight line. Consequently, if $\mathbf{r}(t)$ is a (true) space curve then $\hat{\mathbf{r}}(t)$ is a (true) space curve and conversely.

Concerning the RRMF condition, one can verify that $\mathbf{r}(t)$ is a RRMF curve if and only if $\hat{\mathbf{r}}(t)$ is a RRMF curve. Suppose, as previously, that the quaternion polynomials $\mathcal{A}(t) = \alpha(t) + \mathbf{k}\,\beta(t)$ and $\mathcal{B}(t) = \gamma(t) + \mathbf{k}\,\delta(t)$, are associated to the curves $\mathbf{r}(t)$ and $\hat{\mathbf{r}}(t)$, respectively. Furthermore, we have $\mathcal{A}(t) = \mathcal{B}(t)C(t)$, where $C(t) \in \mathbb{C}[t] \setminus \mathbb{C}$. Suppose first that $\hat{\mathbf{r}}(t)$ is a RRMF curve. Then there exists a polynomial $w(t) = w_1(t) + \mathbf{i}\,w_2(t)$ where $w_1(t), w_2(t) \in \mathbb{R}[t]$ with no real factor such that

$$\frac{\mathrm{Im}\,\left(\bar{\gamma}(t)\gamma'(t) + \bar{\delta}(t)\delta'(t)\right)}{|\gamma(t)|^2 + |\delta(t)|^2} = \frac{\mathrm{Im}\,\left(\bar{w}(t)w'(t)\right)}{|w(t)|^2}.$$

Then, we easily verify that the following equality holds:

$$\frac{\mathrm{Im}\,\left(\bar{\alpha}(t)\alpha'(t) + \bar{\beta}(t)\beta'(t)\right)}{|\alpha(t)|^2 + |\beta(t)|^2} = \frac{\mathrm{Im}\,\left(\overline{wC}(t)(wC)'(t)\right)}{|(wC)(t)|^2}.$$

Since $\mathcal{A}(t)$ is primitive, $C(t)$ has no real factor, and so the polynomial $(wC)(t)$ has also no real factor. Hence $\mathbf{r}(t)$ is a RRMF curve. Suppose next that $\mathbf{r}(t)$ is a RRMF curve. Then there exists a polynomial $V(t) = v_1(t) + \mathbf{i}\, v_2(t)$ where $v_1(t), v_2(t) \in \mathbb{R}[t]$ with $\gcd(v_1(t), v_2(t)) = 1$ such that

$$\frac{\mathrm{Im}\,\left(\bar{\alpha}(t)\alpha'(t) + \bar{\beta}(t)\beta'(t)\right)}{|\alpha(t)|^2 + |\beta(t)|^2} = \frac{\mathrm{Im}\,\left(\bar{V}(t)V'(t)\right)}{|V(t)|^2}.$$

Consequently,

$$\frac{\mathrm{Im}\,\left(\bar{\gamma}(t)\gamma'(t) + \bar{\delta}(t)\delta'(t)\right)}{|\gamma(t)|^2 + |\delta(t)|^2} = \frac{\mathrm{Im}\,\left(\bar{V}(t)V'(t)\right)}{|V(t)|^2} - \frac{\mathrm{Im}\,\left(\bar{C}(t)C'(t)\right)}{|C(t)|} = \frac{\mathrm{Im}\,\left(\overline{V\bar{C}}(t)(V\bar{C})'(t)\right)}{|(V\bar{C})(t)|^2}.$$

Furthermore, $(V\bar{C})(t)$ has no real factor. Hence $\hat{\mathbf{r}}(t)$ is a RRMF curve.

The above discussion is summarized as follows:

Proposition 2. *Let $\mathbf{r}(t)$, $\hat{\mathbf{r}}(t)$ be PH curves with $\mathbf{r}(t) \succeq \hat{\mathbf{r}}(t)$. Then, we have:*

1. $\mathbf{r}(t)$ *and* $\hat{\mathbf{r}}(t)$ *have the same Frenet and Euler-Rodrigues frames at each* t*.*
2. $\mathbf{r}(t)$ *is planar, straight line and true space curve if and only if* $\hat{\mathbf{r}}(t)$ *is likewise, respectively.*
3. $\mathbf{r}(t)$ *is an RRMF curve if and only if* $\hat{\mathbf{r}}(t)$ *is an RRMF curve.*
4. $\kappa_1(t)\tau_2(t) = \kappa_2(t)\tau_1(t)$*, where* $\kappa_1(t)$*,* $\kappa_2(t)$ *and* $\tau_1(t)$*,* $\tau_2(t)$ *are the curvature and torsion of* $\mathbf{r}(t)$*,* $\hat{\mathbf{r}}(t)$ *respectively.*

6. RRMF curves of degrees 5 and 7

In this section we study RRMF curves of degrees 5 and 7 whose hodographs are non-primitive and possess the specific geometrical property of having the ERF as an RMF (ERF = RMF). More precisely, in case of the RRMF curves of degree 7, we find the necessary and sufficient conditions under which an RRMF curve of degree 7 with a non-primitive hodograph has ERF as an RMF, and we prove that these curves are generated only by RRMF quintic curves of Class II with primitive hodographs. Recall that RRMF quintics of Class II are those which satisfy condition (9) with linear polynomials $a(t)$ and $b(t)$ [12]. For the case of the RRMF quintic curves, we study only two sets: the RRMF quintic curves of Class II with non-primitive hodographs and the RRMF quintics with ERF = RMF with non-primitive hodographs as well.

6.1. RRMF curves of degree 5 with non-primitive hodographs

The simplest example of two PH curves $\mathbf{r}(t)$, $\hat{\mathbf{r}}(t)$ such that $\mathbf{r}(t) \succeq \hat{\mathbf{r}}(t)$ concerns the case of a quintic $\mathbf{r}(t)$ defined by a quadratic quaternion polynomial $\mathcal{A}(t)$ that admits a factorization of the form $\mathcal{A}(t) = \mathcal{B}(t)(t - z)$, where $z \in \mathbb{C}$. The PH quintic $\mathbf{r}(t)$ is generated by the PH cubic $\hat{\mathbf{r}}(t)$ defined by the hodograph $\hat{\mathbf{r}}'(t) = \mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t)$. We shall give necessary and sufficient conditions under which an RRMF quintic with non-primitive hodograph is of Class II. Moreover, we prove that quintics with non-primitive hodograph with ERF = RMF do not exist.

Consider the quaternion polynomial (7) where $\alpha(t), \beta(t)$ are considered to be in normal form [12, Lemma 1], i.e., $\alpha(t)$ is a monic quadratic polynomial and $\beta(t)$ a linear polynomial. We assume that $\mathcal{A}(t)$ has one non-real complex root. Then $\alpha(t) = (t - z_1)(t - z_2)$ and $\beta(t) = c(t - z_2)$ with $c \in \mathbb{C}$.

Proposition 3. *The polynomial $\mathcal{A}(t)$ defines a non-primitive hodograph of a RRMF quintic of Class II, if and only if $Im(z_1) = 0$. In this case, the polynomial $w(t) = a(t) + \mathbf{i}\, b(t)$, with $a(t) = t - Re(z_2)$ and $b(t) = -Im(z_2)$, satisfies condition (10). Furthermore, $\mathcal{A}(t)$ does not define a non-primitive hodograph of an RRMF quintic curve with ERF = RMF.*

Proof. The polynomial $\mathcal{A}(t)$ defines a hodograph of a RRMF quintic of Class II if and only if there exists a complex polynomial $w(t) = a(t) + \mathbf{i}\, b(t)$ with $a(t) = t - a_0$, $b(t) = b_0$ and $\gcd(a(t), b(t)) = 1$ such that (10) is valid.

By plugging $z_i = a_i + \mathbf{i}\, b_i$, $(i = 1, 2)$ into $\alpha(t)$ and $\beta(t)$ and substituting $\alpha(t)$ and $\beta(t)$ into the left part of (10), we obtain

$$\frac{\operatorname{Im}(\alpha c \alpha' + \beta c \beta')}{|\alpha|^2 + |\beta|^2} = \frac{(b_1 + b_2)t^2 - 2(b_1 a_2 + a_1 b_2)t - b_2|z_1|^2 - b_1|z_2|^2 + |c|^2 b_2}{[(t - a_1)^2 + b_1^2 + |c|^2][(t - a_2)^2 + b_2^2]}.$$

Thus, (10) yields:

$$(b_1 + b_2)\frac{t^2 - 2\,\dfrac{b_1 a_2 + a_1 b_2}{b_1 + b_2}t - \dfrac{b_2|z_1|^2 + b_1|z_2|^2 - |c|^2 b_2}{b_1 + b_2}}{[(t - a_1)^2 + b_1^2 + |c|^2][(t - a_2)^2 + b_2^2]} = \frac{-b_0}{(t - a_0)^2 + b_0^2}. \tag{17}$$

The denominator of the left side is a real polynomial of degree 4 and of the right side a polynomial of degree 2. Hence, the numerator of the left side is required to be of degree 2 and so $b_1 + b_2 \neq 0$. We have the following two cases: either

$$t^2 - 2\,\frac{b_1 a_2 + a_1 b_2}{b_1 + b_2}t - \frac{b_2|z_1|^2 + b_1|z_2|^2 - |c|^2 b_2}{b_1 + b_2} = (t - a_1)^2 + b_1^2 + |c|^2$$

and

$$\frac{b_1 + b_2}{(t - a_2)^2 + b_2^2} = \frac{-b_0}{(t - a_0)^2 + b_0^2}$$

or

$$t^2 - 2\frac{b_1 a_2 + a_1 b_2}{b_1 + b_2}t - \frac{b_2|z_1|^2 + b_1|z_2|^2 - |c|^2 b_2}{b_1 + b_2} = (t - a_2)^2 + b_2^2$$

and

$$\frac{b_1 + b_2}{(t - a_1)^2 + b_1^2 + |c|^2} = \frac{-b_0}{(t - a_0)^2 + b_0^2}.$$

Thus (17) holds if and only if either

$$\tfrac{b_1 a_2 + a_1 b_2}{b_1 + b_2} = a_1, \quad -\tfrac{b_2|z_1|^2 + b_1|z_2|^2 - |c|^2 b_2}{b_1 + b_2} = b_1^2 + |c|^2 + a_1^2,$$
$$b_0^2 = b_2^2, \quad -b_0 = b_1 + b_2, \quad a_0 = a_2$$

or

$$\tfrac{b_1 a_2 + a_1 b_2}{b_1 + b_2} = a_2, \quad -\tfrac{b_2|z_1|^2 + b_1|z_2|^2 - |c|^2 b_2}{b_1 + b_2} = b_2^2 + a_2^2,$$
$$b_0^2 = b_1^2 + |c|^2, \quad b_0 = -(b_1 + b_2), \quad a_0 = a_1.$$

Combining the third and fourth equation of the first system, we obtain $(b_1 + b_2)^2 = b_2^2$, and so, $b_1 = 0$ or $b_1 = -2\, b_2$. By setting $b_1 = 0$, we get $(a_0, b_0) = (a_2, -b_2)$. If $b_1 = -2\, b_2$, then $b_2 = 0$ which is a contradiction, since $\mathcal{A}(t)$ is primitive. Now, by the fourth and the third equation of the second system we have $|c|^2 = b_2^2 + 2\, b_1\, b_2$ and from the first equation we get $a_1 = a_2$. Substituting the last two relations into the second we obtain $b_1 = 0$ or $b_1 = -2\, b_2$. For $b_1 = 0$ the second system has the solution $(a_0, b_0) = (a_2, -b_2)$, and by substituting $b_1 = -2\, b_2$ into the second equation, we obtain $3\, b_2^2 + |c|^2 = 0$, which is a contradiction. Hence, condition (17) holds if and only if $b_1 = 0$.

Now, a quintic curve generated by the quaternion polynomial (7), has ERF = RMF if and only if we have

$$(b_1+b_2)t^2 - 2(b_1a_2+a_1b_2)t + b_2|z_1|^2 + b_1|z_2|^2 + |c|^2 b_2 = 0$$

which is equivalent to the following system:

$$b_1 = -b_2, \quad b_1a_2 + a_1b_2 = 0, \quad b_1|z_2|^2 + b_2|z_1|^2 + |c|^2 b_2 = 0.$$

From the second equation we get $b_2 = 0$ or $a_1 = a_2$. But since $\mathcal{A}(t)$ is primitive, $b_2 \neq 0$ and hence we study only the case $a_1 = a_2$. Substituting the last relation and $b_1 = -b_2$ into the third equation of the system we obtain that $\mathcal{A}(t)$ is a real polynomial which is a contradiction. □

Example 1. Choosing the values $c = 1$, $z_1 = 0$ and $z_2 = 1 - \mathbf{i}$, we have $\mathcal{A}(t) = \alpha(t) + \mathbf{k}\,\beta(t)$, where $\alpha(t) = t^2 - t + \mathbf{i}$ and $\beta(t) = t - 1 + \mathbf{i}$. The polynomial $\mathcal{A}(t)$ defines a non-primitive hodograph of an RRMF quintic of Class II. We can easily verify that

$$\frac{\mathrm{Im}\,(\alpha c \alpha' + \beta c \beta')}{|\alpha|^2 + |\beta|^2} = \frac{-t^2 - 1}{t^4 - 2\,t^3 + 3\,t^2 - 2\,t + 2} = -\frac{1}{(t-1)^2 + 1}\,,$$

and the complex polynomial $w(t)$ satisfying condition (10) is $w(t) = t - 1 + \mathbf{i}$. The resulting hodograph is

$$\mathbf{r}'(t) = (x'(t),\, y'(t),\, z'(t)) = (t^2 - 2\,t + 2)(t^2 - 1,\; t,\; 0),$$

and its components define a curve with a non-primitive hodograph and which satisfies (3), where $\sigma(t) = (t^2+1)(t^2 - 2\,t + 2)$.

Proposition 4. *We have the following:*

1. *If* $\hat{\mathbf{r}}(t)$ *is an RRMF curve then* $\mathbf{r}(t)$ *is an RRMF quintic of Class II.*
2. *The curve* $\mathbf{r}(t)$ *is an RRMF quintic of Class II with non-primitive hodograph if and only if it is planar.*

Proof. 1. Let $\mathcal{A}(t) = \alpha(t) + \mathbf{k}\,\beta(t)$ and $\mathcal{B}(t) = \gamma(t) + \mathbf{k}\,\delta(t)$. If $\hat{\mathbf{r}}(t)$ is an RRMF cubic curve, then $\hat{\mathbf{r}}(t)$ is planar. Thus, [3] implies that $\hat{\mathbf{r}}(t)$ has ERF = RMF and so, we have

$$\frac{\mathrm{Im}\,\big(\bar{\gamma}(t)\gamma'(t) + \bar{\delta}(t)\delta'(t)\big)}{|\gamma(t)|^2 + |\delta(t)|^2} = 0\,.$$

Since $\alpha(t) = \gamma(t)(t-z)$ and $\beta(t) = \delta(t)(t-z)$, eq. (10) yields

$$\frac{\mathrm{Im}\,\Big[\big(\bar{\gamma}(t)\gamma'(t) + \bar{\delta}(t)\delta'(t)\big)\,\overline{|t-z|}^{\,2} + \big(|\gamma(t)|^2 + |\delta(t)|^2\big)\,\overline{|t-z|}\Big]}{\big(|\gamma(t)|^2 + |\delta(t)|^2\big)\,|t-z|^2} = \frac{\mathrm{Im}\,\big(\overline{t-z}\big)}{|t-z|^2}.$$

From the last follows that the curve $\mathbf{r}(t)$ satisfies condition (10) with $w(t) = t - z$ and so $\mathbf{r}(t)$ is an RRMF quintic of Class II.
2. Suppose that $\mathbf{r}(t)$ is an RRMF quintic of Class II with non-primitive hodograph. Since $\mathbf{r}(t)$ is an RRMF quintic, Proposition 2(3) implies that $\hat{\mathbf{r}}(t)$ is an RRMF cubic curve, i.e., a planar curve. Further, Proposition 2(2) implies that $\mathbf{r}(t)$ is planar. Conversely, if $\mathbf{r}(t)$ is a

planar RRMF quintic of Class II, then by [12, Prop 2] a parameterization of $\mathcal{A}(t)$ is given as follows

$$\mathcal{A}(t) = t^2 + u_1 t - (u_1 + r)r + \mathbf{i}\,[v_1 t + (u_1 + r)v_1] + \mathbf{j}\,(p_1 t + v_1 q_1 - p_1 r) + \mathbf{k}\,(q_1 t - v_1 p_1 - q_1 r)$$

where r, u_1, v_1, p_1, q_1 are the free real variables. We can easily see that $\mathcal{A}(t)$ satisfies the conditions of Corollary 2, and thus $\mathbf{r}(t)$ has a non-primitive hodograph. □

Remark 3. Combining Propositions 4(3) and 2(2), we have that all RRMF quintics curves of Class II generated by cubics are planar.

6.2. RRMF curves of degree 7 with non-primitive hodographs

In this section, we study the PH curves of degree 7 which have a rotation-minimizing ERF and the corresponding cubic polynomial $\mathcal{A}(t) \in \mathbb{H}[t] \setminus \mathbb{C}[t]$ has a complex right factor. We shall prove that these curves are generated only by RRMF quintic curves of Class II. Furthermore, we shall give a parametrization of these curves.

Suppose now that $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$ defines a PH curve of degree 7. By [12, Lemma 1], we may write

$$\begin{aligned} u(t) &= t^3 + u_2 t^2 + u_1 t + u_0, & v(t) &= v_2 t^2 + v_1 t + v_0,\\ p(t) &= p_2 t^2 + p_1 t + p_0, & q(t) &= q_2 t^2 + q_1 t + q_0. \end{aligned}$$

In [11], necessary and sufficient conditions are given for a PH curve of degree 7 having ERF = RMF, in terms of the Hopf map form. For our purposes, we need necessary and sufficient conditions given in terms of the coefficients of $u(t)$, $v(t)$, $p(t)$ and $q(t)$ which are provided in the next lemma.

Lemma 4. *The PH curve of degree 7 defined by $\mathcal{A}(t)$ has the ERF as RMF if and only if*

$$\begin{aligned} v_1 = v_2 &= 0, & u_2 v_0 + p_0 q_2 - p_2 q_0 &= 0,\\ u_1 v_0 + p_0 q_1 - p_1 q_0 &= 0, & 3v_0 + p_1 q_2 - p_2 q_1 &= 0. \end{aligned} \tag{18}$$

Proof. It is easily verified that the relations (18) of Lemma 4 follow by substituting $u(t), v(t)$, $p(t), q(t)$ into (12) and by elementary operations. □

Suppose that there is $\mathcal{B}(t) \in \mathbb{H}[t] \setminus \mathbb{C}[t]$ and $z \in \mathbb{C} \setminus \mathbb{R}$ such that

$$\mathcal{A}(t) = \mathcal{B}(t)(t - z). \tag{19}$$

$\mathcal{A}(t)$ generates the hodograph $\mathbf{r}'(t)$ of a PH curve of degree 7 and $\mathcal{B}(t)$ the hodograph $\hat{\mathbf{r}}'(t)$ of the quintic one. We have the following two cases:

First Case: $\ell(\mathbf{r}'(t)) = 1$. Write $\mathcal{B}(t) = U(t) + \mathbf{i}\,V(t) + \mathbf{j}\,P(t) + \mathbf{k}\,Q(t)$. By [12, Lemma 1], we may assume

$$\begin{aligned} U(t) &= t^3 + u_2' t^2 + u_1' t + u_0', & V(t) &= v_2' t^2 + v_1' t + v_0',\\ P(t) &= p_2' t^2 + p_1' t + p_0', & Q(t) &= q_2' t^2 + q_1' t + q_0'. \end{aligned}$$

Substituting

$$\mathcal{B}(t) = t^2 + u_1' t + u_0' + \mathbf{i}\,(v_1' t + v_0') + \mathbf{j}\,(p_1' t + p_0') + \mathbf{k}\,(q_1' t + q_0')$$

and $z = a + b\,\mathbf{i}$ into (19), we obtain:

$$\begin{aligned}\mathcal{A}(t) \,=\, & t^3 + (u_1' - a)\,t^2 + (u_0' - au_1' - bv_1')\,t - au_0' - bv_0' \\ & + \mathbf{i}\,[\,(v_1' + b)\,t^2 + (v_0' - av_1' + bu_1')\,t + bu_0' - av_0'\,] \\ & + \mathbf{j}\,[\,p_1't^2 + (p_0' - ap_1' + bq_1')\,t + bq_0' - ap_0'\,] \\ & + \mathbf{k}\,[\,q_1't^2 + (q_0' - aq_1' - bp_1')\,t - aq_0' - bp_0'\,],\end{aligned}$$

If $\mathcal{A}(t)$ generates an RRMF curve of degree 7 with ERF = RMF, then conditions (18) are satisfied by $\mathcal{A}(t)$ and thus we have

$$\begin{gathered} v_1' + b = 0, \quad v_0' - av_1' + bu_1' = 0, \quad p_1'q_0' - p_1'^2 b - p_0'q_1' - bq_1'^2 = 3(bu_0' - av_0'), \\ bu_0'u_1' - au_1'v_0' - abu_0' + a^2v_0' + bq_0'q_1' - ap_0'q_1' + aq_0'p_1' + bp_0'p_1' = 0 \end{gathered}$$

and

$$\begin{gathered} bu_0'^2 - au_0'v_0' - abu_0'u_1' + a^2u_1'v_0' - b^2u_0'v_1' + abv_0'v_1' + bq_0'^2 - b^2q_0'p_1' \\ + a^2p_0'q_1' + bp_0'^2 - a^2q_0'p_1' + b^2p_0'q_1' = 0. \end{gathered}$$

The first two equations imply $b = -v_1'$ and $a = v_0'/v_1' - u_1'$ (with $b = -v_1' \neq 0$ since $z \in \mathbb{C}\setminus\mathbb{R}$) and, substituting to the next three, we take the necessary and sufficient conditions for the coefficients of $\hat{\mathbf{r}}'(t)$ in order to generate an RRMF curve of degree 7 with ERF = RMF:

$$\begin{aligned} & p_1'q_0' - p_0'q_1' + v_1'(p_1'^2 + q_1'^2) = -3\left(v_1'u_0' + v_0'u_1' - \frac{v_0'^2}{v_1'}\right), \\ & -2u_0'u_1'v_1' + \frac{u_1'v_0'^2}{v_1'} + 2u_1'^2v_0' + u_0'v_0' + \frac{v_0'^3}{v_1'^2} - v_1'(q_1'q_0' + p_1'p_0') \\ & - \left(\frac{v_0'}{v_1'} - u_1'\right)p_0'q_1' + (v_0' - u_1')q_0'p_1' = 0, \\ & \left(\frac{v_0' - v_1'u_1'}{v_1'}\right)^2 (u_1'v_0' + p_0'q_1' - p_1'q_0') + (v_0' - v_1'u_1')\left(-\frac{u_0'v_0'}{v_1'} + u_1'u_0' - v_0'v_1'\right) \\ & + v_1'^2(p_0'q_1' - q_0'p_1') + v_1'(p_0'^2 + q_0'^2) - u_0'v_1'(u_0' + v_1'^2) = 0. \end{aligned} \tag{20}$$

Now, we shall also determine the set of curves $\mathbf{r}'(t)$. Working as in the proof of item 1 in Proposition 4, we deduce that $\mathcal{B}(t)$ generates an RRMF quintic curve $\hat{\mathbf{r}}'(t)$ of Class II. Hence, we obtain that each RRMF curve of degree 7 with ERF = RMF having a non-primitive hodograph is generated by an RRMF quintic of Class II. For the converse, we can see that relations (22) and (23) of [12, Prop. 2] verify (20) and we deduce that each RRMF quintic of Class II generates an RRMF curve of degree 7 with ERF = RMF. Hence, from (20), [12, Prop. 2], and from the fact that each RRMF curve of degree 7 with ERF = RMF is generated by an RRMF quintic of Class II and vice versa follows that we can represent the set of RRMF curves of degree 7 with ERF = RMF, using (22), (23) of [12, Prop. 2], in terms of the coefficients of the RRMF quintic of Class II, as follows:

$$\begin{aligned} & u_0 \,=\, (u_1' + a)(v_1'^2 + a^2), && u_1 \,=\, -2u_1'a - a^2 + v_1'^2, && v_0 \,=\, v_1 = v_2 = 0, \\ & p_0 \,=\, p_1'(v_1'^2 + a^2), && p_1 \,=\, -2p_1'a, && q_0 \,=\, q_1'(v_1'^2 + a^2), \\ & q_1 \,=\, -2q_1'a, && p_2 \,=\, p_1', && q_2 \,=\, q_1', \end{aligned} \tag{21}$$

or

$$\begin{aligned}
u_0 &= (u_1'+a)(a^2+v_1'^2)+\frac{4av_1'^2(p_1'^2+q_1'^2)}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\,,\\
u_1 &= -2u_1'a-a^2+v_1'^2-\frac{4v_1'^2(p_1'^2+q_1'^2)}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\,,\quad u_2=u_1'-a,\\
v_0 &= \frac{4v_1'^3(p_1'^2+q_1'^2)}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\,,\quad v_1=0,\quad v_2=0,\\
p_0 &= p_1'(v_1'^2+a^2)-4v_1'^2\frac{v_1'[(u_1'+2a)q_1'+3v_1'p_1']+a[(u_1'+2a)p_1'-3v_1'q_1']}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\,,\\
p_1 &= -2p_1'a+\frac{4v_1'^2[(u_1'+2a)p_1'-3v_1'q_1']}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\,,\quad p_2=p_1',\\
q_0 &= q_1'(v_1'^2+a^2)+4v_1'^2\frac{v_1'[(u_1'+2a)p_1'-3v_1'q_1']-a[(u_1'+2a)q_1'+3v_1'p_1']}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\,,\\
q_1 &= -2q_1'a+\frac{4v_1'^2[(u_1'+2a)q_1'+3v_1'p_1']}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\,,\quad q_2=q_1',
\end{aligned}\tag{22}$$

where a, u_1', v_1', p_1', q_1' are free variables with $v_1'\neq 0$.

Second Case: $\ell(\mathbf{r}'(t))=2$.
Then $\mathcal{B}(t)=(t-\mathcal{Q})(t-w)$, where $\mathcal{Q}\in\mathbb{H}\setminus\mathbb{C}$ and $w=c+\mathbf{i}\,d\in\mathbb{C}\setminus\mathbb{R}$. Now, (19) yields

$$\mathcal{A}(t)=(t-\mathcal{Q})(t-w)(t-z),\tag{23}$$

where $z=a+\mathbf{i}\,b$. If $\mathcal{A}(t)$ generates a PH curve of degree 7 with ERF = RMF then item 3 of Proposition 2 implies that $\mathcal{B}(t)$ generates an RRMF quintic curve. Thus, the cubic PH curve which is generated by $t-\mathcal{Q}$ is also an RRMF with ERF = RMF [3]. Let $\mathcal{A}(t)=\alpha(t)+\mathbf{k}\,\beta(\mathbf{t})$ and $\mathcal{B}(t)=\phi(t)+\mathbf{k}\,\psi(t)$. Then, by (23) and (10), we obtain

$$\begin{aligned}
\frac{\operatorname{Im}\left(\bar\alpha(t)\alpha'(t)+\bar\beta(t)\beta'(t)\right)}{|\alpha(t)|^2+|\beta(t)|^2} &= \frac{\operatorname{Im}\left[(\bar\phi(t)\phi'(t)+\bar\psi(t)\psi'(t))|t-w|^2|t-\mathbf{z}|^2\right]}{(|\phi(t)|^2+|\psi(t)|^2)|t-w|^2|t-\mathbf{z}|^2}\\
&\quad+\frac{\operatorname{Im}\,(|\phi(t)|^2+|\psi(t)|^2)\,(|t-\bar w||t-z|^2+|t-\bar z||t-w|^2)}{(|\phi(t)|^2+|\psi(t)|^2)|t-w|^2|t-z|^2}\,.
\end{aligned}$$

Since the PH curve of degree 7 and the cubic curve have ERF = RMF the above relation is equal to

$$\frac{\operatorname{Im}(|t-\bar w||t-z|^2+|t-\bar z||t-w|^2)}{|t-w|^2|t-z|^2}=\operatorname{Im}\frac{1}{t+w}+\operatorname{Im}\frac{1}{t+z}=0.$$

By substituting z and w we get $d=-b$ and $c=a$ and thus w and z are conjugate. Hence $\mathcal{A}(t)$ it is not primitive which is a contradiction.

In view of Proposition 2, the above results can be summarized as follows.

Proposition 5. *Let* $\mathbf{r}(t)$ *be a RRMF curve of degree 7 with ERF = RMF having* $\ell(\mathbf{r}'(t))>0$. *Then,* $\ell(\mathbf{r}'(t))=1$, *and* $\mathbf{r}(t)$ *is generated only by a quintic RRMF curve* $\hat{\mathbf{r}}(t)$ *of Class II and conversely. The curve* $\mathbf{r}(t)$ *is planar (true spacial) if and only if* $\hat{\mathbf{r}}(t)$ *is planar (true spacial). The set of polynomials defining the planar curves* $\mathbf{r}(t)$ *is expressed by equations (21) — in terms of the coefficients of the* $\hat{\mathbf{r}}(t)$ *— and the set of true spatial curves is represented by equations (22).*

Example 2. We consider the quaternion polynomial

$$\mathcal{A}(t)=\left(t-2+\sqrt{5}+\frac{5+2\sqrt{5}}{20}\,\mathbf{i}+\frac{25-3\sqrt{5}}{40}\,\mathbf{j}+\frac{5-\sqrt{5}}{8}\,\mathbf{k}\right)$$
$$\left(t+\frac{15-2\sqrt{5}}{20}\,\mathbf{i}+\frac{15+3\sqrt{5}}{40}\,\mathbf{j}+\frac{3+\sqrt{5}}{8}\,\mathbf{k}\right)(t-\,\mathbf{i}).$$

It is easily seen that $\mathcal{A}(t)$ satisfies (18) and thus it generates an RRMF curve of degree 7 with ERF = RMF. The components $x'(t)$, $y'(t)$, $z'(t)$ of $\mathbf{r}'(t)$ and the parametric speed $\sigma(t)$ are:

$$\begin{aligned}
x'(t) &= t^6+(2\sqrt{5}-4)\,t^5+\frac{40-21\sqrt{5}}{5}\,t^4+\frac{14\sqrt{5}-30}{5}\,t^3\\
&\quad+\frac{78-39\sqrt{5}}{5}\,t^2+\frac{4\sqrt{5}-10}{5}\,t+\frac{43-18\sqrt{5}}{5},\\
y'(t) &= 2\,t^5+\frac{13\sqrt{5}-15}{5}\,t^4+\frac{10-2\sqrt{5}}{5}\,t^3+\frac{12\sqrt{5}-14}{5}\,t^2-\frac{2\sqrt{5}}{5}\,t+\frac{1-\sqrt{5}}{5},\\
z'(t) &= -2\,t^5+(6-2\sqrt{5})\,t^4+\frac{14\sqrt{5}-30}{5}\,t^3\\
&\quad+\frac{55-21\sqrt{5}}{5}\,t^2+\frac{14\sqrt{5}-20}{5}\,t+\frac{25-11\sqrt{5}}{5},\\
\sigma(t) &= t^6+(2\sqrt{5}-4)\,t^5+\frac{60-21\sqrt{5}}{5}\,t^4+(4\sqrt{5}-8)\,t^3\\
&\quad+\frac{105-42\sqrt{5}}{5}\,t^2+(2\sqrt{5}-4)\,t+\frac{50-21\sqrt{5}}{5}.
\end{aligned}$$

The hodograph $\mathbf{r}'(t)$, which is generated by $\mathcal{A}(t)$, is non-primitive since $\mathcal{A}(t)$ has a right complex factor and the curve $\mathbf{r}(t)$ is generated by another curve $\hat{\mathbf{r}}(t)$ with its hodograph $\hat{\mathbf{r}}'(t)$ defined by the polynomial

$$\mathcal{B}(t)=\left(t-2+\sqrt{5}+\frac{5+2\sqrt{5}}{20}\,\mathbf{i}+\frac{25-3\sqrt{5}}{40}\,\mathbf{j}+\frac{5-\sqrt{5}}{8}\,\mathbf{k}\right)$$
$$\left(t+\frac{15-2\sqrt{5}}{20}\,\mathbf{i}+\frac{15+3\sqrt{5}}{40}\,\mathbf{j}+\frac{3+\sqrt{5}}{8}\,\mathbf{k}\right).$$

Since $[\mathbf{r}'(t)\times\mathbf{r}''(t)]\cdot\mathbf{r}'''(t)\neq 0$, for each t, $\mathbf{r}(t)$ is a true space curve.

The next example shows how by using equations (21) and (22), we may generate planar or true space RRMF curves of degree 7 with ERF = RMF.

Example 3. Putting $a=1$, $u_1'=2$, $v_1'=-1$, $p_1'=0$, $q_1'=1$ in (22), we get

$$u_0=\frac{80}{13},\quad u_1=-\frac{54}{13},\quad u_2=1,\quad v_0=-\frac{2}{13},\quad p_0=\frac{2}{13},$$
$$p_1=\frac{6}{13},\quad p_2=0,\quad q_0=\frac{12}{13},\quad q_1=-\frac{16}{13},\quad q_2=1,$$

and hence we have:

$$u(t)=t^3+t^2-\frac{54}{13}\,t+\frac{80}{13},\quad v(t)=-\frac{2}{13},\quad p(t)=\frac{6}{13}\,t+\frac{2}{13},\quad q(t)=t^2-\frac{16}{13}\,t+\frac{12}{13}$$

which satisfy (12). The resulting hodograph components are

$$\begin{aligned}
x'(t) &= t^6+2\,t^5-\frac{108}{13}\,t^4+\frac{84}{13}\,t^3+\frac{4392}{169}\,t^2-\frac{8280}{169}\,t+\frac{6256}{169},\\
y'(t) &= 2\,t^5-\frac{6}{13}\,t^4-\frac{116}{13}\,t^3+\frac{4120}{169}\,t^2-\frac{50080}{2197}\,t+\frac{24976}{2197},\\
z'(t) &= -\frac{12}{13}\,t^4-\frac{16}{13}\,t^3+\frac{604}{169}\,t^2-\frac{9800}{2197}\,t-\frac{4064}{2197},
\end{aligned}$$

and the hodograph defines a true space curve since the torsion of the curve is not zero, for each t. The relations (23) of [12, Proposition 2] imply that the polynomials defining the quintic curve are

$$u'(t) = t^2 + 2t - \frac{41}{13}, \quad v'(t) = -t - 3, \quad p(t) = -\frac{7}{13}, \quad q(t) = t - \frac{5}{13}.$$

7. Conclusion

In this paper a characterization of regular PH curves with non-primitive hodographs which are generated by a primitive quaternion polynomial has been given. As it is proved, these are the curves having an associated quaternion polynomial with a right complex root. Through this work it turns out that there exist RRMF curves $\mathbf{r}(t)$ which are produced by others $\mathbf{r}'(t)$ of lower degree. The geometrical properties of these two sets of curves $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are studied, as well. Finally, the characterization of the case of RRMF curves of 7 degree with non-primitive hodograph and having the ERF as an RMF is presented. Since the above set of RRMF curves of degree 7 produced by RRMF quintics is of Class II, this last set of curves is also studied.

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