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Construction of a Nine-Point Quadric Surface

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Abstract. The fundamental issue of constructing a nine-point quadric was frequently discussed by mathematicians in the 19th century. They failed to find a simple linear geometric dependence that would join ten points of a quadric (similar to Pascal's theorem, which joins six points of a conic section). Nevertheless, they found different algorithms for a geometrically accurate construction (using straightedge and compass or even using straightedge alone) of a quadric that passes through nine given points. While the algorithms are quite complex, they can be implemented only with the help of computer graphics. The paper proposes a simplified computer-based realization of J. H. Engel's well-known algorithm, which makes it possible to determine the nine-point quadric and its axes of symmetry. The proposed graphics algorithm can be considered an alternative to the algebraic solution of the stated problem.

Key Words: biquadratic curves, pencil of quadrics, pencil of conic sections, spatial configuration of Desargues, geometrically accurate construction, computer graphics.

MSC 2010: 51M15, 51M35, 51N20

1. Introduction

Michel Chasles (1793–1880), a French mathematician, said in 1830: *"Despite the important achievements in the theory of quadratic surfaces, it should be noted that these achievements are a very small part of what this theory seems to be capable of"* [1]. This is still relevant. For example, the fundamentally important "Problem of the Tenth Point" remains unsolved in the theory of second-order surfaces (quadrics). It is a matter of a curve which is likely to join ten points of a quadric (similar to B. Pascal's theorem, which joins six points of a conic section), but has not been found yet.

If quadrics have such a curve, it can be regarded as a kind of "projective equivalent" of the algebraic equation of a quadric. The attempts to find the general relation between ten arbitrary points of a quadric undertaken by German and French mathematicians in the 19th century failed to yield any significant results. Much later, the interest in this problem

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completely disappeared. But the efforts were not futile. Different graphical solutions to the problem of constructing a nine-point quadric were found [2, 8, 9]. Unfortunately, the algorithms proved to be very complicated (especially in comparison with B. Pascal's algorithm, which makes it possible to solve a similar problem for conic sections) and practically inapplicable. At the end of the 20th century, there were no fundamental changes, but computer graphics appeared. Computer graphics tools enable making graphical constructions of any complexity. In particular, second-order curves have become as simple and accurately plotted as a straight line and a circle [5, 6]. The new tools may revive the interest in the theory of second-order surfaces.

2. Problem statement

In three-dimensional Euclidean space, let the points 1, 2, ..., 9 be given. Nine points of general position determine a single quadric Θ that passes through these points. These points should not be on the same biquadratic curve, since any number of points if marked on such a curve does not uniquely determine a second-order surface. Three given points should not belong to the same straight line, since the entire straight line and all its points belong to the unknown surface in this case. Six given points should not belong to the same second-order curve, since all points on this curve belong to the desired surface.

Let us draw a line t of general position through one of the points 1, 2, ..., 9. The *"Problem of the Tenth Point"* is formulated as follows: construct the second point of intersection of the straight line t with the quadric Θ (find the tenth point of the quadric) using straightedge.

The problem can be formulated less strictly: construct the quadric Θ (find a graphical algorithm for constructing a set of quadric points) using compass and straightedge.

The stated problem can be extended with the task to construct the principal diameters of the desired quadric Θ. The algebraic solution of this subtask is reduced to solving the characteristic equation of third order. The equivalent geometric solution is reduced to constructing three points of intersection of two conic sections with one common point being known. This is a third-degree problem which cannot be solved using compass and straightedge.

It is noteworthy that the issue of constructing a nine-point quadric, being of fundamental importance, was frequently discussed by mathematicians in the 19th century. A brief review of the solutions is presented in [9]. Let us consider the Rohn and Papperitz' algorithm (1896) [8] and J. H. Engel's algorithm (1889) [2].

3. Rohn and Papperitz' algorithm

Nine points of general position 1, 2, ..., 9 are marked in three-dimensional space. The task is to construct a second-order surface Θ passing through these points.

We mark planes $\alpha(123)$, $\beta(456)$ and $\gamma(789)$. The straight lines $m = \alpha \cap \beta$, $n = \beta \cap \gamma$ and $l = \alpha \cap \gamma$ intersect at the point P. According to [8], the solution is reduced to constructing conic sections $a^2(1,2,3)$, $b^2(4,5,6)$ and $g^2(7,8,9)$ that intersect mutually on the straight lines m, n and l. These conic sections completely determine the desired surface Θ (Figure 1).

Let us simultaneously consider the planes α and β. We mark arbitrary points M, N on the straight line $m = \alpha \cap \beta$. Then, we construct the conic sections a^2 through the points $1, 2, 3, M, N$ in the plane α and b^2 through the points $4, 5, 6, M, N$ in the plane β (Figure 2).

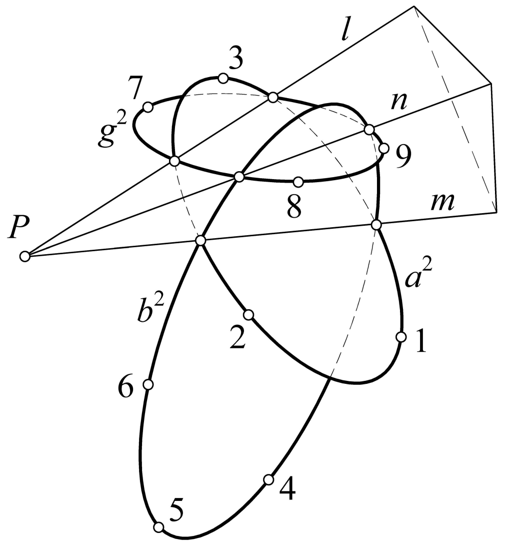

Figure 1: Conics a^2, b^2, g^2 determine the quadric $\Theta(1, 2, \dots, 9)$

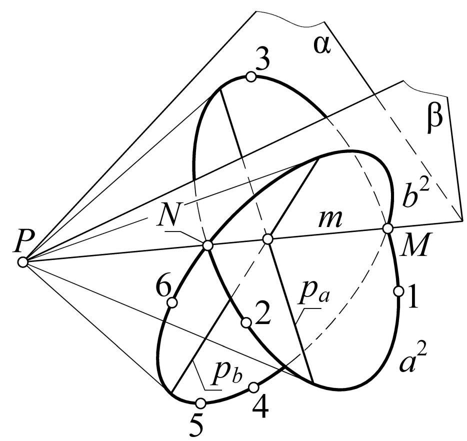

Figure 2: Conjugate conics $a^2 \sim b^2$ and conjugate polars $p_a \sim p_b$

Definition. Two conic sections a^2, b^2 lying in the respective planes α and β are called *conjugate* (symbol $a^2 \sim b^2$) if they intersect at the points lying on the straight line $m = \alpha \cap \beta$. For any point $P \in m$ let p_a, p_b be the polars of P with respect to the conic sections a^2, b^2. Then the polars of conjugate conic sections are also called *conjugate* (symbol $p_a \sim p_b$).

Two conjugate polars intersect at a point on the line m. The set of conjugate conic sections $a^2 \sim b^2$ corresponds to the set of conjugate polars $p_a \sim p_b$.

After specifying arbitrary points M, N on the straight line m, we can draw three pairs of conjugate conic sections $a^2 \sim b^2$, $a'^2 \sim b'^2$, $a''^2 \sim b''^2$ in the planes α, β. The three pairs of conjugate conic sections correspond to three pairs of conjugate polars $p_a \sim p_b$, $p'_a \sim p'_b$, $p''_a \sim p''_b$. The pairs of conjugate polars intersect on the line m and form a Desargues configuration with the center $O_{\alpha\beta}$ and the axis m. The center $O_{\alpha\beta}$ is the intersection of the straight lines which join the corresponding points $(p_a \cap p'_a) \leftrightarrow (p_b \cap p'_b)$, $(p'_a \cap p''_a) \leftrightarrow (p'_b \cap p''_b)$, and $(p_a \cap p''_a) \leftrightarrow (p_b \cap p''_b)$.

The pairs of planes $\alpha \leftrightarrow \gamma$ (with the center $O_{\alpha\gamma}$ and the axis $l = \alpha \cap \gamma$) and $\beta \leftrightarrow \gamma$ (with the center $O_{\beta\gamma}$ and the axis $n = \beta \cap \gamma$) are perspectively related in a similar way. Thus, we obtained the following important result.

Let an arbitrary plane δ pass through one of the centers, for example, through the center $O_{\alpha\beta}$. The plane δ intersects the planes α, β along the lines $p_\alpha = \delta \cap \alpha$, $p_\beta = \delta \cap \beta$. These straight lines can be considered as conjugate polars (with respect to a pole P) of two conjugate conic sections $d^2_\alpha(1, 2, 3) \sim d^2_\beta(4, 5, 6)$. The conic section d^2_α passing through the points 1, 2, 3 is completely determined by the pole P and the polar p_α. The conic section d^2_β passing through the points 4, 5, 6 is completely determined by the pole P and the polar p_β.

Thus, the arbitrary plane δ incident to the center $O_{\alpha\beta}$ generates a pair of conjugate conic sections passing through the given points $(1, 2, 3)$ and $(4, 5, 6)$ in the planes α, β. Consequently, the plane $\Delta(O_{\alpha\beta} O_{\alpha\gamma} O_{\beta\gamma})$ generates three pairs of conjugate conic sections $a^2 \sim b^2$, $a^2 \sim g^2$, $b^2 \sim g^2$ passing respectively through the given points $(1, 2, 3)$, $(4, 5, 6)$, and $(7, 8, 9)$ in the planes α, β, and γ. These conic sections determine the desired surface Θ. The problem is solved.

4. J. H. Engel's algorithm

Nine points of general position 1, 2, ..., 9 are marked in the three-dimensional space. The task is to construct a second-order surface Θ_0 passing through these points.

We introduce a pencil of second-order surfaces (quadrics) passing through the points 1, 2, ..., 8. Following J. H. Engel, let us refer to this pencil of quadrics as *"pencil no. 1"*. All the quadrics of pencil no. 1 intersect along a biquadratic curve passing through the points 1, 2, ..., 8. The desired quadric Θ_0 belongs to this pencil no. 1.

Let us intersect the pencil no. 1 with the plane $\alpha(129)$. We obtain a pencil ψ of second-order curves with base points $1, 2, S, T$ (the position of the points S, T is yet unknown). The pencil ψ contains a conic section $c^2 = \Theta_0 \cap \alpha$ passing through the point 9 and through the base points $1, 2, S, T$ of the pencil ψ. If we find the unknown base points S, T of the pencil ψ, we will completely define the conic section c^2. Then the task of constructing the quadric Θ_0 will be almost solved, since the conic c^2 belongs to the desired quadric Θ_0.

Thus, the problem of constructing the quadric Θ_0 is reduced to finding the base points S, T of the pencil ψ. To construct the points S, T, two arbitrary quadrics Θ and Θ_0' have to be selected from the pencil of quadrics no. 1 and the conic sections $\theta^2 = \Theta \cap \alpha$ and $\theta'^2 = \Theta' \cap \alpha$ have to be found. The conics θ^2 and θ'^2 intersect at the known points 1, 2 and at the desired points S and T.

The pencil of quadrics no. 1 is completely determined by the points $1, 2, \ldots, 8$. How can we select an arbitrary quadric Θ from this pencil and how we find its intersection with the plane $\alpha(129)$? Let us consider an auxiliary problem, following J. H. Engel.

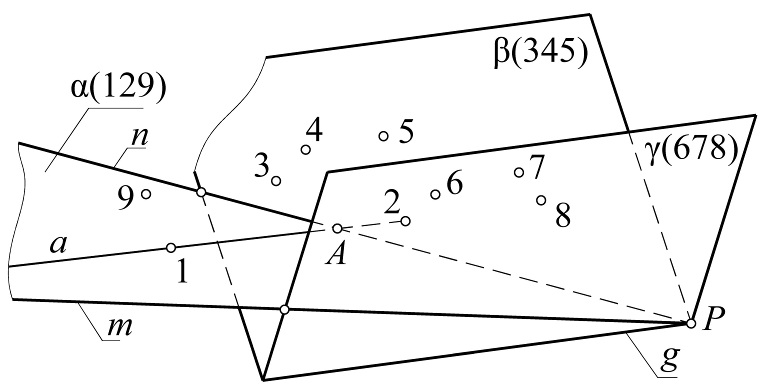

Figure 3: Initial data for Engel's algorithm

Auxiliary problem. Construct an arbitrary quadric Θ passing through eight given points $1, 2, \ldots, 8$ and find the conic section $\theta^2 = \Theta \cap \alpha$.

Solution of the auxiliary problem. We select the planes $\alpha(129)$, $\beta(345)$, $\gamma(678)$ and draw a straight line $a = 12$ (Figure 3). We mark two arbitrary points F, G on the straight line $g = \beta \cap \gamma$. Temporarily, we exclude point 2 from the consideration. A single quadric Φ_G passes through the nine points $1, 3, \ldots, 8, F, G$.

After we have fixed the position of the point F, we change the position of the point G on the line g. We obtain a pencil of quadrics $\{\Phi_G\}$ passing through eight of the fixed points $1, 3, 4, 5, 6, 7, 8, F$ and through the ninth moving point G. Following J. H. Engel, we refer to this pencil of quadrics as *"pencil no. 2"*. To solve the auxiliary problem, a quadric passing through the excluded point 2 has to be selected from the pencil no. 2.

Let the moving point G take successive positions $G, G', G'', \dots$ on the straight line g. The quadrics Φ_G, $\Phi_{G'}$, $\Phi_{G''}$ of pencil no. 2 pass respectively through the points G, G', G'', … and intersect the straight line a in the fixed point 1 and in the points A, A', A'', … We establish a projective correspondence

$$g(G, G', G'', \dots) \;\overline{\wedge}\; a(A, A', A'', \dots)$$

between the ranges of points g and a (Theodor Reye's theorem; the proof of the theorem is considered in Section 7). In this projective correspondence, the point 2 on the straight line a corresponds to a certain point G_2 on the straight line g. The quadric of pencil no. 2, which passes through the point G_2, also passes through the point 2.

To establish the projective correspondence between the ranges of points g and a, we need to select three arbitrary quadrics from pencil no. 2 and mark the points of intersection of these quadrics with the straight lines g and a. Let us select an arbitrary quadric $\Phi_G(1, 3, 4, 5, 6, 7, 8, F, G)$ of pencil no. 2 by marking an arbitrary point G on the straight line g. We obtain conics $k^2_\beta(3, 4, 5, F, G)$ and $k^2_\gamma(6, 7, 8, F, G)$ on the quadric when we intersect it with the planes $\beta(345)$ and $\gamma(678)$. The conic sections k^2_β, k^2_γ intersect the plane $\alpha(129)$ at the points $\{N_1, N_2\} = n \cap k^2_\beta$ and $\{M_1, M_2\} = m \cap k^2_\gamma$ (Figure 4, left). The conic section $k^2_\alpha = \Phi_G \cap \alpha$ is completely determined by the points $1, N_1, N_2, M_1, M_2$. The conic k^2_α intersects the line a at the points 1 and A (Figure 4, right). The point A on the straight line a and the point G on the straight line g are corresponding in the projective correspondence $g \;\overline{\wedge}\; a$.

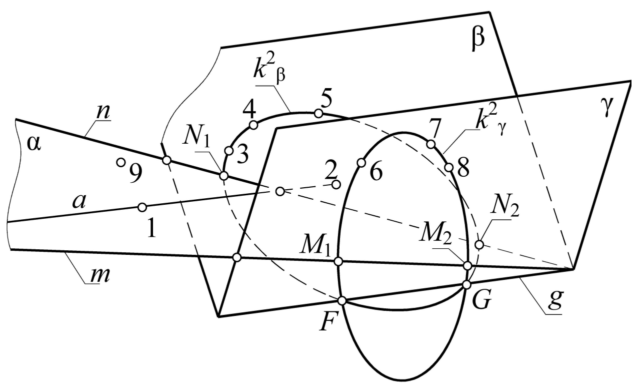

Figure 4: Left: Quadric Φ_G of pencil no. 2 determined by point 1 and the conics k^2_β, k^2_γ;
Right: section k^2_α of the quadric Φ_G with the plane $\alpha(129)$

Having selected three arbitrary quadrics Φ_G, $\Phi_{G'}$, $\Phi_{G''}$ from pencil no. 2, we find three pairs of projectively corresponding points A, A', A'' and G, G', G'' on the straight lines a and g. In the projectivity $g(G, G', G'', \dots) \;\overline{\wedge}\; a(A, A', A'', \dots)$, we find the point G_2 on the straight line g corresponding to the point 2 on the straight line a. The quadric of pencil no. 2 passing through the points $1, 3, 4, 5, 6, 7, 8, F, G_2$ also passes through the point 2; therefore it is the quadric Θ of the pencil no. 1. The quadric Θ is completely determined by points $1, 2, 3, \dots, 8$ and point F on the straight line g.

In order to find the section $\theta^2 = \Theta \cap \alpha$, we have to construct the conics $d^2_\beta(3, 4, 5, F, G_2) = \Theta \cap \beta$ and $d^2_\gamma(6, 7, 8, F, G_2) = \Theta \cap \gamma$ and mark their intersection points with the plane α:

$$\{N_{\Theta 1}, N_{\Theta 2}\} = n \cap d^2_\beta \quad \text{and} \quad \{M_{\Theta 1}, M_{\Theta 2}\} = m \cap d^2_\gamma.$$

The conic θ^2 passes through the points 1, 2 and $N_{\Theta 1}, N_{\Theta 2}, M_{\Theta 1}, M_{\Theta 2}$. The auxiliary problem is solved.

Having marked another arbitrary point $F' \neq F$ on the straight line g, we solve the auxiliary problem again. We select another arbitrary quadric $\Theta'(1, 2, \ldots, 8, F')$ from pencil no. 1 and find its section $\theta'^2 = \Theta' \cap \alpha$. The conics θ^2, θ'^2 intersect at the base points $1, 2, S, T$ of the pencil ψ. The conic section c^2 that belongs to the desired quadric Θ_0 passes through the points $1, 2, 9$ and the points S and T. The main problem is solved.

5. Simplified computer version of J. H. Engel's algorithm

Nine points of general position 1, 2, ..., 9 are marked in three-dimensional space. The task is to construct a second-order surface Φ_9 passing through these points. We propose to use a computer program to simplify the design solution of the problem [6]. The program creates a geometrically accurate (= "with a compass and straightedge") construction of data like center, principal diameters, asymptotes, and foci of a second-order curve determined by i points and j tangents, where $i + j = 5$. Having determined these data, the program plots a smooth second-order curve which passes through the given points and is tangent to the given tangents.

We select the planes $\alpha(123)$, $\beta(456)$, and $\gamma(789)$ and mark the lines $m = \alpha \cap \beta$, $n = \beta \cap \gamma$ and $l = \alpha \cap \gamma$.

Operation 1. Fix an arbitrary point F on the straight line m. Mark another arbitrary point G that does not coincide with the point F on the line m. Plot the conic sections $k_\alpha^2(1, 2, 3, F, G)$ and $k_\beta^2(4, 5, 6, F, G)$ in the planes α, β. The conic sections k_α^2, k_β^2 and the point 7 determine the quadric Φ_G. The quadric Φ_G passes through the points F, G and through all given points, except the points 8, 9. Plot the intersection k_γ^2 of the quadric Φ_G and the plane γ: $k_\gamma^2 = \Phi_G \cap \gamma$. The conic section k_γ^2 is completely determined by the point 7 and by two pairs of points $\{L_1, L_2\} = k_\alpha^2 \cap l$ and $\{N_1, N_2\} = k_\beta^2 \cap n$ (Figure 5, left).

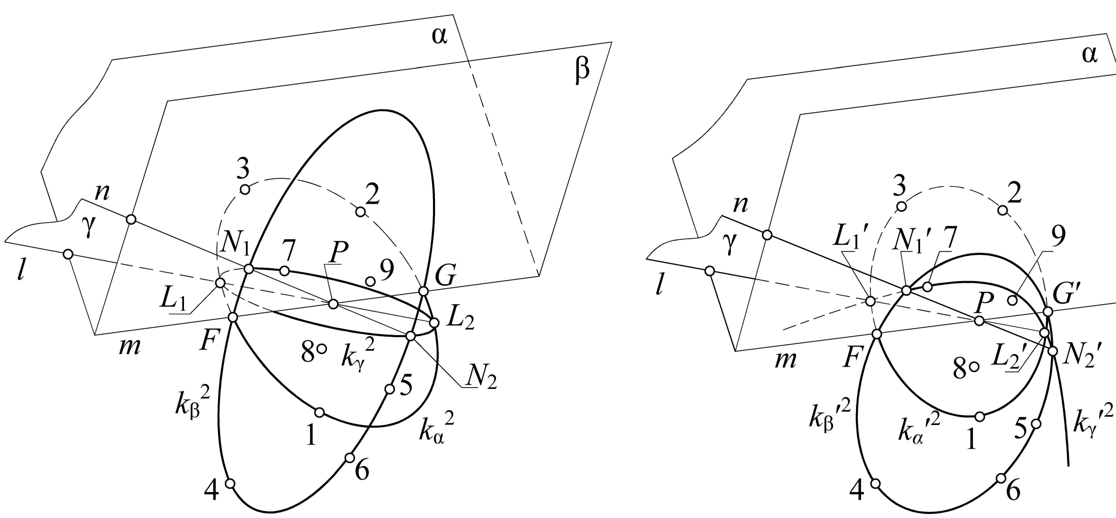

Figure 5: Construction of the quadrics Φ_G (left) and $\Phi_{G'}$ (right)

Operation 2. We mark another arbitrary point G' on the straight line g that does not coincide with the points F, G. We plot conic sections $k_\alpha'^2(1, 2, 3, F, G')$, $k_\beta'^2(4, 5, 6, F, G')$ in the planes α and β. These conic sections and point 7 determine the quadric $\Phi_{G'}$. The quadric $\Phi_{G'}$ passes through the points F, G' and through all given points, except the points 8, 9. We plot the intersection k'^2_γ of the quadric $\Phi_{G'}$ with the plane γ: $k'^2_\gamma = \Phi_{G'} \cap \gamma$. The conic section k'^2_γ is completely determined by the point 7 and by two pairs of points $\{L'_1, L'_2\} = k'^2_\alpha \cap l$ and $\{N'_1, N'_2\} = k'^2_\beta \cap n$ (Figure 5, right).

The quadrics Φ_G and $\Phi_{G'}$ determine a pencil of quadrics $\{\Phi_G, \Phi_{G'}, \dots\}$ that intersect along the biquadratic curve $f(F, 1, 2, \dots, 7)$. The conic sections k^2_γ and k'^2_γ span a pencil γ_F of conic sections $\gamma_F(k^2_\gamma, k'^2_\gamma, \dots)$ in the plane γ. There is a one-to-one correspondence between the quadrics of the pencil $\{\Phi_G, \Phi_{G'}, \dots\}$ and the conics of the pencil γ_F.

Operation 3. Mark the base points $7, U, V, W$ of the pencil γ_F at the intersection of the conic sections k^2_γ and k'^2_γ. The construction of the intersection points U, V, W of the conic sections k^2_γ and k'^2_γ with the known common point 7 is a third-degree problem that cannot be solved using straightedge and compass. The simplified computer version of J. H. Engel's algorithm is based on a computer program [6] which plots the smooth curves $k^2_\gamma(7, L_1, L_2, N_1, N_2)$, $k'^2_\gamma(7, L'_1, L'_2, N'_1, N'_2)$ and marks their intersection points U, V, W (Figure 6).

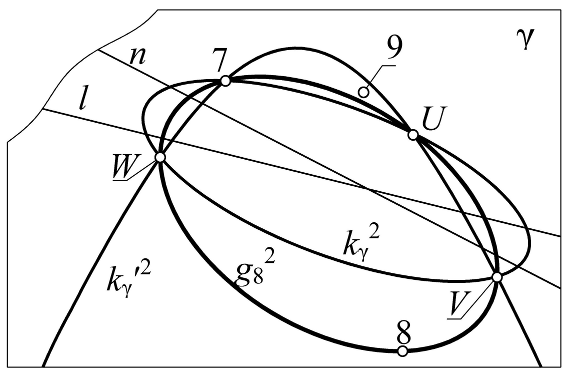

Figure 6: Construction of the conic g^2_8 of the pencil $\gamma_F(7, U, V, W)$

Select the conic g^2_8 that passes through point 8 from the pencil $\gamma_F(7, U, V, W)$ (Figure 6). The single quadric Φ_8 of the pencil $\{\Phi_G, \Phi_{G'}, \dots\}$ corresponds to the conic section g^2_8. The quadric Φ_8 passes through the fixed point F and through all given points other than point 9.

Thus, an arbitrary quadric Φ_8 passing through the eight given points $1, 2, ..., 8$ is selected from the pencil $\{\Phi_G, \Phi_{G'}, \dots\}$. The section $g^2_8 = \Phi_8 \cap \gamma$ passing through the points 7, 8 is found.

Operation 4. Having marked another arbitrary point F' that does not coincide with the point F on the line m, repeat the operations 1, 2 and 3. Obtain a quadric Φ'_8 passing through the fixed point F' and through the eight given points $1, 2, \dots, 8$. Find the intersection g'^2_8 of the quadric Φ'_8 with the plane γ. The conic g'^2_8 passes through points $7, 8$.

The quadrics Φ_8 and Φ'_8 determine a pencil of quadrics $\{\Phi_8, \Phi'_8, \dots\}$ that intersect along a biquadratic curve passing through the points $1, 2, ..., 8$. The conic sections g^2_8 and g'^2_8 form a pencil of conic sections $\psi(g^2_8, g'^2_8, \dots)$ with base points $7, 8, S, T$ in the plane γ. The points S and T are determined as the points of intersection of the conics g^2_8 and g'^2_8 with the known common points $7, 8$. There is a one-to-one correspondence between the quadrics of the pencil $\{\Phi_8, \Phi'_8, \dots\}$ and the conics of the pencil ψ.

Operation 5. Pass a conic g^2_9 through point 9 and through the base points $7, 8, S, T$ of the pencil ψ. This conic belongs to the desired quadric Φ_9. The problem is solved.

6. Construction of principal axes and symmetry planes

A quadric Θ is determined by nine points of general position. The task is to construct the principal axes of this quadric.

Following one of the above algorithms, we find a conic section θ^2 of the quadric Θ. The presence of a smooth conic θ^2 makes it possible to perform auxiliary constructions that are necessary to construct the principal axes of the quadric.

We find the center O of the quadric and select a bundle of lines and planes $O(d,\Sigma)$ that are conjugate with respect to Θ. The diametral plane Σ which is conjugate to an arbitrary diameter d passes through the midpoints of the chords parallel to d. In the bundle $O(d,\Sigma)$, there are three pairs of orthogonal diameters d and three pairs of respectively orthogonal diametral planes Σ. These are the desired principal axes and symmetry planes of the quadric Θ.

In addition, let us consider the orthogonal correspondence in the bundle O: for each plane Σ there is a perpendicular n passing through the point O. Hence, we obtain in the bundle O a projective correspondence Λ_O between the two-parameter set of straight lines d and the lines n. Three pairs of coincident straight lines $d \equiv n$ indicate the three principal diameters of the quadric Θ.

Let us plot four pairs of straight lines $d_1 \sim n_1$, $d_2 \sim n_2$, $d_3 \sim n_3$, $d_4 \sim n_4$ corresponding in Λ_O. We pass an arbitrary secant plane Π. The plane Π intersects the corresponding straight lines at the corresponding points $D_1 \sim N_1$, $D_2 \sim N_2$, $D_3 \sim N_3$, $D_4 \sim N_4$. We obtain superimposed point fields $\Pi = \Pi_D = \Pi_N$. Four pairs of points determine the collineation $\Delta(\Pi_D \leftrightarrow \Pi_N)$. The task is to find the three double points of this collineation.

According to [4], we mark an arbitrary point D_0 in the plane Π_D to construct the double collineation points Δ. Let us plot a pencil of lines $D_0(D_0D_1, D_0D_2, D_0D_3, D_0D_4)$. In the plane Π_N, we find the point N_0 that corresponds to the point D_0 in the collineation Δ. We plot a pencil of straight lines $N_0(N_0N_1, N_0N_2, N_0N_3, N_0N_4)$. The pencils D_0 and N_0 are projective. The intersection points of the corresponding pencil rays form a certain conic section h^2.

Let us assume that the curve h^2 belongs to the field Π_D. In the field Π_N, we find the conic h^2_N corresponding to the conic h^2 in the collineation Δ. The curves h^2 and h^2_N intersect at the point N_0 and at the three double points X, Y, Z of the collineation $\Delta(\Pi_D \leftrightarrow \Pi_N)$. The points $\{X, Y, Z\} \subset h^2 \cap h^2_N$ cannot be constructed using straightedge and compass, since this is a third-degree problem. A computer program is used to solve it in a constructive way [6]. The principal axes of the quadric Θ pass through the center O and through the points X, Y, Z. The symmetry planes are orthogonal to the principal axes. The problem is solved.

7. T. Reye's theorem

Theorem (T. Reye). *There is a pencil of quadrics $\Phi\{\varphi_1, \varphi_2, \varphi_3, \dots\}$ that intersect along a biquadratic curve f. Let us mark two arbitrary points A, B on the curve f. We pass an arbitrary straight line a of general position through the point A. We pass a line b of general position through the point B. The line a intersects the quadrics of the pencil Φ at the point A and at the points $A_1, A_2, A_3, \dots$ The line b intersects the quadrics of the pencil Φ at the point B and at the points $B_1, B_2, B_3, \dots$*

Then the ranges $a(A_1, A_2, A_3, \dots)$ and $b(B_1, B_2, B_3, \dots)$ are projective.

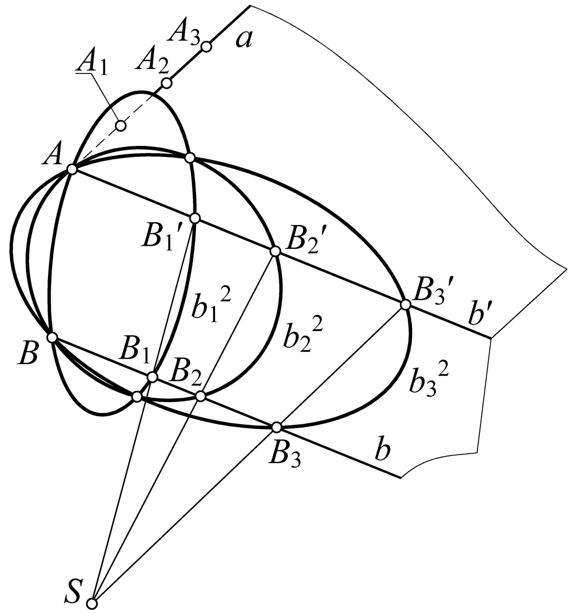

Figure 7: To the proof of T. Reye's theorem

Proof. Let us pass a straight line b' parallel to the straight line b through the point A. The quadrics $\varphi_1, \varphi_2, \varphi_3, \ldots$ of the pencil Φ intersect the plane $b \| b'$ along a pencil β of conic sections $b_1^2, b_2^2, b_3^2, \ldots$ The straight lines b and b' can be regarded as a degenerate conic section passing just through two base points A, B of the pencil β. The conics of the pencil β form point ranges $b(B_1, B_2, B_3, \ldots)$ and $b'(B_1', B_2', B_3', \ldots)$ when intersecting the lines b, b'. According to the generalized Theorem of Desargues [4, p. 130], the pairs of corresponding points $B_1 \sim B_1'$, $B_2 \sim B_2'$, $B_3 \sim B_3'$, ... of these ranges belong to the same involution with the center S (Figure 7). Consequently, the quadrics of the pencil Φ intersect the straight lines $b \| b'$ along the perspective point ranges

$$b(B_1, B_2, B_3, \ldots) \ \overline{\barwedge} \ b'(B_1', B_2', B_3', \ldots).$$

We obtain a pencil of conics $\alpha(a_1^2, a_2^2, \ldots)$ in the section of the pencil of quadrics Φ cut by the plane ab'. The conics of α form point ranges $a(A_1, A_2, A_3, \ldots)$ and $b'(B_1', B_2', B_3', \ldots)$ when intersecting the straight lines a and b'. Let us show that the ranges a and b' are projective.

We pass an arbitrary straight line a' through a base point of the pencil α (not coinciding with the point A) in the plane ab'. According to the generalized Theorem of Desargues, all the conics of the pencil α intersect with the straight lines a, a' along the perspectively corresponding ranges of points $a \ \overline{\barwedge}\ a'$. This theorem is also true for the straight lines a' and b': $a' \ \overline{\barwedge}\ b'$. Consequently, the point ranges $a(A_1, A_2, A_3, \ldots)$ and $b'(B_1', B_2', B_3', \ldots)$ located on the straight lines a and b' that pass through the same base point of the pencil of conics are projective (but not perspective).

Thus, the quadrics of the pencil Φ intersect the lines a and b' along projective ranges of points $a(A_1, A_2, A_3, \ldots) \barwedge b'(B_1', B_2', B_3', \ldots)$. It has been shown that the ranges $b(B_1, B_2, B_3, \ldots)$ and $b'(B_1', B_2', B_3', \ldots)$ are perspective. Hence, the point ranges $a(A_1, A_2, A_3, \ldots)$ and $b(B_1, B_2, B_3, \ldots)$ are projective, too. T. Reye's theorem is proved. J. H. Engel's algorithm (see Section 4) is based on this theorem. □

8. Conclusion

The development of computer graphics at the end of the 20th century resulted in a "projective computational" method of geometric modeling that combines the advantages of synthetic and analytic research methods. The new method combines the high accuracy of computers, whose operations are based on coordinate calculations, and the geometric simplicity of projective algorithms, where only the incidences of points and straight lines are important. This combination may put an end to the "ideological confrontation" between Analytic Geometry (R. Descartes, 1596–1650) and Projective Geometry (G. Desargue, 1591–1661).

Modern computer graphics software allows us to constructively implement complex graphics algorithms, including projective algorithms for constructing a nine-point quadric. The algebraic solution to this problem consists in calculating the coefficients in the quadratic equation of the desired quadric, starting from a system of nine linear equations for nine unknown coefficients. The equivalent geometric solution must also be linear, that is, it must be obtained with a straightedge alone and even without a compass. The Rohn and Papperitz' algorithm [8] meets this requirement. J. H. Engel's algorithm fails to completely meet the requirement [2], while the simplified computer version of J. H. Engel's algorithm fails to meet it at all (Section 5).

It is noteworthy that [2] not only considers the algorithm based on T. Reye's theorem, but also proposes two alternate problem solutions with the help of a straightedge alone. Obviously, these alternate solutions are not the simplest. These are very complex constructions that meet the scientifically justified requirement of equivalence between graphical and algebraic solutions.

The simplified algorithm for the quadric's construction is implemented using a computer program [6]. This program performs a geometrically accurate compass-and-straightedge construction of a second-order curve metric determined by a set of five linear incidences (points or tangents) [5]. The points and tangents can be either real or imaginary [3]. Having determined the metric, the program plots a smooth second-order curve that satisfies the specified incidence conditions. The application of such a program does not contribute to solving the theoretical "Problem of the Tenth Point", but it makes it possible to compile a relatively simple graphical algorithm for quadric construction, which is an alternative to the algebraic solution.

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