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A Proof of Pohlke's Theorem with an Analytic Determination of the Reference Trihedron

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Abstract

By elementary arguments of linear algebra and vector algebra we give here a proof of Pohlke's fundamental theorem on oblique axonometry. We also present explicit formulae for the reference trihedrons (Pohlke matrices) and the corresponding directions of projection.

Key Words: Pohlke's theorem, oblique axonometry

MSC 2010: 51N10, 51N05

1. Introduction

The famous Pohlke's fundamental theorem of oblique axonometry asserts that

> *three arbitrary straight line segments OP_1, OP_2, OP_3 in a plane, originating from a point O and which are not contained in a line, can be considered as the parallel projection of three edges OQ_1, OQ_2, OQ_3 of a cube.*

Or, with the words of H. Steinhaus [14, p. 170],

> *one can draw any three (not all parallel) segments from one point, complete the figure with parallel segments, and consider it as a (generally oblique) projection of a cube.*

K.W. Pohlke formulated this theorem in 1853 and published it in 1860, without demonstration, in the first part of his textbook on Descriptive Geometry [11]. The first elementary rigorous proof was given by H.A. Schwarz [12] in 1864, at that time a student of Pohlke.

Subsequently, as remarked by D.J. Struik [16, p. 240], several proofs have been given, synthetic and analytic, none of which is simple because they also give the method by which one can construct the direction of projection. See among others [4, 5, 8, 10, 15, 7, 1, 9, 3, 2, 13].

In general these proofs are more geometric than that presented here, but do not give explicit formulae. For instance, in [10, 2, 13] they concentrate more on the properties of

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the affine transformation between the orthonormal n-frame in space and its given image. In particular, the singular value decomposition and the directions of principal distortions are of importance. Furthermore, often different coordinate frames are used in space and for the given image. Here, instead, a fixed coordinate frame is used and an argument of linear algebra leads to simple explicit formulae for the reference trihedron and the direction of projection.

We prove Pohlke's theorem by reducing it to a particular case in which the direction of projection coincides with that of one of the edges of the cube. To achieve this simplification we restate Pohlke's theorem as a problem of linear algebra and then we make a simple observation based on the fact that for a matrix *row*-rank and *column*-rank are equal.

More precisely, if we introduce a cartesian system of coordinates such that

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad P_i = \begin{pmatrix} x_i \\ y_i \\ 0 \end{pmatrix}, \quad (1 \le i \le 3) \tag{1.1}$$

are points of the plane $\{z = 0\}$, then Pohlke's theorem can be reformulated as follows:

Theorem 1.1. *If the matrix*

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix} = \left(A^1,\, A^2,\, A^3\right) \tag{1.2}$$

has rank 2*, then there exist a matrix B with orthogonal columns B^1, B^2, B^3 of equal norm,*

$$B = \begin{pmatrix} x'_1 & x'_2 & x'_3 \\ y'_1 & y'_2 & y'_3 \\ z'_1 & z'_2 & z'_3 \end{pmatrix}, \tag{1.3}$$

and a parallel projection Π onto the plane $\{z = 0\}$ such that $\Pi(B^i) = A^i$ $(1 \le i \le 3)$.

Now (see Lemma 2.1) the columns A^1, A^2, A^3 of A can be obtained by a parallel projection of the columns B^1, B^2, B^3 of B if and only if the rows A_1, A_2, A_3 of A can be obtained by a parallel projection of the rows B_1, B_2, B_3 of B. But, since A_3 has zero entries, the direction of this last projection is the same of B_3.

By this observation (see Sections 3 and 4) we bypass the problem of the direction of projection and we easily get an explicit expression for B. In particular, for $1 \le i \le 3$, we have

$$\|B^i\| = \frac{\sqrt{2}\,\|A_1 \times A_2\|}{\sqrt{\|A_1\|^2 + \|A_2\|^2 + \sqrt{\left(\|A_1\|^2 + \|A_2\|^2\right)^2 - 4\|A_1 \times A_2\|^2}}}, \tag{1.4}$$

(see (4.5), (4.6) and the example at the end of Section 4).

The fact, that the proof becomes easier if one has some information about the direction of projection, is particularly evident when Π is, *a priori*, the orthogonal projection. Indeed, we can reformulate the Gauss' theorem of orthogonal axonometry as follows:

Proposition 1.2. *Let A be the matrix of* (1.2) *and let $\Pi_\perp : \mathbf{R}^3 \to \{z = 0\}$ be the orthogonal projection. Then, there exists a matrix B with nonzero orthogonal columns of equal norm such that $\Pi_\perp(B^i) = A^i$ $(1 \le i \le 3)$ if and only if*

$$\|A_1\| = \|A_2\| \ne 0 \quad \textit{with} \quad A_1 \perp A_2\,. \tag{1.5}$$

Proof. In fact, if (1.5) holds and $\rho = \|A_1\| = \|A_2\|$, we may define $B_3 = (z_1', z_2', z_3')$ as

$$B_3 = \rho^{-1} A_1 \times A_2 \quad \text{or} \quad B_3 = -\rho^{-1} A_1 \times A_2. \tag{1.6}$$

Then, $B_3 \perp A_1, A_2$ and $\|B_3\| = \rho$. Hence, setting $B_1 = A_1$, $B_2 = A_2$, the matrix

$$B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1' & z_2' & z_3' \end{pmatrix} \tag{1.7}$$

has orthogonal rows of norm ρ. Then, B is a multiple of an orthogonal matrix and its columns B^i $(1 \le i \le 3)$ are orthogonal and of norm ρ. Finally, we clearly have $\Pi_{\perp}(B^i) = A^i$. Conversely, if there exists $B = (B^1, B^2, B^3)$ such that $\Pi_{\perp}(B^i) = A^i$ $(1 \le i \le 3)$, then $B_1 = A_1$, $B_2 = A_2$. If, in addition, the columns B^1, B^2, B^3 are nonzero, orthogonal and of equal norm then B is a multiple of an orthogonal matrix. Hence (1.5) holds. □

Remark 1.3. In Proposition 1.2 the row B_3 is necessarily given by (1.6). Hence, in case of orthogonal projection "Pohlke's problem" (namely, that of finding a matrix B and a projection Π with the required properties) has exactly two solutions. On the other hand, having obtained one solution, in case of non-orthogonal projection we immediately get four distinct solutions by reflection in the image plane $\{z = 0\}$ and in a plane orthogonal to the direction of projection.

Definition 1.4. Let A be a matrix of rank 2 as in (1.2). We say that a matrix B is a *Pohlke matrix for A* if B has orthogonal columns of equal norm and there exists a parallel projection $\Pi\colon \mathbf{R}^3 \to \{z = 0\}$ such that $\Pi(B^i) = A^i$, $1 \le i \le 3$.

Taking into account Remark 1.3, we deduce from the explicit formula (4.5)

Corollary 1.5. *Under the assumption of Theorem 1.1, in case of an oblique projection, i.e., a non-orthogonal projection, there are exactly four distinct Pohlke matrices.*

2. Some facts from linear algebra

We prove here two simple results of linear algebra. The first is the key lemma that we need in the proof of Pohlke's theorem. The second one is a standard fact concerning the existence of orthogonal transition matrices between two given bases of $\mathbf{R}^3$; we need it for computing the matrix B and the direction of projection.

Let A be the real 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \left(A^1, A^2, A^3\right) \tag{2.1}$$

where A_i, A^i $(1 \le i \le 3)$ are, respectively, row and column vectors of A.

We indicate with $\mathbb{V}_A$ and $\mathbb{V}^A$ the row and column spaces of A, namely

$$\mathbb{V}_A = \operatorname{span}\left\{A_1, A_2, A_3\right\}, \quad \mathbb{V}^A = \operatorname{span}\left\{A^1, A^2, A^3\right\}. \tag{2.2}$$

If $\operatorname{rank}(A) = 2$, then $\mathbb{V}^A$ and $\mathbb{V}_A$ are 2-dimensional subspaces (i.e., planes through the origin) of $\mathbf{R}^3$. Given any column vector U, $U \not\parallel \mathbb{V}^A$, we may consider the parallel projection $\Pi^U :$ $\mathbf{R}^3 \to \mathbb{V}^A$ in the direction of U by setting

$$\Pi^U(V) = \mathbb{V}^A \cap \{V + t\,U : t \in \mathbf{R}\} \tag{2.3}$$

for any column vector $V \in \mathbf{R}^3$. Since $\mathbb{V}^A$ is a plane through the origin, Π^U is a *linear* map.

In the same way we can define the parallel projection $\Pi_W\colon\ \mathbf{R}^3 \to \mathbb{V}_A$ in the direction of a given row vector W, if $W \nparallel \mathbb{V}_A$. When the direction of projection is not specified, we write Π^* (Π_*) for column (row) projections onto $\mathbb{V}^A$ ($\mathbb{V}_A$).

Lemma 2.1. *Let A and B be 3×3 matrices such that rank$(A) = 2$ and rank$(B) = 3$. Then the rows of A can be obtained by a parallel projection $\Pi_*\colon\ \mathbf{R}^3 \to \mathbb{V}_A$ of the rows of the matrix B if and only if the columns of A can be obtained by a parallel projection $\Pi^*\colon\ \mathbf{R}^3 \to \mathbb{V}^A$ of the columns of the matrix B.*

Proof. Let Π_* be a parallel projection such that $\Pi_*(B_i) = A_i$ $(1 \le i \le 3)$. Then

$$\text{row-rank}\,(B - A) = \text{row-rank}\begin{pmatrix} B_1 - A_1 \\ B_2 - A_2 \\ B_3 - A_3 \end{pmatrix} = 1, \tag{2.4}$$

because the row vectors $B_i - A_i$ are parallel to the direction of the projection Π_*. Since row-rank and column-rank of a matrix are equal [6, p. 81], this implies that

$$\text{column-rank}\,(B - A) = \text{column-rank}\,\big(B^1 - A^1,\ B^2 - A^2,\ B^3 - A^3\big) = 1. \tag{2.5}$$

Thus there exists a column vector $U \in \mathbf{R}^3$ such that

$$(B^i - A^i) \parallel U \quad \text{for} \quad 1 \le i \le 3\,. \tag{2.6}$$

Moreover, observe that

$$U \notin \mathbb{V}^A \quad (\text{that is } U \nparallel \mathbb{V}^A) \tag{2.7}$$

because rank$(B) = 3$, while rank$(A) = 2$.

Hence, for all column vector $V \in \mathbf{R}^3$ the line $\{V + Ut : t \in \mathbf{R}\}$ has one and only one intersection with the plane $\mathbb{V}^A$. This permits us to define the projection $\Pi^* : \mathbf{R}^3 \to \mathbb{V}^A$ in the direction of the column vector U by setting

$$\Pi^*(V) = \mathbb{V}^A \cap \{V + Ut : t \in \mathbf{R}\} \quad \text{for} \quad V \in \mathbf{R}^3. \tag{2.8}$$

Since we clearly have $\Pi^*(B^i) = A^i$ $(1 \le i \le 3)$, this proves the first part of the lemma. The converse can be proved similarly. □

Having proved Lemma 2.1, and taking into account that if B is a square matrix then

B has nonzero orthogonal columns of equal norm $\iff$ B is a nonzero multiple of an orthogonal matrix $\iff$ B has nonzero orthogonal rows of equal norm,

we can immediately restate Definition 1.4 in the following equivalent form:

Definition 2.2. Let A be a 3×3 matrix of rank 2. We say that a 3×3 matrix B is a *Pohlke matrix for A* if B is a multiple of an orthogonal matrix and there exist parallel projections $\Pi_*\colon\ \mathbf{R}^3 \to \mathbb{V}_A$ and $\Pi^*\colon\ \mathbf{R}^3 \to \mathbb{V}^A$ such that $\Pi_*(B_i) = A_i$ and $\Pi^*(B^i) = A^i$ for $1 \le i \le 3$.

Definition 2.3. We denote by E_i, $1 \le i \le 3$, the standard base of row vectors

$$E_1 = (1, 0, 0), \quad E_2 = (0, 1, 0), \quad E_3 = (0, 0, 1). \tag{2.9}$$

Lemma 2.4. *Let* $\{A'_1, A'_2, A'_3\}$ *and* $\{\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3\}$ *be two sets of linearly independent row vectors (i.e., two bases of* $\mathbf{R}^3$*) such that*

$$A'_i \cdot A'_j = \widetilde{A}_i \cdot \widetilde{A}_j \qquad (1 \le i,\, j \le 3)\,. \tag{2.10}$$

Then, there exists a unique orthogonal transition matrix $\mathcal{T}$ *such that* $A'_i = \widetilde{A}_i \mathcal{T}$ $(1 \le i \le 3)$.

Proof. Uniqueness is quite obvious. To determine $\mathcal{T}$, let us define the nonsingular matrices

$$\mathcal{G} = \begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \end{pmatrix}, \quad \widetilde{\mathcal{G}} = \begin{pmatrix} \widetilde{A}_1 \\ \widetilde{A}_2 \\ \widetilde{A}_3 \end{pmatrix}. \tag{2.11}$$

We have $E_i\,\mathcal{G} = A'_i$ and $E_i\,\widetilde{\mathcal{G}} = \widetilde{A}_i$, for $1 \le i \le 3$. Then, setting

$$\mathcal{T} = \widetilde{\mathcal{G}}^{-1}\,\mathcal{G}\,, \tag{2.12}$$

it is clear that $\widetilde{A}_i\,\mathcal{T} = \widetilde{A}_i\,\widetilde{\mathcal{G}}^{-1}\,\mathcal{G} = E_i\,\mathcal{G} = A'_i$ for $1 \le i \le 3$.

Finally, to see that $\mathcal{T} = \widetilde{\mathcal{G}}^{-1}\mathcal{G}$ is orthogonal (that is $\mathcal{T}\mathcal{T}^t = I$, where $\mathcal{T}^t$ is the transposed of $\mathcal{T}$) we observe that (2.10) is equivalent to $\mathcal{G}\,\mathcal{G}^t = \widetilde{\mathcal{G}}\,\widetilde{\mathcal{G}}^t$. Then

$$\mathcal{G}\,\mathcal{G}^t = \widetilde{\mathcal{G}}\,\widetilde{\mathcal{G}}^t \quad\Longrightarrow\quad \widetilde{\mathcal{G}}^{-1}\,\mathcal{G}\,\mathcal{G}^t\,(\widetilde{\mathcal{G}}^{-1})^t = I \quad\Longrightarrow\quad \widetilde{\mathcal{G}}^{-1}\,\mathcal{G}\,(\widetilde{\mathcal{G}}^{-1}\,\mathcal{G})^t = I\,. \qquad \square$$

3. Proof of Pohlke's theorem

We prove here Pohlke's theorem as stated in Theorem 1.1. With the notations of the previous sections, in the following we suppose

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ 0 \end{pmatrix} \quad \text{with} \quad \operatorname{rank}(A) = 2\,. \tag{3.1}$$

By Lemma 2.1, it is sufficient to find a matrix

$$B = \begin{pmatrix} x'_1 & x'_2 & x'_3 \\ y'_1 & y'_2 & y'_3 \\ z'_1 & z'_2 & z'_3 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \tag{3.2}$$

with orthogonal rows B_1, B_2, B_3 of equal norm, and a parallel projection $\Pi_W : \mathbf{R}^3 \to \mathbb{V}_A$, with $W \notin \mathbb{V}_A$, such that $\Pi_W(B_i) = A_i$ for $1 \le i \le 3$. This means that, for $1 \le i \le 3$,

$$A_i = B_i + k_i W \quad \text{with} \quad k_i \in \mathbf{R}\,. \tag{3.3}$$

Having $A_3 = (0,0,0)$, W must be parallel to B_3. Then, it is enough to prove that there exist B_1, B_2, B_3 orthogonal and of equal norm such that

$$\begin{cases} A_1 = B_1 + k_1 B_3 \\ A_2 = B_2 + k_2 B_3 \end{cases} \quad \text{with} \quad k_1,\, k_2 \in \mathbf{R}\,. \tag{3.4}$$

Since B_3 defines the direction of projection, the vectors A_1, A_2 are located, relatively to the frame (B_1, B_2, B_3), in the B_1B_3-plane and in the B_2B_3-plane, respectively. Furthermore, A_1, A_2 include a given angle, say γ, and their orthogonal projections into the B_1B_2-plane must show the same lengths.

Taking into account these facts, to solve (3.4) we first consider the following:

Auxiliary problem: *Let $\{E_1, E_2, E_3\}$ be the standard base of row vectors of Definition 2.3, and let $\gamma \in (0, \pi)$, $\lambda > 0$ be given parameters. We look for α, $\beta \in \mathbf{R}$ such that:*

$$\begin{cases} \dfrac{\|E_1 + \alpha E_3\|}{\|E_2 + \beta E_3\|} = \lambda\,, \\[3ex] \dfrac{(E_1 + \alpha E_3) \cdot (E_2 + \beta E_3)}{\|E_1 + \alpha E_3\|\,\|E_2 + \beta E_3\|} = \cos\gamma\,. \end{cases} \tag{3.5}$$

Before proving the solvability of (3.5) it is worthwhile to define a quantity:

Definition 3.1. For $(\gamma, \lambda) \in (0, \pi) \times (0, +\infty)$ we set

$$\eta = \eta(\lambda, \gamma) := \frac{\lambda^2 + 1 + \sqrt{(\lambda^2 + 1)^2 - 4\lambda^2 \sin^2\gamma}}{2\lambda^2 \sin^2\gamma}\,. \tag{3.6}$$

Then, for $(\gamma, \lambda) \in (0, \pi) \times (0, +\infty)$, we have:

$$\eta(\lambda, \gamma) \geq \eta\left(\lambda, \frac{\pi}{2}\right) = \frac{\lambda^2 + 1 + |\lambda^2 - 1|}{2\lambda^2} = \begin{cases} 1/\lambda^2 & \text{if} \quad 0 < \lambda \leq 1, \\ 1 \quad \text{if} & \lambda \geq 1, \end{cases} \tag{3.7}$$

with strict inequality if $\gamma \neq \frac{\pi}{2}$. In particular, $\eta(\gamma, \lambda)$ satisfies the following:

$$\begin{array}{llll} (i) & \eta \geq 1\,,\ \eta\lambda^2 \geq 1 & \text{for all} & (\gamma, \lambda) \in (0, \pi) \times (0, +\infty)\,, \\ (ii) & \eta = 1 & \Leftrightarrow & (\gamma, \lambda) \in \left\{\frac{\pi}{2}\right\} \times [1, +\infty)\,, \\ (iii) & \eta\lambda^2 = 1 & \Leftrightarrow & (\gamma, \lambda) \in \left\{\frac{\pi}{2}\right\} \times (0, 1]\,. \end{array} \tag{3.8}$$

Finally, for simplicity of writing, we also introduce a "signum" function:

$$\operatorname{sgn}(t) := \begin{cases} 1 & \text{if} \quad t \geq 0 \\ -1 & \text{if} \quad t < 0 \end{cases} \tag{3.9}$$

Lemma 3.2. *Assume that $(\gamma, \lambda) \in (0, \pi) \times (0, +\infty)$. Then the real solutions of (3.5) are*

$$(\alpha, \beta) = \pm\left(\sqrt{\eta\,\lambda^2 - 1}\,,\ \operatorname{sgn}(\cos\gamma)\sqrt{\eta - 1}\right). \tag{3.10}$$

In particular, for $(\gamma, \lambda) \neq (\frac{\pi}{2}, 1)$ the system (3.5) has two distinct real solutions.

Proof. It is clear that the system (3.5) is equivalent to the following

$$\begin{cases} 1 + \alpha^2 = \lambda^2(1 + \beta^2) \\ \alpha\beta = \sqrt{1 + \alpha^2}\,\sqrt{1 + \beta^2}\,\cos\gamma\,. \end{cases} \tag{3.11}$$

Multiplying the first equation of (3.11) by $(1+\beta^2)$, one easily sees that

$$\alpha^2\beta^2 = \lambda^2(1+\beta^2)^2 - (\lambda^2+1)(1+\beta^2) + 1\,. \tag{3.12}$$

On the other hand, squaring the other equation of (3.11), we have

$$\alpha^2\beta^2 = \lambda^2\,(1+\beta^2)^2\,\cos^2\gamma\,. \tag{3.13}$$

Hence, from (3.12) and (3.13), we deduce that $1+\beta^2$ satisfies the second order equation

$$\sin^2\gamma\,\lambda^2(1+\beta^2)^2 - (\lambda^2+1)(1+\beta^2) + 1 = 0\,. \tag{3.14}$$

Solving (3.14), we obtain

$$1+\beta^2 = \frac{\lambda^2+1 \pm \sqrt{(\lambda^2+1)^2 - 4\lambda^2\sin^2\gamma}}{2\lambda^2\sin^2\gamma}\,. \tag{3.15}$$

But, for all $(\gamma,\lambda) \in (0,\pi)\times(0,+\infty)$, we have

$$\begin{aligned} \frac{\lambda^2+1 - \sqrt{(\lambda^2+1)^2 - 4\lambda^2\sin^2\gamma}}{2\lambda^2\sin^2\gamma} \quad &= \quad \frac{2}{\lambda^2+1+\sqrt{(\lambda^2+1)^2 - 4\lambda^2\sin^2\gamma}} \\ &\le \quad \frac{2}{\lambda^2+1+|\lambda^2-1|} \ \le\ 1\,, \end{aligned} \tag{3.16}$$

with equality only when $(\gamma,\lambda) \in \left\{\frac{\pi}{2}\right\}\times(0,1]$. Thus, taking into account $(i),(iii)$ of (3.8), for $(\gamma,\lambda) \notin \{\frac{\pi}{2}\}\times(0,1]$ we have

$$1+\beta^2 = \frac{\lambda^2+1 + \sqrt{(\lambda^2+1)^2 - 4\lambda^2\sin^2\gamma}}{2\lambda^2\sin^2\gamma} \ = \ \eta \ge 1\,, \tag{3.17}$$

$$1+\alpha^2 = \frac{\lambda^2+1 + \sqrt{(\lambda^2+1)^2 - 4\lambda^2\sin^2\gamma}}{2\sin^2\gamma} \ = \ \eta\lambda^2 > 1\,. \tag{3.18}$$

By the second equation of (3.11) $\alpha\beta > 0$ if $\gamma \in (0,\frac{\pi}{2})$, while $\alpha\beta < 0$ if $\gamma \in (\frac{\pi}{2},\pi)$; furthermore, by $(ii),(iii)$ of (3.8) and (3.17), (3.18), we have $\alpha^2 = \eta\lambda^2 - 1 > 0$, $\beta^2 = \eta - 1 = 0$ for $\gamma = \frac{\pi}{2}$, $\lambda > 1$. It follows that for $(\gamma,\lambda) \notin \{\frac{\pi}{2}\}\times(0,1]$ the solutions of (3.5) must satisfy

$$(\alpha,\beta) = \begin{cases} \pm\left(\sqrt{\eta\lambda^2-1}\,,\ \sqrt{\eta-1}\,\right) & \text{if} \quad 0<\gamma<\frac{\pi}{2}\,, \quad \lambda>0, \\ \pm\left(\sqrt{\eta\lambda^2-1}\,,\ \sqrt{\eta-1}\,\right) & \text{if} \quad \gamma=\frac{\pi}{2}\,, \quad \lambda>1, \\ \pm\left(\sqrt{\eta\lambda^2-1}\,,\ -\sqrt{\eta-1}\,\right) & \text{if} \quad \frac{\pi}{2}<\gamma<\pi\,, \quad \lambda>0. \end{cases} \tag{3.19}$$

On the other hand, for $(\gamma,\lambda) \in \{\frac{\pi}{2}\}\times(0,1]$ it is immediate to verify that (3.11) is satisfied if and only if $\alpha = 0$ and $\beta^2 = \dfrac{1}{\lambda^2} - 1$. Thus, by (iii) of (3.8), we can write again

$$(\alpha,\beta) = \pm(\sqrt{\eta\lambda^2-1}\,, \sqrt{\eta-1}\,)\,, \tag{3.20}$$

and (by (ii) of (3.8)) we find two distinct solutions unless $(\gamma,\lambda) = (\frac{\pi}{2},1)$. It is now clear that (3.19), (3.20) can be summarized by formula (3.10) for all $(\gamma,\lambda) \in (0,\pi)\times(0,+\infty)$.

Finally, it is straightforward to verify that (3.10) gives, effectively, solutions of both equations of system (3.5) an that for $(\gamma,\lambda) \ne (\frac{\pi}{2},1)$ there are two distinct solutions. $\square$

Now, to prove the solvability of (3.4), we apply Lemma 2.4 and Lemma 3.2. To begin with, we set the values of the parameters $\gamma \in (0, \pi)$, $\lambda > 0$ of the system (3.5):

$$\gamma := \arccos\left(\frac{A_1 \cdot A_2}{\|A_1\|\|A_2\|}\right), \qquad \lambda := \frac{\|A_1\|}{\|A_2\|}. \tag{3.21}$$

Applying Lemma 3.2, we choose (α, β) as a real solution of the system (3.5), and then we introduce the “scale" factor $\varrho \in (0, +\infty)$:

Definition 3.3.

$$\varrho := \frac{\|A_1\|}{\sqrt{1+\alpha^2}} = \frac{\|A_1\|}{\lambda\sqrt{\eta}} = \frac{\|A_2\|}{\sqrt{\eta}} = \frac{\|A_2\|}{\sqrt{1+\beta^2}}. \tag{3.22}$$

Next, we look for an isometry which transforms the vectors $\varrho(E_1+\alpha E_3)$ and $\varrho(E_2+\beta E_3)$ into A_1 and A_2, respectively. To this aim, there are only two possibilities: we consider the sets of linearly independent row vectors $\{A_1', A_2', A_3'\}$ and $\{\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3\}$ defined by

$$\begin{aligned} A_1' &= A_1 \\ A_2' &= A_2 \\ A_3' &= \pm\,\varrho^{-1} A_1 \times A_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \widetilde{A}_1 &= \varrho(E_1+\alpha E_3) = (\varrho, 0, \varrho\alpha), \\ \widetilde{A}_2 &= \varrho(E_2+\beta E_3) = (0, \varrho, \varrho\beta), \\ \widetilde{A}_3 &= \varrho^{-1}\widetilde{A}_1 \times \widetilde{A}_2 = (-\varrho\alpha, -\varrho\beta, \varrho), \end{aligned} \tag{3.23}$$

where for A_3' we may take, indifferently, the sign “ + ” or “ − ”. From (3.21), (3.22) we have

$$A_i' \cdot A_j' = \widetilde{A}_i \cdot \widetilde{A}_j \qquad (1 \le i,\, j \le 3). \tag{3.24}$$

By Lemma 2.4, there exists a unique orthogonal transition matrix $\mathcal{T}$ such that $A_i' = \widetilde{A}_i \mathcal{T}$ $(1 \le i \le 3)$. Since $\mathcal{T}$ is orthogonal, setting

$$B_i := \varrho\, E_i\, \mathcal{T} \quad (1 \le i \le 3), \tag{3.25}$$

we have $\|B_i\| = \varrho$ and $B_i \perp B_j$ for $i \ne j$. Finally, by (3.23), we get

$$\begin{aligned} A_1 &= \widetilde{A}_1 \mathcal{T} = \varrho E_1 \mathcal{T} + \alpha(\varrho E_3 \mathcal{T}) = B_1 + \alpha B_3, \\ A_2 &= \widetilde{A}_2 \mathcal{T} = \varrho E_2 \mathcal{T} + \beta(\varrho E_3 \mathcal{T}) = B_2 + \beta B_3. \end{aligned} \tag{3.26}$$

Thus, we have proved that (3.4) is solvable with $k_1 = \alpha$, $k_2 = \beta$.

4. Determination of Pohlke matrices

The Pohlke matrix B, with the rows B_1, B_2, B_3 defined in (3.25), coincides with $\varrho\mathcal{T}$ where $\mathcal{T}$ is the transition matrix given by Lemma 2.4. Namely, by (2.12) we get

$$B = \varrho\mathcal{T} = \varrho\, \widetilde{\mathcal{G}}^{-1}\mathcal{G}, \tag{4.1}$$

where $\mathcal{G}$, $\widetilde{\mathcal{G}}$ are matrices defined as in (2.11). More precisely, by (3.23) we have:

$$\widetilde{\mathcal{G}} = \varrho \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ -\alpha & -\beta & 1 \end{pmatrix}, \tag{4.2}$$

$$\widetilde{\mathcal{G}}^{-1} = \frac{1}{\varrho(1+\alpha^2+\beta^2)} \begin{pmatrix} 1+\beta^2 & -\alpha\beta & -\alpha \\ -\alpha\beta & 1+\alpha^2 & -\beta \\ \alpha & \beta & 1 \end{pmatrix}. \tag{4.3}$$

For $\mathcal{G}$ there are two possibilities:

$$\mathcal{G} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \frac{x_2y_3 - y_2x_3}{\nu} & \frac{y_1x_3 - x_1y_3}{\nu} & \frac{x_1y_2 - y_1x_2}{\nu} \end{pmatrix} \tag{4.4}$$

with $\nu = -\varrho$ or $\nu = \varrho$. In conclusion, we finally obtain:

$$B = \frac{1}{1+\alpha^2+\beta^2} \begin{pmatrix} 1+\beta^2 & -\alpha\beta & -\alpha \\ -\alpha\beta & 1+\alpha^2 & -\beta \\ \alpha & \beta & 1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \frac{x_2y_3 - y_2x_3}{\nu} & \frac{y_1x_3 - x_1y_3}{\nu} & \frac{x_1y_2 - y_1x_2}{\nu} \end{pmatrix}, \tag{4.5}$$

where (α, β) is any solution of (3.5) and $\nu = -\varrho$ or $\nu = \varrho$.

Remark 4.1. From (4.1) we have $\|B^1\| = \|B^2\| = \|B^3\| = \varrho$, because $\mathcal{T} = \widetilde{\mathcal{G}}^{-1}\mathcal{G}$ is an orthogonal matrix. Taking into account the definition (3.22) of ϱ and (3.6), (3.21) we can easily obtain the explicit formula (1.4) for the norm of the columns of B.

4.1. The direction of projection

To find the direction of the projection corresponding to the Pohlke matrix (4.5) we write explicitly a column of $B - A$. For simplicity, below we compute $(1+\alpha^2+\beta^2)(B_3 - A_3)$:

$$\begin{pmatrix} (1+\beta^2)x_3 - \alpha\beta y_3 - \alpha\frac{x_1y_2 - y_1x_2}{\nu} \\ -\alpha\beta x_3 + (1+\alpha^2)y_3 - \beta\frac{x_1y_2 - y_1x_2}{\nu} \\ \alpha x_3 + \beta y_3 + \frac{x_1y_2 - y_1x_2}{\nu} \end{pmatrix} - \begin{pmatrix} (1+\alpha^2+\beta^2)x_3 \\ (1+\alpha^2+\beta^2)y_3 \\ 0 \end{pmatrix} = k \begin{pmatrix} -\alpha \\ -\beta \\ 1 \end{pmatrix},$$

with $k = \alpha x_3 + \beta y_3 + \frac{x_1y_2 - y_1x_2}{\nu}$. This means that the direction of projection is given by the column vector

$$U = \begin{pmatrix} -\alpha \\ -\beta \\ 1 \end{pmatrix}. \tag{4.6}$$

Remark 4.2. The Pohlke matrix B given by (4.5) corresponds to the orthogonal projection onto the plane $\{z = 0\}$ iff $(\alpha, \beta) = (0, 0)$ iff $(\gamma, \lambda) = (\frac{\pi}{2}, 1)$ iff A_1, A_2 are nonzero, orthogonal rows of equal norm, as we have already seen in Proposition 1.2.

4.2. Multiplicity of Pohlke matrices

By taking (α, β) or $-(\alpha, \beta)$ and $\nu = \varrho$ or $\nu = -\varrho$ in (4.5) we obtain, at most, 4 distinct Pohlke matrices, say $B_{(1)}, B_{(2)}, B_{(3)}, B_{(4)}$. By Remark 1.3 in case of non-orthogonal projection (i.e., when $(\alpha, \beta) \neq (0, 0)$) these matrices must be distinct. In fact, having obtained one Pohlke matrix, say $B_{(1)}$, we get $B_{(2)}$, $B_{(3)}$, $B_{(4)}$ by reflection in the image plane $\{z = 0\}$ and in the plane $\{\alpha x + \beta y - z = 0\}$ orthogonal (by (4.6)) to the direction of projection.

4.3. An example of Pohlke matrices

Let us consider the matrix

$$A = \begin{pmatrix} \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} & \sqrt{2} \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.7}$$

We have

$$\lambda = \frac{\|A_1\|}{\|A_2\|} = \sqrt{2}\,, \quad \cos\gamma = \frac{A_1 \cdot A_2}{\|A_1\|\,\|A_2\|} = \frac{\sqrt{3}}{2\sqrt{2}}\,, \tag{4.8}$$

and then

$$\eta = \frac{\lambda^2 + 1 + \sqrt{(\lambda^2+1)^2 - 4\lambda^2 \sin^2\gamma}}{2\lambda^2 \sin^2\gamma} = 2\,, \qquad \varrho = \frac{\|A_2\|}{\sqrt{\eta}} = 1\,. \tag{4.9}$$

Thus, applying Lemma 3.2, we get

$$(\alpha,\beta) = \pm\Big(\sqrt{\eta\lambda^2 - 1}\,, \operatorname{sgn}(\cos\gamma)\sqrt{\eta - 1}\ \Big) = \pm\big(\sqrt{3}\,,\, 1\big)\,. \tag{4.10}$$

According to (4.3), the matrix $\widetilde{\mathcal{G}}^{-1}$ is given by

$$\widetilde{\mathcal{G}}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -\sqrt{3} & -\delta\sqrt{3} \\ -\sqrt{3} & 4 & -\delta \\ \delta\sqrt{3} & \delta & 1 \end{pmatrix} \quad \text{with} \quad \begin{cases} \delta = 1 & (i) \\ \delta = -1 & (ii) \end{cases} \tag{4.11}$$

Having $A_1 \times A_2 = \big(-\sqrt{2}\,, \sqrt{2}\,, -1\big)$ and $\varrho = 1$, from (4.4) we find:

$$\mathcal{G} = \begin{pmatrix} \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} & \sqrt{2} \\ 1 & 1 & 0 \\ -\frac{\sqrt{2}}{\nu} & \frac{\sqrt{2}}{\nu} & -\frac{1}{\nu} \end{pmatrix} \quad \text{with} \quad \begin{cases} \nu = 1 & (I) \\ \nu = -1 & (II) \end{cases} \tag{4.12}$$

Combining the matrices (4.11), (4.12) we have 4 possibilities, say $B_{(i,I)}$, $B_{(i,II)}$, $B_{(ii,I)}$, $B_{(ii,II)}$, for the Pohlke matrix $B = \varrho\,\widetilde{\mathcal{G}}^{-1}\,\mathcal{G}$:

$$B_{(i,I)} = \frac{1}{5}\begin{pmatrix} \sqrt{6}-1 & 1-\sqrt{6} & 2\sqrt{2}+\sqrt{3} \\ \frac{5+2\sqrt{2}+\sqrt{3}}{2} & \frac{5-2\sqrt{2}-\sqrt{3}}{2} & 1-\sqrt{6} \\ \frac{5-2\sqrt{2}-\sqrt{3}}{2} & \frac{5+2\sqrt{2}+\sqrt{3}}{2} & \sqrt{6}-1 \end{pmatrix}, \tag{4.13}$$

$$B_{(i,II)} = \frac{1}{5}\begin{pmatrix} -\sqrt{6}-1 & \sqrt{6}+1 & 2\sqrt{2}-\sqrt{3} \\ \frac{5-2\sqrt{2}+\sqrt{3}}{2} & \frac{5+2\sqrt{2}-\sqrt{3}}{2} & -\sqrt{6}-1 \\ \frac{5+2\sqrt{2}-\sqrt{3}}{2} & \frac{5-2\sqrt{2}+\sqrt{3}}{2} & \sqrt{6}+1 \end{pmatrix}, \tag{4.14}$$

$$B_{(ii,I)} = \frac{1}{5}\begin{pmatrix} -\sqrt{6}-1 & \sqrt{6}+1 & 2\sqrt{2}-\sqrt{3} \\ \frac{5-2\sqrt{2}+\sqrt{3}}{2} & \frac{5+2\sqrt{2}-\sqrt{3}}{2} & -\sqrt{6}-1 \\ \frac{-5-2\sqrt{2}+\sqrt{3}}{2} & \frac{-5+2\sqrt{2}-\sqrt{3}}{2} & -\sqrt{6}-1 \end{pmatrix}, \tag{4.15}$$

$$B_{(ii,II)} = \frac{1}{5}\begin{pmatrix} \sqrt{6}-1 & 1-\sqrt{6} & 2\sqrt{2}+\sqrt{3} \\ \frac{5+2\sqrt{2}+\sqrt{3}}{2} & \frac{5-2\sqrt{2}-\sqrt{3}}{2} & 1-\sqrt{6} \\ \frac{-5+2\sqrt{2}+\sqrt{3}}{2} & \frac{-5-2\sqrt{2}-\sqrt{3}}{2} & 1-\sqrt{6} \end{pmatrix}. \qquad (4.16)$$

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