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A Study of Paths Formed by Directed Arcs and Line Segments

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Abstract. In this paper, we study paths formed by directed arcs and line segments in $\mathbb{R}^2$. The central angle and radius of the arc that we consider are $\pi/4$ and 1, respectively, and the length of the line segments is 1. We are interested in the properties of the paths that are formed by these elements. We derive the conditions that determine which locations can be reached with these paths, and a method for constructing a path to an arbitrary location within a specified distance tolerance. We also show that the number of elements in a closed path is even. Further, we show that any open path can be closed by connecting eight or fewer identical paths.

Key Words: Path, arc, line segment

MSC 2010: 51M04, 68U05

1. Introduction

Arcs and line segments are simple geometric elements of a path in $\mathbb{R}^2$. Among these, the $\pi/4$ arc, which is 1/8 of a circle, and unit length line segments can be used conveniently as elements of orderly aligned paths. These elements are observed in the paths formed by popular toys such as *Plarail train tracks* [4], as well as in geometrical patterns in 2D drawings. Figure 1 shows a photo from an exhibition of our research. The exhibits were of the elements of toy-train rails.

In this paper, we focus on the paths formed by directed $\pi/4$ arcs and unit-length straight-line segments. Figure 2 shows a tree consisting of paths made from five elements starting from the same point with the same direction. Although the terminal points of paths vary widely, it seems there exists some constraints on the locations which can be reached by these paths.

Here, we are interested in analytically determining these constraints, and what properties the paths have. This research presents a framework that describes the basic geometrical elements and operations that define the geometries constructed by these elements. This

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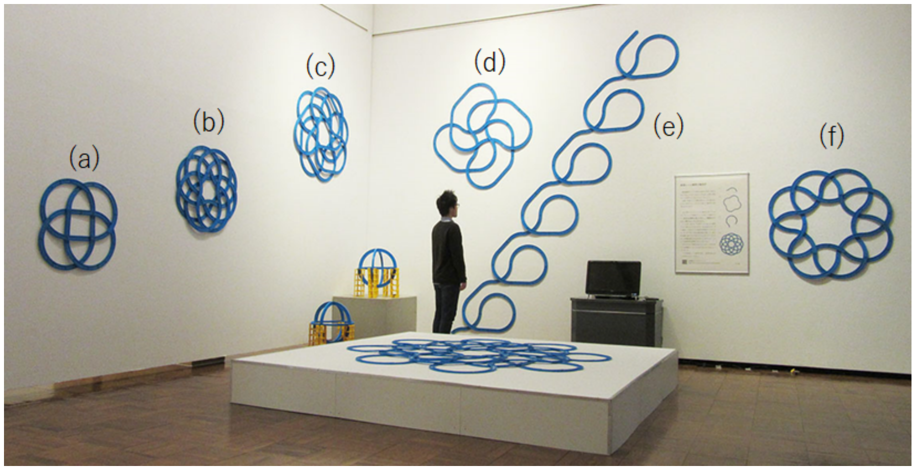

Figure 1: Exhibition of geometrical objects made with toy-train rails. The designs are based on the theory described in this paper. In terms of the notation of rails in Section 2, these are as follows: (a) $(\mathrm{SL}^6)^4$, (b) $(\mathrm{L}^6\mathrm{R})^8$, (c) $(\mathrm{RSLR}^4\mathrm{SR})^8$, (d) $(\mathrm{L}^2\mathrm{R}^3\mathrm{SRSR}^4)^4$, (e) $(\mathrm{RL}^2\mathrm{R}^6\mathrm{SR}^3\mathrm{S})^6$, (f) $(\mathrm{R}^5\mathrm{L}^2)^8$.

framework is common with that presented in [1, 3], which studied the coordinates of points that are obtained by folding a sheet of paper.

In Section 2, we define the notation used in this paper. In Section 3, we consider expressions for determining the terminal points, the positions that can be reached by the paths. We derive constraints for the variables $m_x, n_x, m_y, n_y \in \mathbb{Z}$ when the terminal point is expressed by $(m_x + n_x\sqrt{2}/2,\ m_y + n_y\sqrt{2}/2)$. Then, we show that the terminal point can take any position within a specified distance tolerance. In Section 4, closed paths are discussed, and we show

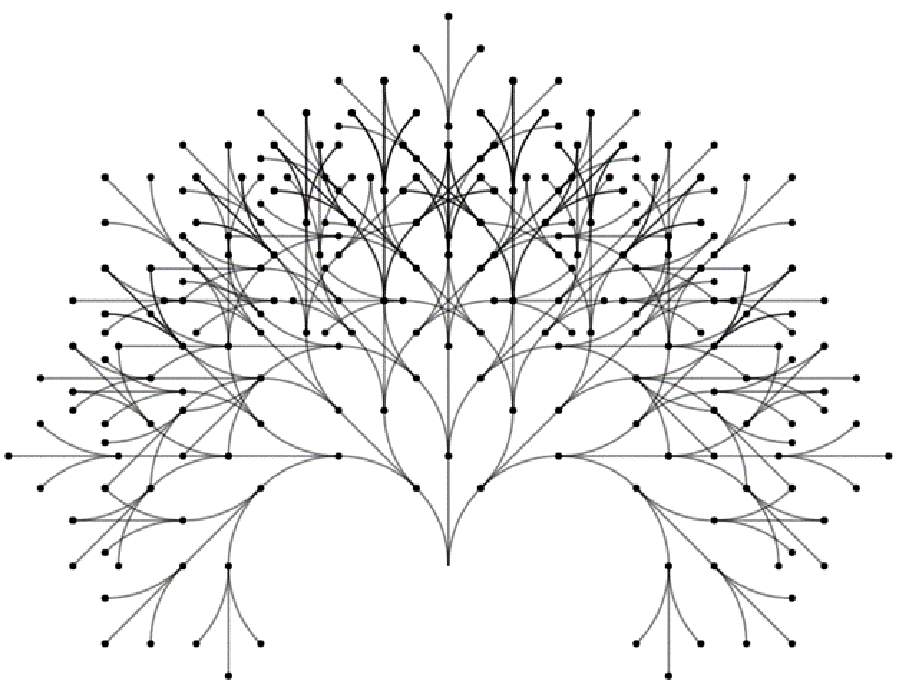

Figure 2: Tree of five-element paths

that the number of elements in a closed path is always even.

In Section 5, ways to close a path are discussed. We show that any open path can be closed by connecting eight or fewer identical paths if the direction angles at the start point and the terminal point are not the same. From this result, it is straightforward to design geometrically attractive closed paths with rotational symmetry. We summarize our results in Section 6.

2. Preliminaries

The central angle and radius of the arc that we consider in this paper are $\pi/4$ and 1, respectively, and the lengths of the line segments are 1. The elements are directed, i.e., have a start point and an end point. The start point of an element can connect to the end point of another element only when their direction angles coincide at the point. The direction angle at a point is the angle between the x-axis in Cartesian coordinates and the tangent vector of an element at the point. A path is constructed by connecting elements under this constraint.

Without loss of generality, the start point of a path is placed at $\mathrm{O}(0,0)$, and the direction angle at the start point is set as 0. Regarding arc elements, there are two distinguishable types based on the direction of rotation, i.e., counter clockwise (CCW) and clockwise (CW) (Figure 3). We use the letters L, R, and S to describe a CCW-arc, a CW-arc, and a line element, respectively. A path can then be encoded as a sequence of L, R, and S. For example, the path illustrated in the right of Figure 3 is encoded to SRLLS. When a letter or a substring is repeated within a string, we abbreviate the expression by denoting the number of repetitions in superscript to the right of the letter, or parentheses for the substring. For example, SRLLS is rewritten as $\mathrm{SRL^2S}$, and $(\mathrm{SRL^2S})^4$ expresses the path which is constructed by repeating SRLLS four times.

The *terminal point* of a path is the end point of the last element of the path. If the terminal point can be connected to the start point of the first element, the path is called a *closed path*. Note that the paths we consider may have self-intersections and overlaps.

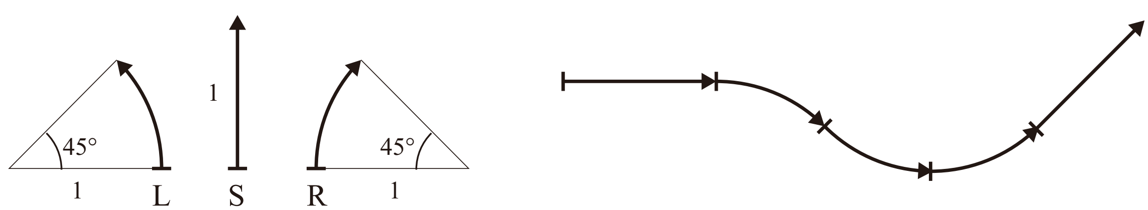

Figure 3: The elements of a path (left), and a path expressed by SRLLS or $\mathrm{SRL^2S}$ (right)

2.1. Terminal point of a path

When an element S is appended to a path, the direction angle at the terminal point does not change. On the other hand, when an R (or L) is appended to a path, the direction angle at the terminal point decreases (or increases) by $\pi/4$.

Because the direction angle at the start point of the first element is set to 0, every direction angle at the start/end point of an element in a path is expressed by $k\pi/4$ $(0 \le k \le 7)$. Table 1 shows the difference of x and y coordinate values between the end point and the start point of each element when the direction angle at the start point is $k\pi/4$. For example, we can see

Table 1: Difference of x and y coordinate values between the end point and the start point of an element

k	S	L	R
0	$(1,\ 0)$	$\left(\frac{\sqrt{2}}{2},\ 1-\frac{\sqrt{2}}{2}\right)$	$\left(\frac{\sqrt{2}}{2},\ -1+\frac{\sqrt{2}}{2}\right)$
1	$\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$	$\left(1-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$	$\left(\frac{\sqrt{2}}{2},\ 1-\frac{\sqrt{2}}{2}\right)$
2	$(0,\ 1)$	$\left(-1+\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$	$\left(1-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$
3	$\left(-\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$	$\left(-\frac{\sqrt{2}}{2}, 1-\frac{\sqrt{2}}{2}\right)$	$\left(-1+\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$
4	$(-1,\ 0)$	$\left(-\frac{\sqrt{2}}{2},\ -1+\frac{\sqrt{2}}{2}\right)$	$\left(-\frac{\sqrt{2}}{2},\ 1-\frac{\sqrt{2}}{2}\right)$
5	$\left(-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$	$\left(-1+\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$	$\left(-\frac{\sqrt{2}}{2},\ -1+\frac{\sqrt{2}}{2}\right)$
6	$(0,\ -1)$	$\left(1-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$	$\left(-1+\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$
7	$\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right)$	$\left(\frac{\sqrt{2}}{2},\ -1+\frac{\sqrt{2}}{2}\right)$	$\left(1-\frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right)$

that the coordinates of the terminal point changes by $(1-\sqrt{2}/2,\ \sqrt{2}/2)$ when an element L is appended to the path whose direction angle at the terminal point is $\pi/4$.

Since the coordinates of the terminal point of a path are calculated by aggregating the values of the corresponding cells in Table 1, the following expression can be introduced to describe the coordinates of the terminal point of a path,

$$\left(m_x + n_x\frac{\sqrt{2}}{2},\ m_y + n_y\frac{\sqrt{2}}{2}\right), \tag{1}$$

where $m_x, n_x, m_y, n_y \in \mathbb{Z}$.

Here, a question arises as to whether there exists a path that can reach positions expressed in the form of equation (1) for any m_x, n_x, m_y, and n_y, or if there exist some constraints for reachable locations. We will discuss this and show that such constraints exist.

Lemma 1. *The direction angle at the terminal point of a path can change in increments of $\pi/4$ by appending elements without changing the location of the terminal point.*

Proof. Appending a path component RSR^{5}SR (Figure 4) increases the direction angle by $\pi/4$ without changing the location of the terminal point. This is confirmed by summing the corresponding coordinates in Table 1. The change in the direction angle is confirmed by the number of Rs in the component. Seven Rs decrease the direction angle by $7\pi/4$, which is equivalent to increasing it by $\pi/4$. We name this component the *angle-adjust component*. By repeatedly connecting the angle-adjust components as necessary, the direction angle can be changed arbitrarily in increments of $\pi/4$ without changing the location of the terminal point. □

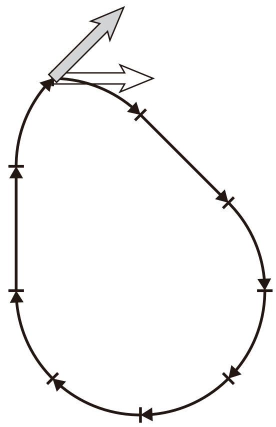

Figure 4: The angle-adjust component which increases the direction angle by $\pi/4$ without changing the location of the terminal point. The white and gray arrows show the direction vector at the start point and the terminal point, respectively.

As mentioned above in the proof, the direction angle of a path can be changed by any multiple of $\pi/4$ by using the angle-adjust component. If using the smallest number of elements is preferred, the components listed in Figure 5 can be used. All such simple components were identified using a computer with an exhaustive search.

Here, we introduce the term *general path* to refer to a path which is constructed by connecting elements without considering continuity of direction angle between successive elements while restricting the direction angle at the start point of each element to multiples of $\pi/4$. Figure 6 shows an example of a general path. To distinguish from a general path, let the

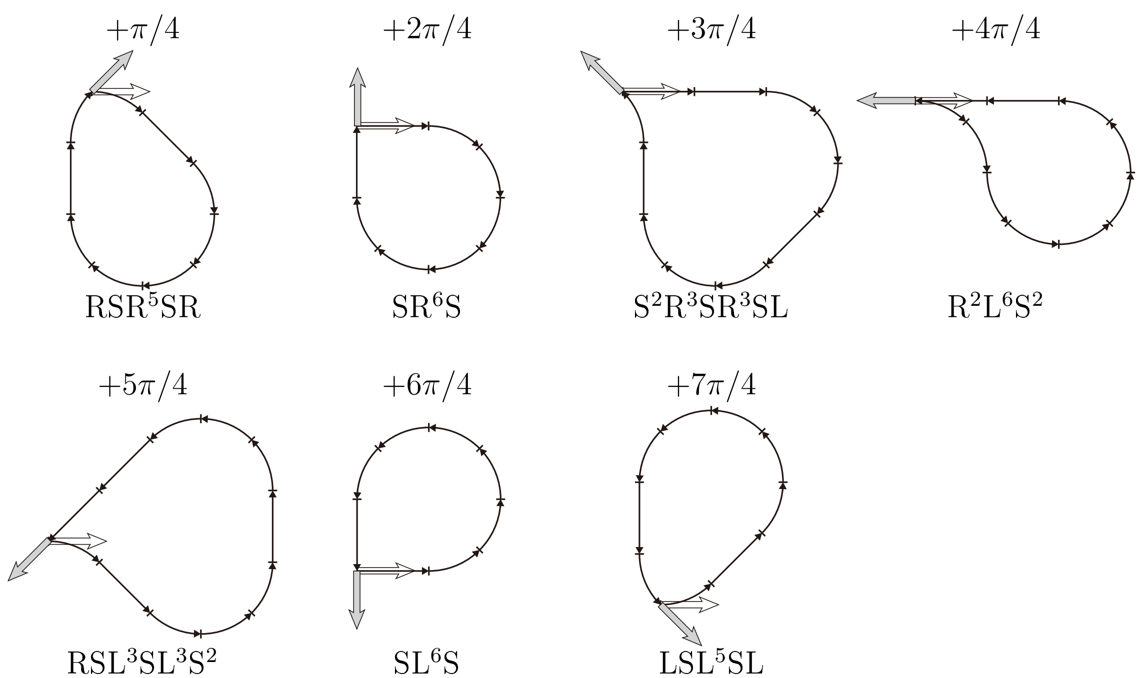

Figure 5: The seven components which change the direction angle by $k\pi/4$ $(1 \leq k \leq 7)$ without changing location of the terminal point with the smallest number of elements.

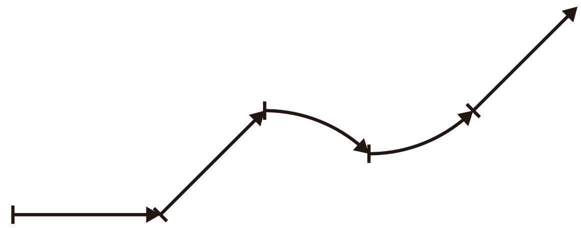

Figure 6: Example of a general path

Figure 7: A general path having an arc element (left) can be converted to LGP (right) by replacing the arc with two connected line elements

paths that we have considered until now be called *smooth paths*.

Lemma 2. *Any point that a general path can reach can also be reached by a smooth path.*

Proof. By inserting the angle-adjust component(s) as necessary at the points where the direction angles are not continuous on a general path, the general path can be converted to a smooth path without changing the location of the terminal point. Therefore, for any general path, there exist at least one smooth path whose terminal point coincides with its terminal point. □

From the definition of a general path, the set of general paths is a superset of the set of smooth paths. Therefore, it can be said that any location reached by a smooth path can also be reached by a general path. From this and Lemma 2, it is derived that the set of locations that smooth paths can reach is identical with the set of locations that general paths can reach, despite smooth paths being a subset of general paths.

Lemma 3. *Any point that a general path can reach can also be reached by a 'line element only general path' (LGP), which is a general path consisting of line elements only.*

Proof. An arc element in a general path can be replaced with two connected line elements whose inner angle is $\pi/4$, while retaining the path's connectivity. The start and end points of the first line element are placed at the start point of the arc element and the circle center of the arc, respectively. The start and end points of the second line element are placed at the circle center of the arc and the end point of the arc, respectively, as illustrated in Figure 7. All arc elements in a general path can be replaced with line elements with this manner. As the result, a general path can be converted to a LGP while retaining the location of the terminal point. □

From Lemmas 2 and 3, the following lemma is derived.

Lemma 4. *The set of locations that can be reached by a smooth path is the same with as set of locations that can be reached by a LGP.*

We discuss the possible locations of the terminal points of LGPs below.

Lemma 5. *When the location of the terminal point of a LGP is expressed by*

$$\left(m_x + n_x\frac{\sqrt{2}}{2},\ m_y + n_y\frac{\sqrt{2}}{2}\right) \tag{2}$$

where $m_x, m_y, n_x, n_y \in \mathbb{Z}$, the necessary and sufficient conditions for n_x and n_y are described as

$$|n_x \pm n_y| \equiv 0 \pmod 2,$$

while there are no restrictions on m_x and m_y.

Proof. The location of the terminal point of an LGP is expressed by

$$\sum_{0\le k\le 7} a_k\,\mathbf{e}_k, \tag{3}$$

where $\mathbf{e}_k$ is a vector of $(\cos(k\pi/4), \sin(k\pi/4))$, and a_k is the number of line elements in the path whose direction angle is $k\pi/4$.

Since $\mathbf{e}_{4+k} = -\mathbf{e}_k \quad (0 \le k \le 3)$, the expression (3) can be rewritten as

$$(a_0 - a_4)\,\mathbf{e}_0 + (a_1 - a_5)\,\mathbf{e}_1 + (a_2 - a_6)\,\mathbf{e}_2 + (a_3 - a_7)\,\mathbf{e}_3.$$

The location of the terminal point of a path is expressed by

$$a\,\mathbf{e}_0 + b\,\mathbf{e}_2 + c\,\mathbf{e}_1 + d\,\mathbf{e}_3 = \left(a + (c-d)\sqrt{2}/2,\ b + (c+d)\sqrt{2}/2\right), \tag{4}$$

where $a, b, c, d \in \mathbb{Z}$. The pairs of $\{\mathbf{e}_0, \mathbf{e}_2\}$ and $\{\mathbf{e}_1, \mathbf{e}_3\}$ are vectors independent of each other. The values of elements of $\mathbf{e}_0$ and $\mathbf{e}_2$ are integers, and the elements of $\mathbf{e}_1$ and $\mathbf{e}_3$ are irrational numbers. Therefore, the values a, b, c and d are independent of each other.

By referring expressions in (4), the variables in expression (2) are described as

$$\begin{aligned} m_x &= a, & n_x &= c - d,\\ m_y &= b, & n_y &= c + d. \end{aligned}$$

Then, the following condition is derived.

$$|n_x \pm n_y| \equiv 0 \pmod 2,$$

since

$$n_x + n_y = 2c, \quad -n_x + n_y = 2d.$$

A path whose terminal point is given by expression (4) for any a, b, c and d is constructible with the following algorithm by denoting a line segment with $k\pi/4$ direction angle as S_k.

(1) If $a > 0$, append S_0 $|a|$ times.
(2) If $a < 0$, append S_4 $|a|$ times.
(3) If $b > 0$, append S_2 $|b|$ times.
(4) If $b < 0$, append S_6 $|b|$ times.
(5) If $c > 0$, append S_1 $|c|$ times.
(6) If $c < 0$, append S_5 $|c|$ times.
(7) If $d > 0$, append S_3 $|d|$ times.
(8) If $d < 0$, append S_7 $|d|$ times. $\square$

From Lemmas 4 and 5, the following Theorem is immediately derived.

Theorem 1. *The locations the terminal point of a path can take is expressed by*

$$\left(m_x + n_x\frac{\sqrt{2}}{2},\ m_y + n_y\frac{\sqrt{2}}{2}\right) \quad |n_x \pm n_y| \equiv 0\,(\mathrm{mod}\ 2),$$

where $m_x, m_y, n_x, n_y \in \mathbb{Z}$.

Based on Theorem 1, there exist coordinates unreachable from the terminal point of any path, e.g., $(0, \sqrt{2}/2)$. However, a path which reaches any specified position within any distance tolerance is constructible.

Theorem 2. *For any* $\mathbf{q} \in \mathbb{R}^2$ *and* $\varepsilon > 0$*, there exists a path having its terminal point* $\mathbf{p}$ *such that* $|\mathbf{p} - \mathbf{q}| < \varepsilon$.

Proof. We introduce the grid system G whose grid line distance is 1, and G' obtained by rotating G by $\pi/4$ where $\mathbf{e}_0 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ are basis vectors of G, and $\mathbf{e}_1 = (\sqrt{2}/2,\ \sqrt{2}/2)$ and $\mathbf{e}_3 = (-\sqrt{2}/2,\ \sqrt{2}/2)$ are basis vectors of G' (Figure 8).

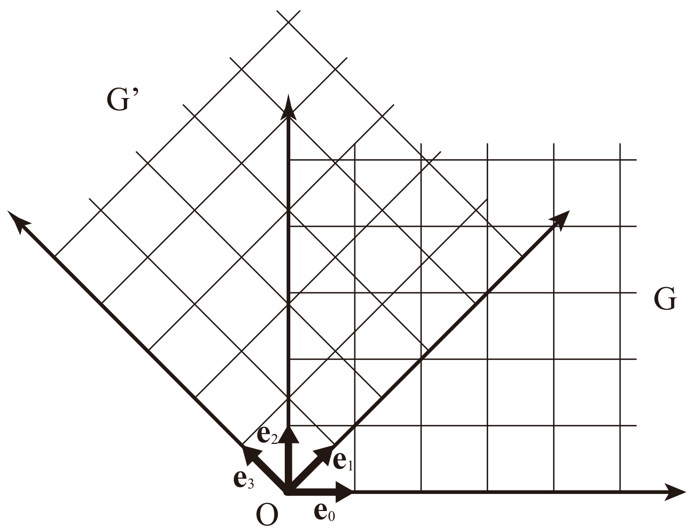

Figure 8: Grid systems G and G'

The path is constructed by the following steps (Figure 9).

(1) Move the origin of G' to $\mathbf{q}$.

(2) Find a pair of grid points $\mathbf{v} \in G$ and $\mathbf{v}' \in G'$ such that $|\mathbf{v} - \mathbf{v}'| < \varepsilon$. A pair of points which satisfies this constraint always exists because the fractional part of $n\sqrt{2}$ $(n \in \mathbb{Z})$ is dense in $[0, 1]$ [2].

(3) Construct an LGP (P_1) in G whose start point and terminal point is the origin of G and $\mathbf{v}$, respectively.

(4) Construct a LGP (P_2) in G' whose start point and terminal point is $\mathbf{v}'$ and the origin of G', respectively.

(5) Make the LGP (P_3) by connecting P_1 and P_2 by moving the start point of P_2 to the terminal point of P_1.

(6) The path is constructed by converting P_3 to a smooth path by inserting the angle-adjust component(s) as necessary. □

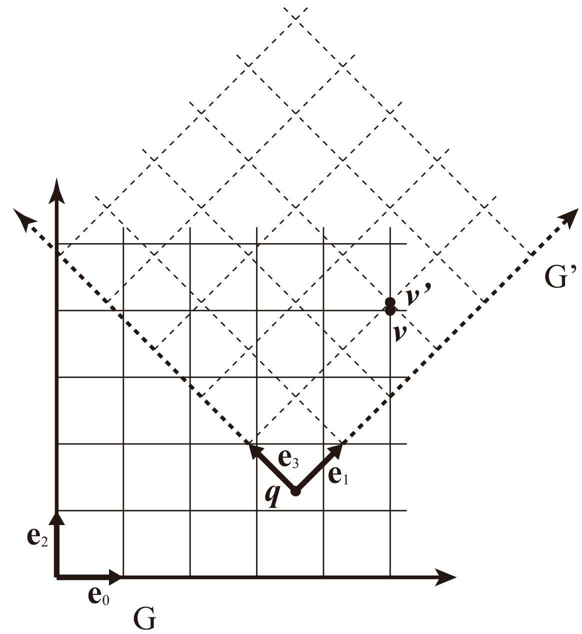

Figure 9: The path is constructed by connecting a path O → **v** on G, and **v**′ → O on G'

3. Closed path

In this section, we consider the number of elements in closed paths.

Lemma 6. *The number of arc elements in a closed path is positive and even.*

Proof. A path consisting only of line elements cannot be closed. Therefore, the number of arc elements in a closed path is non-zero. An R element decreases the direction angle by $\pi/4$ and an L element increases it by $\pi/4$, and the direction angle at the terminal point coincides with the direction angle at the start point. Therefore, $|n_R - n_L| \equiv 0 \pmod 8$, where n_R and n_L are the numbers of R and L elements in the path.

From the following equation, the sum of the numbers of R and L elements in a closed path, i.e., the number of arc elements, is even.

$$n_R + n_L = n_R - n_L + 2n_L = \pm 8m + 2n_L = 2(n_L \pm 4m)$$

where m is an integer. □

Lemma 7. *The number of line elements in a closed path is even.*

Proof. Any closed path can be converted to LGP by replacing each arc element by two connected line elements, as shown in Figure 7. The number of line elements in the LGP is $n_L + 2n_A$, where n_L and n_A are the numbers of line elements and arc elements in the original closed path, respectively. Since the terminal point of the LGP is located at the start point $(0, 0)$, the following equation holds:

$$\sum_{0 \le k \le 7} a_k \, \mathbf{e}_k = (0, 0),$$

where a_k and $\mathbf{e}_k$ are the same which were used in expression (3). To satisfy this equation, the following relations hold:

$$a_0 = a_4, \quad a_2 = a_6, \quad a_1 = a_5, \quad a_3 = a_7.$$

Therefore,

$$\sum_{0\le k\le 7} a_k = 2 \sum_{0\le k\le 3} a_k \,.$$

It says that the number of elements in the LGP is even. Hence,

$$n_L + 2n_A \equiv 0 \pmod 2.$$

Finally, it is derived that n_L, the number of line elements in the original closed path, is even. □

From Lemmas 6 and 7, the following theorem is immediately derived.

Theorem 3. *The number of elements in a closed path is even.*

4. Path closing

In this section, we discuss the closing of an open path.

Theorem 4. *Any open path can be closed by adding elements.*

Proof. A *reverse component*, which changes the direction by π without changing the location, is constructed by connecting four angle-adjust components. A closed path is created by adding the reverse components at the start point and the terminal point, and by reproducing the original open path in the reverse direction. □

Theorem 5. *Any open path can be closed by repeatedly connecting the identical path at most eight times if the direction angles at the terminal points is not* 0.

Proof. If identical paths are repeatedly connected, the start point of each path lies on a common circle with identical angle intervals, as illustrated in Figure 10. If a pair n and m of positive integers satisfies

$$n\theta = 2m\pi, \tag{5}$$

where θ is the difference between direction angles at the start point and the terminal point ($\theta = k\pi/4,\ (1 \le k \le 4)$), then the path can be closed by repeatedly connecting the original path n times. The relation between θ, eq. (5), and the minimum number of n such that m exists is shown in Table 2. Since the maximum number of the minimum number of n is 8, an open path can be closed by connecting eight or fewer identical paths. □

Table 2: Relation between θ, n, and m

θ	*Equation*	*min value of n*
$\frac{1}{4}\pi$	$n = 8m$	$n = 8$
$\frac{2}{4}\pi$	$n = 4m$	$n = 4$
$\frac{3}{4}\pi$	$3n = 8m$	$n = 8$
π	$n = 2m$	$n = 2$

The triangles illustrated in Figure 10 form a regular octagon and a square, when $\theta = \pi/4$ and $\theta = 2\pi/4$, respectively. When $\theta = 3\pi/4$, the triangle forms a star regular polygon described by the Schläfli symbol 8/3 [5] (Figure 11).

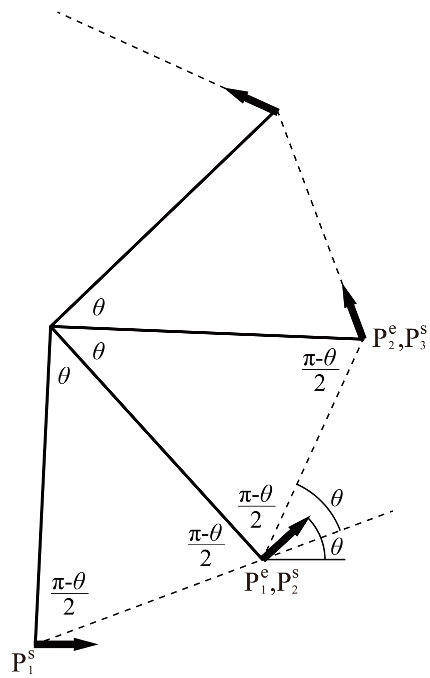

Figure 10: The start point P_i^s and the end point P_i^e of the i-th connected path are located on a common circle. The angle bisectors of $\angle P_{i-1}^s P_i^s P_{i+1}^s$ meet at a point and form isosceles triangles with the inner angle θ.

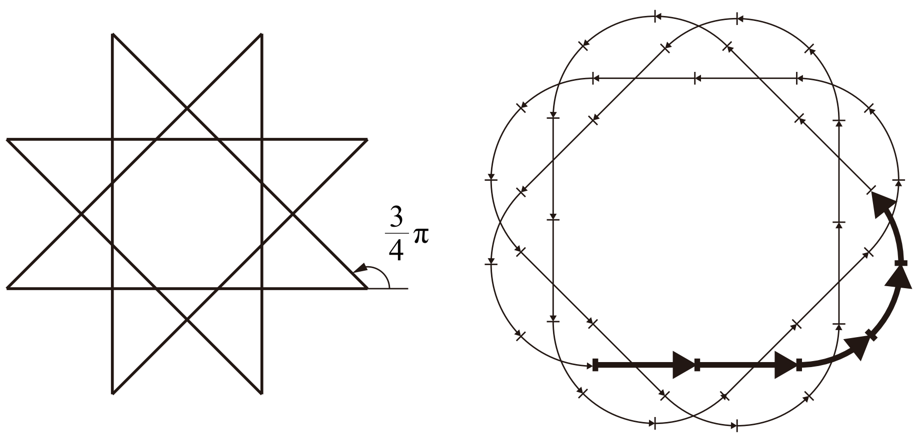

Figure 11: A regular star polygon {8/3} (left), and a closed path created by repeating the sequence SSLLL eight times (right). The initial path is drawn in bold.

5. Summary

We studied paths formed by directed $\pi/4$ arcs and line segments. We discussed the possible locations that can be reached by the terminal points of such paths and the number of elements in closed paths. Further, we discussed ways to close an open path.

Determining the shortest path to close a given open path remains an open problem. The problem restricted to non-self-intersecting paths will result in far more complex solutions than the case considered in this paper. In future work, we are also interested in fully enumerating the possible configurations of closed paths with a specified number of elements.

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