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Theorems on Two Tetrahedrons Intersecting a Sphere

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Abstract. In this article, we describe three-dimensional theorems of two tetrahedrons intersecting a sphere. These theorems can be considered as generalizations of the two-dimensional Pascal's hexagon and Steiner's theorems. We first restructure the original version of the two-dimensional Pascal's hexagon theorem, and prove it synthetically using a simple lemma. In the proving process, we found the essential nature of Pascal's theorem that leads to the synthetic generalization in three-dimensional space. In order to focus on visual representations, we only use a synthetic method in the generalization process.

Key Words: triangles and tetrahedrons in perspective, extension of Pascal's hexagon and Steiner's theorems

MSC 2010: 51M04, 51M35

1. Introduction

In a Euclidean version, Pascal's hexagon theorem states that if a hexagon is inscribed in a circle, the intersections of opposite sides are collinear [3, 8]. Because of its beautiful harmonic properties, this theorem is widely known as one of the greatest theorems in Euclidean geometry. Many mathematicians have examined this theorem, and found additional collinear and concurrent points [6, 7].

Pascal's theorem may be stated as follows: "If two triangles in inverted position (t and t') where six points of intersections lie on a circle, then opposite sides of t and t' meet in three collinear points" (Figure 1a). Analogously, Chasles [1] extended Pascal's theorem to three-dimensional space as follows: "If the four triads of edges of a tetrahedron T passing through the four vertices are cut by the four planes of a tetrahedron T' and the twelve points of intersections lie on a sphere, then the opposite planes of T and T' meet in four lines which are coplanar or hyperbolic" (Figure 1b). This was proved by Salmon [5] and rediscovered by Court [2]. Although Pascal's theorem was clearly generalized in three dimensions, something seems to be missing since a hyperboloid suddenly appears. Notice that in the two-dimensional

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theorem, two triangles t and t' are in perspective position and opposite sides meet in three collinear points. Thus, in the completely general three-dimensional theorem, two tetrahedrons T and T' are also expected to be in perspective position and the opposite planes should meet in four coplanar lines.

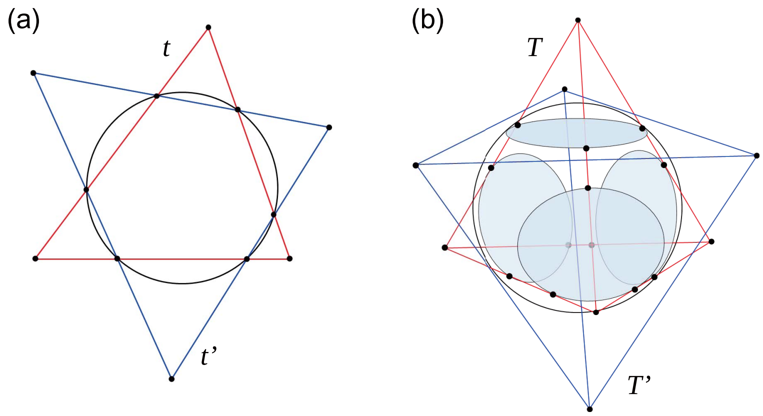

Figure 1: (a) Two triangles t and t' in inverted position with six points of intersection on a circle. (b) Four triads of edges of a tetrahedron T passing through the four vertices are cut by the four planes of inverted tetrahedron T'.

In this article, we generalize Pascal's and Steiner's theorem to three-dimensional space in a way that satisfies the above requirements. We first restructure the original version of the two-dimensional Pascal's theorem. The three diagonal segments of the hexagon and the two triangles in perspective position are the focus of the restructuring process. We will then prove this restructured theorem using a simple lemma (Lemma 1). In the process of proving the two-dimensional Pascal's theorem using Lemma 1, we found the essential nature of Pascal's theorem. This easily leads to a three-dimensional generalization of the two-dimensional Pascal's and Steiner's theorems.

2. Restructuring the original Pascal's hexagon theorem

An Euclidean version of Pascal's hexagon theorem can be stated as follows: If the six points P_1, Q_1, P_2, Q_2, P_3, and Q_3 lie on a circle c, the intersections of Q_2P_1 with Q_1P_2, P_2Q_3 with P_3Q_2, and Q_1P_3 with Q_3P_1 are on a line (see Figure 2a). In order to generalize the theorem, we restructure the theorem as the following.

Theorem 1. *Let l_1, l_2, and l_3 be mutually intersecting chords of a circle c (see Figure 2b). Let lines a_1 and b_1 be lines joining the endpoints of l_2 and l_3 such that they do not intersect l_1. Similarly, let a_2 and b_2 be lines joining the endpoints of l_3 and l_1, and let a_3 and b_3 be lines joining the endpoints of l_1 and l_2. Then, $a_1 \cap b_1 = X_1$, $a_2 \cap b_2 = X_2$, and $a_3 \cap b_3 = X_3$ are on a line.*

We will prove the theorem later. This restructured Pascal's hexagon theorem is identical to the original theorem because the segments l_1, l_2, and l_3 in Figure 2b can be considered as the segments P_1Q_1, P_2Q_2, and P_3Q_3, respectively, in Figure 2a. Here, the line containing X_1, X_2, and X_3 is called the *Pascal line*. Steiner's theorem [7], related to Pascal's hexagon theorem, can be stated as the following.

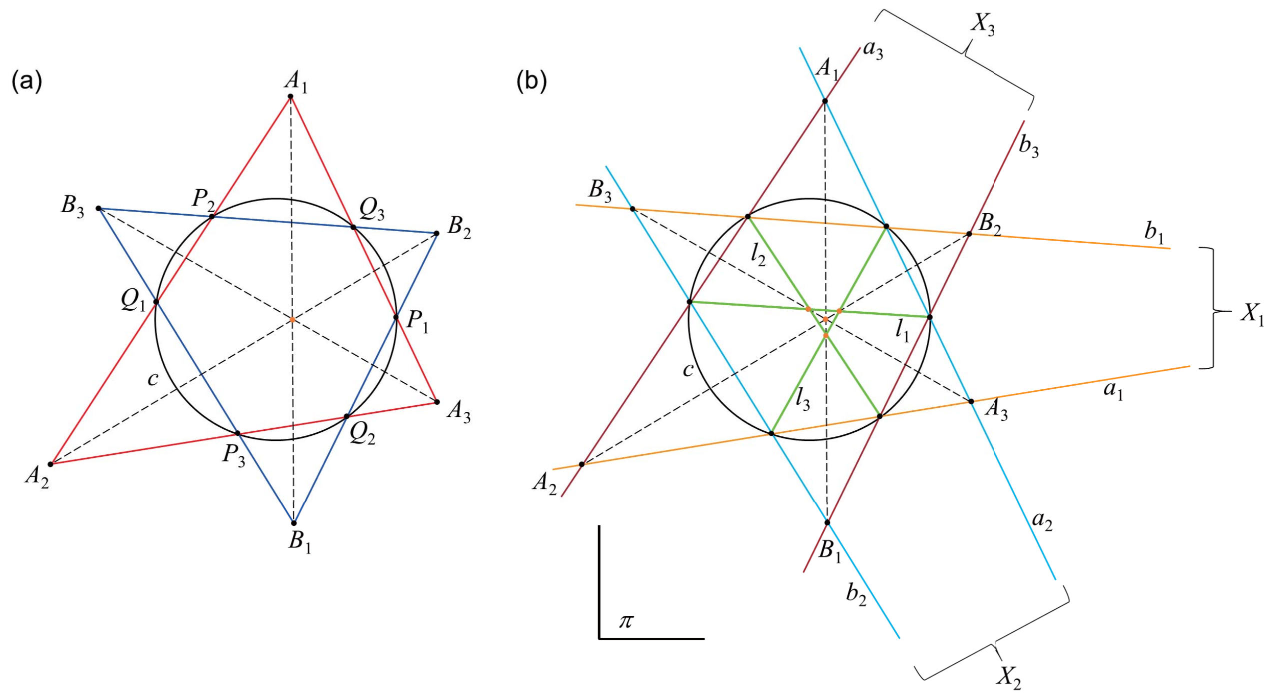

Figure 2: (a) Configuration of Pascal's hexagon and Steiner's theorems. (b) Restructured form of Pascal's hexagon theorem.

Theorem 2 (Steiner's theorem)**.** *For the configuration of Theorem 1, let* $A_1A_2A_3$ *and* $B_1B_2B_3$ *be triangles whose sides are denoted by* a_1, a_2, a_3 *and* b_1, b_2, b_3*, respectively, as shown in Figures 2a and 2b. Then, the three lines* A_1B_1*,* A_2B_2*, and* A_3B_3 *are concurrent.*

Theorem 2 can be easily deduced from Theorem 1. If X_1, X_2, and X_3 are collinear the two triangles $A_1A_2A_3$ and $B_1B_2B_3$ share the same center of perspective. This center is called the *Steiner point*. In the case shown in Figure 2b, we also have the following corollary.

Corollary 1. *For the configuration of Theorem 2, the following triples of lines* $\{l_2, l_3, A_1B_1\}$*,* $\{l_3, l_1, A_2B_2\}$*, and* $\{l_1, l_2, A_3B_3\}$ *are concurrent (see Figure 2b).*

The following lemma will be very important for synthetic proofs of the theorems and corollary given above, and further of generalizations to three dimensions.

Lemma 1. *Let* c_a *and* c_b *be intersection circles of a sphere* S *and two planes. If* c_a *and* c_b *intersect each other, then there exist two cones* T_O *and* T_U *which include* c_a *and* c_b *(see Figure 3).*

Proof. On the sphere S, let P_1, Q_1, P_2, and Q_2 be points on the great circle c_g, and furthermore c_a and c_b be circles orthogonal to c_g with $P_1, Q_1 \in c_\mathrm{b}$ and $P_2, Q_2 \in c_\mathrm{a}$ (Figure 4). Let O be the intersection of the lines P_1P_2 and Q_1Q_2, and consider the additional sphere S_O which includes c_b and O. Let O' (South pole) be the point of the sphere diametrically opposite to the point of O (North pole), and σ be the plane tangent to sphere S_O at O'.
We now consider the inverse stereographic projection of the sphere S_O with center O into the plane σ. If we suppose that P_1, Q_1, and c_b are projected onto P_1', Q_1', and c_b' in σ then we know that c_b' is a circle in σ because of the properties of inverse stereographic

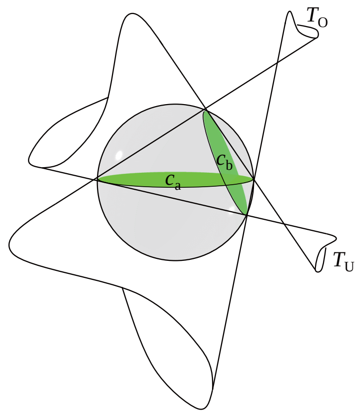

Figure 3: Two cones T_{O} and T_{U} including the intersection circles C_{a} and C_{b} between the sphere and two planes.

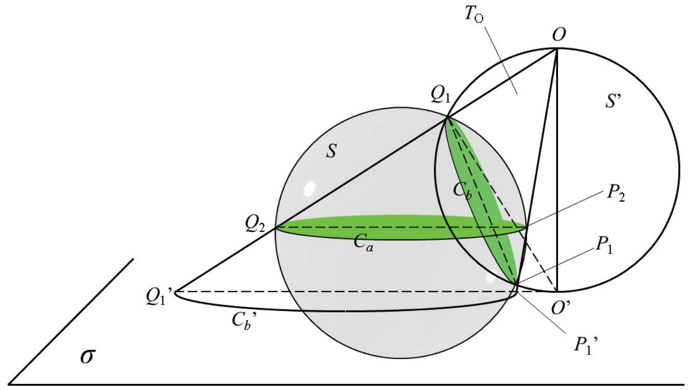

Figure 4: Figure for the proof of Lemma 1

projection. Furthermore, from properties of the angles of circumference we know that $\measuredangle OO'Q_1 = \measuredangle OP_1Q_1 = \measuredangle OQ_2P_2$. We also learn from similar triangles OQ_1O' and $OO'Q_1'$ that $\measuredangle OO'Q_1 = \measuredangle OQ_1'O'$. Thus, we have $\measuredangle OQ_2P_2 = \measuredangle OQ_1'O'$, which indicates that c_{a} is parallel to σ.
Since c_{a} and c_{b}' are circles with respective diameters Q_2P_2 and $Q_1'P_1'$ in parallel planes, we see that the projection of c_{a} from O into σ gives a circle c_{b}'. This shows that there exists a cone T_{O} with vertex O which passes through c_{a} and c_{b}. Similarly, there also exists a second cone T_{U} with vertex U which includes c_{a} and c_{b} (see Figure 3), where U is the point of intersection between Q_2P_1 with Q_1P_2). This completes the proof of Lemma 1. □

Remark 1. Lemma 1 holds also in the case when c_{a} and c_{b} do not intersect. However, we do not discuss the case since it will not be applied in the theorems in this paper. In addition, notice that each plane which is tangent to both circles must be a tangent plane of one of these cones.

Definition 1. Let c_1, c_2, and c_3 be intersection circles of the sphere S with three planes such that the circles are mutually intersecting at six points. Let $\{G_1, G_1'\} = c_1 \cap c_3$, $\{G_2, G_2'\} = c_1 \cap c_2$ and $\{G_3, G_3'\} = c_2 \cap c_3$, as shown in Figure 5.
Notice that two types of configurations can be considered. One is the *Type I configuration* where the common point of the three planes, which is the intersection point of the lines G_1G_1', G_2G_2' and G_3G_3', is inside of S (Figure 5a), while the other is the *Type II configuration* where this point is outside of S or at infinity (Figure 5b).
Now, we define *circular triangles* on S with sides defined by three mutually intersection circles, not necessarily great circles. For example, the circular triangles $G_1G_2G_3$ and $G_1'G_2'G_3'$ have the curved sides c_1, c_2, c_3 and the incircles α and α', which are shown in the Figures 5a and 5b.

Remark 2. Note that the circular triangles $G_1G_2G_3$ and $G_1'G_2'G_3'$ as well as α and α' are opposite pairs of triangles and incircles, and there exist four such pairs of circular triangles $G_1G_2G_3$ and $G_1'G_2'G_3'$, $G_1G_2G_3'$ and $G_1'G_2'G_3$, $G_1G_2'G_3$ and $G_1'G_2G_3'$, and the pair $G_1'G_2G_3$ and $G_1G_2'G_3'$, i.e., in total eight circular triangles, who share the circular sides c_1, c_2, c_3. Therefore, there exist eight incircles of the circular triangles defined by c_1, c_2, c_3. Each plane spanned by any incircle is tangent to all three circles c_1, c_2, c_3.

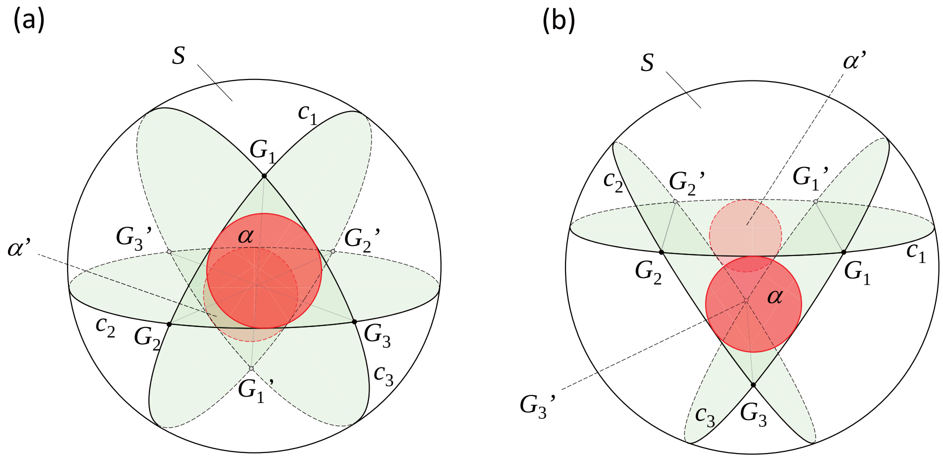

Figure 5: (a) Three mutually intersecting circles c_1, c_2, c_3 on the sphere S in Type I configuration, and (b) in Type II configuration, together with one pair of opposite incircles.

We we use Lemma 1 in order to prove Theorem 1.

Proof of Theorem 1. Assume that all the elements (circles, lines, etc.) in Figure 2b lie in the plane π. In Figure 6, let S be a sphere centered in π, and let c be the great circle that is the intersection of S and π. Let c_1, c_2, and c_3 be intersection circles of S with planes orthogonal to π. Let l_1, l_2, and l_3 be the segments defined by the intersections of c_1, c_2, and c_3, respectively, with π. Applying Lemma 1, we know that there exist cones T_1, T_2, and T_3 with external apices X_1, X_2, and X_3 , which include c_2, c_3, c_3, c_1, and c_1, c_2, respectively,

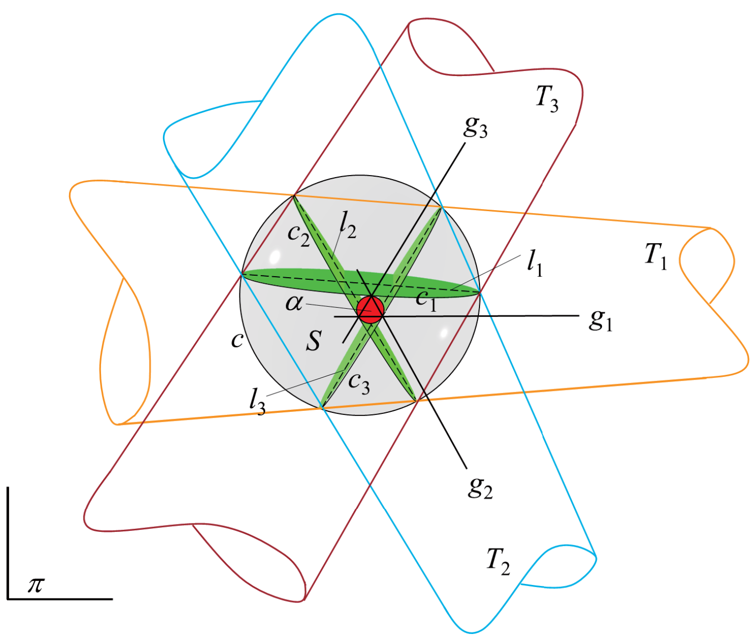

Figure 6: Three circles on the sphere together with three cones, each connecting two circles out of the three. For simplicity, the complete intersections between the cones are not displayed in the figure.

as shown in Figure 6. For simplicity, the complete intersections between the cones are not desplayed in the figure.

Now, let us consider the circular triangles on S defined by c_1, c_2, and c_3, which lie completely on one side of π. The configuration of c_1, c_2, and c_3 is of Type II, and there are two circular triangles that satisfy this condition. Consider one of the circular triangles and its incircle denoted by α. As mentioned in Remark 2, the plane spanned by α is tangent to c_1, c_2, and c_3. Notice that the plane of α is also tangent to T_1, T_2, and T_3, and it includes the generators g_1, g_2, and g_3 of T_1, T_2, and T_3 which pass through the vertices X_1, X_2, and X_3, respectively. Here, for example, g_1 is the line which passes through the points of contact between the plane of α and c_2 and c_3 and through the apex X_1. Thus, the vertices X_1, X_2, and X_3 are on the intersection line of π with the plane spanned by α. It follows that X_1, X_2, and X_3 are collinear, which completes the proof of Theorem 1. $\square$

Proof of Theorem 2 and Corollary 1. Applying the converse of Desargues' theorem, we notice that A_1B_1, A_2B_2, and A_3B_3 are concurrent since the two triangles $A_1A_2A_3$ and $B_1B_2B_3$ are in perspective position with respect to (w.r.t. in brief) the *Pascal line* $X_1X_2X_3$. We also notice that the lines l_2, l_3 and A_1B_1 are concurrent since the two triangles $A_1P_2Q_3$ and $B_1Q_2P_3$ are also perspective w.r.t. the Pascal line $X_1X_2X_3$, which automatically proves the concurrency of l_3, l_1 and A_2B_2, and of l_1, l_2 and A_3B_3, due to the permutation symmetry. This completes the proofs of Theorem 2 and Corollary 1. $\square$

One must notice above, that Lemma 1 and Desargues' theorem in two dimensions are quite important and essential for Pascal's and Steiner's theorems and the corollary.

3. Extension of Pascal's hexagon theorem and Steiner's theorem to three dimensions

Theorem 3 (Extension of Pascal's theorem to three dimensions)**.**
Let c_1, c_2, c_3, and c_4 be the circles of intersection between a sphere S and four planes such that the circles are mutually intersecting, and let all triads of the circles out of $\{c_1, c_2, c_3, c_4\}$ be arranged in Type I configurations.

Let α_k, β_k be pairs of intersection circles of S with planes tangent to c_l, c_m and c_n, where (k, l, m, n) is any permutation of $(1, 2, 3, 4)$. Let the triad of circles $\alpha_l, \alpha_m, \alpha_n$ be on the same side of the plane of c_k, while the triad of circles $\beta_l, \beta_m, \beta_n$ is on the other side of the plane c_k. That is, (α_1, β_1), (α_2, β_2), (α_3, β_3), and (α_4, β_4) are pairs of incircles of circular triangles defined by (c_2, c_3, c_4), (c_3, c_4, c_1), (c_4, c_1, c_2), and (c_1, c_2, c_3), respectively (Figures 7a and 7b).

Now we consider the two tetrahedrons whose planes are spanned by the circles $\alpha_1, \alpha_2, \alpha_3$, α_4 and by $\beta_1, \beta_2, \beta_3$, β_4. If the plane spanned by α_i, $i = 1, \ldots, 4$, intersects the opposite plane spanned by β_i along a line denoted by p_i (see Figure 7c) then all lines $p_1, \ldots, p_4$ lie in the same plane (see Figure 7).

Proof. Due to Lemma 1, we know that there exist two external tangent cones, denoted by T_{kl}, which include the pair of circles (c_k, c_l). The cone T_{kl} is tangent to the pairs of circles (α_m, β_m) and (α_n, β_n), and not tangent to the pairs of circles (α_k, β_k) and (α_l, β_l). That is, when we specify $k = 2$ and $l = 3$, for example, then T_{23} is a cone passing through c_2 and c_3, which is tangent to both pairs (α_1, β_1) and (α_4, β_4), but not tangent to the pairs (α_2, β_2) and $(\alpha_3, \beta_3$.

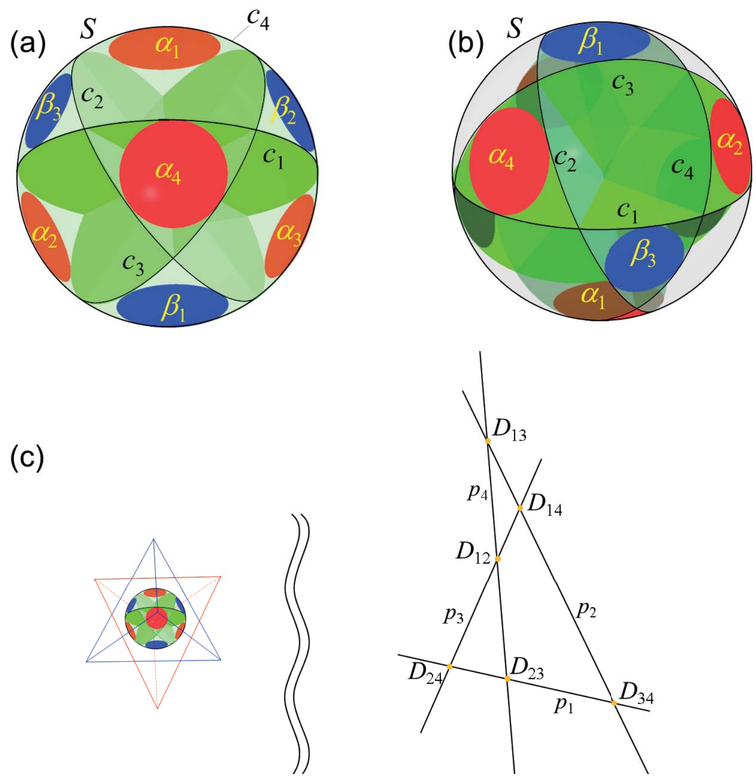

Figure 7: (a) Configuration of the extension of Pascal's theorem to three dimensions. (b) View from a different angle. (c) Lines of intersections of planes with the opposite faces of the tetrahedron.

Let D_{kl} denote the vertex of T_{kl}. Then, we know (note Remark 2) that D_{23} is coplanar with α_1, β_1, α_4 and β_4. Therefore, D_{23} lies on the lines p_1 and p_4 where the planes spanned by the respective pairs (α_1, β_1) and (α_4, β_4) intersect. Consequently, we find that $D_{23} = p_1 \cap p_4$. Similarly, we have $D_{34} = p_1 \cap p_2$, $D_{24} = p_1 \cap p_3$, $D_{12} = p_3 \cap p_4$, $D_{14} = p_2 \cap p_3$, and $D_{13} = p_2 \cap p_4$. This indicates that the lines p_1, p_2, p_3, and p_4 intersect each other at six points, all in the same plane. This completes the proof of Theorem 3. $\square$

Remark 3. In the proof of Theorem 3 we proved the coplanarity of six lines. This argument is valid only when the four lines p_1, p_2, p_3, p_4 mutually intersect at six different points. Note that the three points D_{12}, D_{13}, D_{23} lie on p_4, and they must be mutually different, since each point is on a line joining two points out of three points of contact between α_4 and the circles c_1, c_2, c_3. Also, p_1 and p_4 must be different lines meeting at D_{23}, since the planes spanned by α_1 and α_4 are different. These two facts show that the four lines p_1, p_2, p_3, p_4 intersect mutually at six different points.

Theorem 4 (Extension of Steiner's theorem to three dimensions)**.**
Following the notation in Theorem 3, let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be two tetrahedrons whose faces lie in the planes spanned by $\alpha_1, \alpha_2, \alpha_3$, and α_4 as well as $\beta_1, \beta_2, \beta_3$, and β_4 (see Figure 8). Then, the lines A_1B_1, A_2B_2, A_3B_3, and A_4B_4 are concurrent.

The plane through $p_1, \ldots, p_4$, as described in Theorem 3, can be considered as an extension of the *Pascal line* in Theorem 1; so we refer to this plane as a *Pascal plane*. Furthermore, the

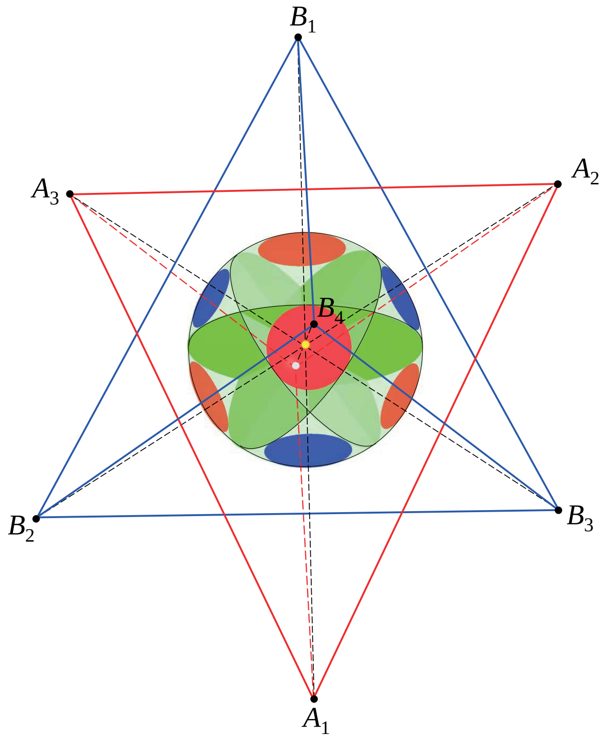

Figure 8: Steiner's theorem extended to three dimensions.

point of concurrency in Theorem 4 can be considered as an extension of the Steiner point in Theorem 2; so we refer to this point as an *extended Steiner point*. The extension of Steiner's theorem to three dimensions in Theorem 4 can be automatically proved by using the following lemma.

Lemma 2 (Desargues' theorem in three dimensions)**.**
If two tetrahedrons share a center of perspectivity, then the lines of intersection between corresponding faces lie in the same plane. Conversely, if two tetrahedrons have lines of intersection

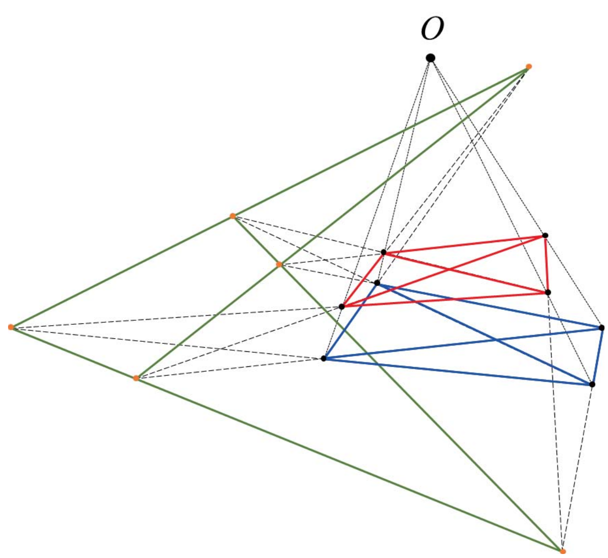

Figure 9: Extended theorem of Desargues applied to the red and the blue tetrahedron.

between corresponding faces in the same plane, then the tetrahedrons share a center of perspectivity.

The above lemma, which is illustrated in Figure 9, is quite easy to prove; a proof was given in our previous paper in this journal [4]. Note that the concurrency at the Steiner point, as stated in Theorem 2, can be proved by the two-dimensional Desargues' theorem, while the concurrency at the extended Steiner point, stated in Theorem 4, can be proved by the three-dimensional Desargues' theorem (Lemma 2).

In Corollary 1, we saw that A_1B_1 passes through the point of intersection between l_2 and l_3, as depicted in Figure 2b. Thus, one may wonder whether A_1B_1 passes through point of intersection between the three planes spanned by circles c_2, c_3, and c_4 (Figure 8). In fact, it does not, since Lemma 2 cannot be applied to this configuration. Instead, applying Lemma 2, we have the following result, which is a three-dimensional extension of Corollary 1.

Corollary 2. *Referring to the configuration of Theorem 4, let A_{12}, A_{13}, and A_{14} be the points where the lines A_1A_2, A_1A_3, and A_1A_4 intersect the plane $B_2B_3B_4$, and let B_{12}, B_{13}, and B_{14} be the points where the lines B_1B_2, B_1B_3, and B_1B_4 intersect the plane $A_2A_3A_4$. Analogously, we define the new points A_{21}, A_{23}, A_{24}, B_{21}, B_{23}, B_{24}, A_{31}, A_{32}, A_{34}, B_{31}, B_{32}, B_{34}, A_{41}, A_{42}, A_{43}, B_{41}, B_{42}, and B_{43}.*
Then all pairs of lines taken out from the four-line sets $\{A_1B_1, A_{12}B_{12}, A_{13}B_{13}, A_{14}B_{14}\}$, $\{A_2B_2, A_{21}B_{21}, A_{23}B_{23}, A_{24}B_{24}\}$, $\{A_3B_3, A_{31}B_{31}, A_{32}B_{32}, A_{34}B_{34}\}$, and $\{A_4B_4, A_{41}B_{41}, A_{42}B_{42}, A_{43}B_{43}\}$ are concurrent.

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References

[1] M. Chasles: *Aperçu Historique sur l'Origine et le Développment des Méthodes en Géométrie*. Gauthier-Villars, Paris 1889.

[2] N.A. Court: *Pascal's theorem in space*. Duke Math. J. **20**, 417–421 (1953).

[3] R.A. Johnson: *Modern Geometry*. Houhton Miffline, Boston/MA 1929.

[4] K. Morita: *Some Theorems on Kissing Circles and Spheres*. JGG **15**, 159–168 (2011).

[5] G. Salmon: *A Treatise of Analytic Geometry of Three Dimensions*. 1919.

[6] G. Salmon: *A Treatise on Conic Sections*. Chelsea 1960.

[7] J. Steiner: Ann. Math. **18**, 339–340 (1827-1828).

[8] D. Wells: *The Penguin Dictionary of Curious and Interesting Geometry*. Penguin, London 1991.

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