

The Plücker Quadric's Ambient Space: Alternative Interpretation and its Application

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Abstract. It is well-known that there exists a bijection between the set of lines of the projective 3-dimensional space P^3 and all real points of the so-called Plücker quadric Ψ . Moreover one can identify each point of the Plücker quadric's ambient space with a linear complex of lines in P^3 . Within this paper we give an alternative interpretation for the points of P^5 as lines of an Euclidean 4-space E^4 , which are orthogonal to a fixed direction. By using the quaternionic notation for lines, we study straight lines in P^5 which correspond in the general case to cubic 2-surfaces in E^4 . We show that these surfaces are geometrically connected with circular Darboux 2-motions in E^4 , as they are basic surfaces of the underlying line-symmetric motions.

Moreover we extend the obtained results to line-elements of the Euclidean 3-space E^3 , which can be represented as points of a cone over Ψ sliced along the 2-dimensional generator space of ideal lines. We also study straight lines of its ambient space P^6 and show that they correspond to ruled surface strips composed of the mentioned 2-surfaces with circles on it.

Finally we present an application of this interpretation in the context of interactive design of ruled surfaces and ruled surface strips/patches based on the algorithm of De Castel'jau.

Key Words: Plücker quadric, line-element, Euclidean 4-space, circular Darboux 2-motion, De Castel'jau algorithm

MSC 2010: 51M30, 53A17

1. Introduction

Details about the following basics in line-geometry can be found in [22]. Let us consider two distinct real points P and Q of the projective 3-space P^3 , which possess the following homogenous coordinates:

$$(\bar{p}_0 : \bar{p}_1 : \bar{p}_2 : \bar{p}_3), \quad (\bar{q}_0 : \bar{q}_1 : \bar{q}_2 : \bar{q}_3) \quad (1)$$

with $\bar{p}_i, \bar{q}_i \in \mathbb{R}$ for $i = 0, \dots, 3$. Then the line l spanned by P and Q can be represented by the following homogeneous 6-tuple

$$(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12}) \quad (2)$$

with $l_{ij} = \bar{p}_i \bar{q}_j - \bar{p}_j \bar{q}_i$. These are the so-called Plücker coordinates of the line l .

But contrary not each homogeneous 6-tuple corresponds to a line of P^3 , as this set \mathcal{L} of lines in P^3 is 4-dimensional. Only the 6-tuples fulfilling the so-called Plücker condition

$$l_{01}l_{23} + l_{02}l_{31} + l_{03}l_{12} = 0 \quad (3)$$

represent lines of P^3 . Therefore there exists a bijection between the set \mathcal{L} and all real points of the so-called Plücker¹ quadric Ψ of P^5 , which is given by Eq. (3). This bijection $\mathcal{L} \rightarrow \Psi$ is known as *Klein mapping*.

Remark 1. Let us represent an arbitrary real point $U \in P^3$ by $(\bar{u}_0 : \bar{u}_1 : \bar{u}_2 : \bar{u}_3)$. We identify the set of points U determined by $\bar{u}_0 = 0$ with the ideal plane of the projective extended Euclidean 3-space. Then the 2-dimensional generator space $L : l_{01} = l_{02} = l_{03} = 0$ of Ψ corresponds to the set of ideal lines. \diamond

A set \mathcal{C} of lines fulfilling a linear equation

$$c_{01}l_{23} + c_{02}l_{31} + c_{03}l_{12} + c_{23}l_{01} + c_{31}l_{02} + c_{12}l_{03} = 0 \quad (4)$$

is called a linear complex of lines in P^3 . The set \mathcal{C} corresponds to the intersection of a hyperplane γ and the Plücker quadric Ψ . Therefore one can identify \mathcal{C} with the pole C of γ with respect to Ψ , which has homogenous coordinates $(c_{01} : c_{02} : c_{03} : c_{23} : c_{31} : c_{12})$. This bijection between the points of P^5 and linear complexes of lines in P^3 is known as *extended Klein mapping*.

1.1. Lines and line-elements of Euclidean 3-space

As we want to apply the theoretical results of the study at hand to the interactive design of rational ruled surfaces (cf. Section 5), we restrict to the lines of the Euclidean 3-space E^3 . They are represented by the real points of $\Psi \setminus L$; i.e. the Plücker quadric Ψ sliced along the 2-dimensional generator space L .

The coordinates of a point $P \in E^3$ are given by $\mathbf{p} := (p_1, p_2, p_3)$ with respect to the Cartesian frame $(O; x_1, x_2, x_3)$. In this case the entries of Eq. (2) have the following geometric meaning:

- $\mathbf{l} := (l_{01}, l_{02}, l_{03})$ gives the direction of the line and differs from the zero-vector \mathbf{o} .
- $\widehat{\mathbf{l}} := (l_{23}, l_{31}, l_{12})$ is the so-called moment-vector, which can also be computed by $\mathbf{p} \times \mathbf{l}$, where $P \in l$ holds.

Using the notation $(\mathbf{l}, \widehat{\mathbf{l}}) \in \mathbb{R}$ the Plücker condition of Eq. (3) can be rewritten as $\langle \mathbf{l}, \widehat{\mathbf{l}} \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product.

Remark 2. It is well known [22] that in E^3 a linear line complex \mathcal{C} has the following kinematic interpretation: The set of lines of \mathcal{C} equals the set of path-normals of an instantaneous motion different from the instantaneous standstill. For an instantaneous translation/rotation/screw motion the corresponding point $C \in P^5$ of the linear line complex \mathcal{C} has the property $C \in L$ resp. $C \in \Psi \setminus L$ resp. $C \in P^5 \setminus \Psi$. \diamond

¹This quadric is also known as Klein quadric.

For some applications (e.g. 3D shape recognition and reconstruction [6]) it is superior to study so-called line-elements instead of lines. As these geometric objects consist of a line l and a point P on it, we write them as (l, P) . Moreover we call a ruled surface together with a curve on it a *ruled surface strip*.

According to [13] the Plücker coordinates of lines can be extended for line-elements of E^3 by

$$(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12} : l) \quad (5)$$

with $l := \langle \mathbf{p}, \mathbf{l} \rangle$. In the remainder of the article we abbreviate this homogenous 7-tuple by $(\mathbf{l}, \widehat{\mathbf{l}}, l)\mathbb{R}$. Obviously there is a bijection between the set of line-elements of E^3 and all real points of P^6 located on a cone Λ over Ψ , which is sliced along the 3-dimensional generator space $G: l_{01} = l_{02} = l_{03} = 0$ of Λ . Again the points $(\mathbf{c}, \widehat{\mathbf{c}}, c)\mathbb{R}$ of Λ 's ambient space P^6 can be interpreted as linear complexes of line-elements; i.e. the set of line-elements $(\mathbf{l}, \widehat{\mathbf{l}}, l)\mathbb{R}$ fulfilling the linear equation

$$\langle \mathbf{c}, \widehat{\mathbf{l}} \rangle + \langle \widehat{\mathbf{c}}, \mathbf{l} \rangle + cl = 0. \quad (6)$$

Remark 3. For reasons of completeness it should be noted that the set of line-elements of P^3 is studied in [14] and [10, Sect. 6], respectively. Moreover the set of oriented line-elements is investigated in [11]. \diamond

1.2. Outline

In Section 2 we give an alternative interpretation for the points of Ψ 's ambient space P^5 as the set of lines of the Euclidean 4-space E^4 , which are orthogonal to a fixed direction. The obtained results are extended to line-elements in Section 3. In Section 4 we study 2-surfaces (resp. surface strips) of E^4 , which correspond to straight lines in P^5 (resp. P^6). The connection of these 2-surfaces with circular Darboux 2-motions of E^4 is discussed in Section 4.3. We close the paper (cf. Section 5) with an application of the given interpretation in the context of interactive design of ruled surfaces and ruled surface strips/patches based on the algorithm of De Casteljau.

2. Lines in Euclidean 4-space

The quaternionic representation of lines in E^4 allows a very compact notation. This formulation, which is shortly repeated in the next subsection, was already used by the author in [9] and [10, Sec. 6], respectively.

2.1. Quaternionic representation

$\mathfrak{Q} := q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ with $q_0, \dots, q_3 \in \mathbb{R}$ is an element of the skew field of quaternions \mathbb{H} , where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the quaternion units. The scalar part is q_0 and the pure part equals $q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, which is also denoted by \mathfrak{q} . The conjugated quaternion to $\mathfrak{Q} = q_0 + \mathfrak{q}$ is given by $\widetilde{\mathfrak{Q}} := q_0 - \mathfrak{q}$ and \mathfrak{Q} is called a unit-quaternion for $\mathfrak{Q} \circ \widetilde{\mathfrak{Q}} = 1$, where \circ denotes the quaternion multiplication.

We embed points P of E^4 with coordinates (p_0, p_1, p_2, p_3) with respect to the Cartesian frame $(O; x_0, x_1, x_2, x_3)$ into the set of quaternions by the mapping $\iota: \mathbb{R}^4 \rightarrow \mathbb{H}$ with

$$(p_0, p_1, p_2, p_3) \mapsto \mathfrak{P} := p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} = p_0 + \mathfrak{p}. \quad (7)$$

Let us identify E^3 with the hyperplane $x_0 = 0$.

The so-called *homogenous minimal coordinates* (cf. [10, Def. 3]) of a line $l \in E^4$ can be written in terms of quaternions as

$$(\mathfrak{L}, \mathfrak{m})\mathbb{R} \quad (8)$$

with $\mathfrak{m} := \tilde{\mathfrak{L}} \circ \mathfrak{F}$, where the latter quaternion results from the embedding ι of the pedal point F of the line l with respect to the origin O of the reference frame. Note that \mathfrak{m} is a pure quaternion and that \mathfrak{L} correspond to the direction of the line $l \in E^4$.

There is a bijection between the set of lines of E^4 and the points of P^6 , which is sliced along the 2-space $l_0 = l_1 = l_2 = l_3 = 0$; i.e. $\mathfrak{L} = 0$.

2.2. Alternative Interpretation

In the following we are only interested in the subset \mathcal{M} of lines of E^4 , which are orthogonal to the x_0 -direction. As a consequence \mathfrak{L} has to be a pure quaternion, i.e. the *homogenous minimal coordinates* of a line $l \in \mathcal{M}$ read as

$$(l, \mathfrak{m})\mathbb{R}. \quad (9)$$

Thus there is a bijection between the set \mathcal{M} and the points of P^5 , which is sliced along the 2-space $L : l_1 = l_2 = l_3 = 0$; i.e. $l = 0$. Moreover lines of \mathcal{M} belonging to E^3 (given by $x_0 = 0$) are determined by the fact that \mathfrak{F} , which can be computed as

$$\frac{l \circ \mathfrak{m}}{l \circ \tilde{l}}, \quad (10)$$

is a pure quaternion. Therefore

$$f_0 = \frac{1}{2} \frac{l \circ \mathfrak{m} + \tilde{\mathfrak{m}} \circ \tilde{l}}{l \circ \tilde{l}} = \frac{1}{2} \frac{l \circ \mathfrak{m} + \mathfrak{m} \circ l}{l \circ \tilde{l}} \quad (11)$$

holds and the condition $f_0 = 0$ is equivalent with

$$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0, \quad (12)$$

which is exactly the Plücker condition given in Eq. (2). This completes the alternative interpretation of $P^5 \setminus L$, i.e. the ambient space of the Plücker quadric Ψ sliced along the 2-dimensional generator space L .

2.3. Projection on the Plücker quadric

According to the extended Klein mapping and Rem. 2 every point $(\mathbf{c}, \hat{\mathbf{c}})\mathbb{R}$ of P^5 , which is not located on Ψ , corresponds to the path-normals of a instantaneous screw motion. The Plücker coordinates $(\mathbf{a}, \hat{\mathbf{a}})\mathbb{R}$ of the so-called axis of this instantaneous screw motion can be computed according to [22, Thm. 3.1.9] as

$$\mathbf{a} = \mathbf{c}, \quad \hat{\mathbf{a}} = \hat{\mathbf{c}} - \frac{\langle \mathbf{c}, \hat{\mathbf{c}} \rangle}{\langle \mathbf{c}, \mathbf{c} \rangle} \mathbf{c}. \quad (13)$$

For points $(\mathbf{c}, \hat{\mathbf{c}})\mathbb{R}$ of $\Psi \setminus L$ Eq. (13) simplifies to

$$\mathbf{a} = \mathbf{c}, \quad \hat{\mathbf{a}} = \hat{\mathbf{c}}, \quad (14)$$

which is the axis of the instantaneous rotation. In total, Eq. (13) implies a mapping $\mu : P^5 \setminus L \rightarrow \Psi \setminus L$ with $(\mathbf{c}, \hat{\mathbf{c}})\mathbb{R} \mapsto (\mathbf{a}, \hat{\mathbf{a}})\mathbb{R}$.

In analogy² to [19, Thm. 1] the fibers of this mapping μ can be written as follows:

²In this context see also [24] and [11, Rem. 10].

Theorem 1. *The fiber of point $(\mathbf{c}, \widehat{\mathbf{c}})\mathbb{R} \in P^5 \setminus L$ with respect to the mapping μ is a straight line through $(\mathbf{c}, \widehat{\mathbf{c}})\mathbb{R}$ that intersects the 2-dimensional generator space L in the point $(\mathbf{o}, \mathbf{c})\mathbb{R}$.*

What is the geometric meaning of the mapping μ in terms of our alternative interpretation? In order to clarify this we rewrite the mapping μ in the quaternionic formulation; i.e.

$$\mu : (\mathbf{l}, \mathbf{m})\mathbb{R} \mapsto \left(\mathbf{l}, \mathbf{m} + \frac{1}{2} \frac{\mathbf{l} \circ \mathbf{m} + \mathbf{m} \circ \mathbf{l}}{\mathbf{l} \circ \widetilde{\mathbf{l}}} \right) \mathbb{R} \tag{15}$$

and compute the pedal point F^* of l^* given by $(\mathbf{l}^*, \mathbf{m}^*)\mathbb{R} := \mu(\mathbf{l}, \mathbf{m})\mathbb{R}$. The formula of Eq. (10) implies

$$\mathfrak{F}^* = \frac{\mathbf{l}^* \circ \mathbf{m}^*}{\mathbf{l}^* \circ \widetilde{\mathbf{l}^*}} = \frac{\mathbf{l} \circ \mathbf{m}}{\mathbf{l} \circ \widetilde{\mathbf{l}}} - \frac{1}{2} \frac{\mathbf{l} \circ \mathbf{m} + \mathbf{m} \circ \mathbf{l}}{\mathbf{l} \circ \widetilde{\mathbf{l}}} = \mathfrak{F} - f_0. \tag{16}$$

This shows that the mapping μ corresponds to the orthogonal projection of the line $l \in \mathcal{M}$ onto E^3 (given by $x_0 = 0$). We denote this orthogonal projection $E^4 \rightarrow E^3$ by π .

3. Line-elements in Euclidean 4-space

In this section we extend the results of Section 2 to the line-elements of E^4 .

The so-called *homogenous minimal coordinates* (cf. [10, Def. 4]) of a line-element can be written in terms of quaternions as

$$(\mathfrak{L}, l + \mathbf{m})\mathbb{R} \tag{17}$$

with \mathfrak{L} and \mathbf{m} of Eq. (8) and l of Eq. (5). There is a bijection between the set of line-elements of E^4 and the points of P^7 , which is sliced along the 3-space $l_0 = l_1 = l_2 = l_3 = 0$; i.e. $\mathfrak{L} = 0$.

The subset of line-elements (l, \mathbf{P}) of E^4 , where l is orthogonal to the x_0 -direction, is denoted by \mathcal{N} . As a consequence \mathfrak{L} has to be a pure quaternion, i.e. the *homogenous minimal coordinates* of a line-element $(l, \mathbf{P}) \in \mathcal{N}$ read as:

$$(l, l + \mathbf{m})\mathbb{R}. \tag{18}$$

Analog considerations to Section 2.2 imply the following statement: There is a bijection between the set \mathcal{N} and the points of P^6 , which is sliced along the 3-space $G : l_1 = l_2 = l_3 = 0$; i.e. $l = 0$. Moreover line-elements of \mathcal{N} belonging to E^3 (given by $x_0 = 0$) are located on the cone $\Lambda \setminus G$.

We can also extend the projection μ of Eq. (15) to the set \mathcal{N} by

$$\nu : P^6 \setminus G \rightarrow \Lambda \setminus G, \quad (\mathbf{c}, \widehat{\mathbf{c}}, c)\mathbb{R} \mapsto (\mathbf{a}, \widehat{\mathbf{a}}, c)\mathbb{R}. \tag{19}$$

In analogy to Theorem 1 we get:

Theorem 2. *The fiber of point $(\mathbf{c}, \widehat{\mathbf{c}}, c)\mathbb{R} \in P^6 \setminus G$ with respect to the mapping ν is a straight line through $(\mathbf{c}, \widehat{\mathbf{c}}, c)\mathbb{R}$ that intersects the 3-dimensional generator space G in the point $(\mathbf{o}, \mathbf{c}, 0)\mathbb{R}$.*

The quaternionic notation of the mapping ν is given by

$$\nu : (l, l + \mathbf{m})\mathbb{R} \mapsto \left(l, l + \mathbf{m} + \frac{1}{2} \frac{\mathbf{l} \circ \mathbf{m} + \mathbf{m} \circ \mathbf{l}}{\mathbf{l} \circ \widetilde{\mathbf{l}}} \right) \mathbb{R} \tag{20}$$

and the geometric meaning of ν in terms of our alternative interpretation reads as follows: The mapping ν corresponds to the orthogonal projection π of the line-element $(l, \mathbf{P}) \in \mathcal{N}$ onto E^3 (given by $x_0 = 0$).

3.1. Lower-dimensional analogue

In order to make the things more descriptive we stress the following lower-dimensional analogue: Consider the set \mathcal{Q} of line-elements of E^3 , those lines are orthogonal to the x_3 -direction. Moreover we consider the set \mathcal{P} of line-elements, which are contained in the x_1x_2 -plane. This implies the relation $\mathcal{P} \subset \mathcal{Q}$. If we apply an orthogonal projection η along the x_3 -direction on the x_1x_2 -plane (analogue of π) to line-elements of \mathcal{Q} , we obtain line-elements of \mathcal{P} . This is illustrated in Figs. 1 and 2.

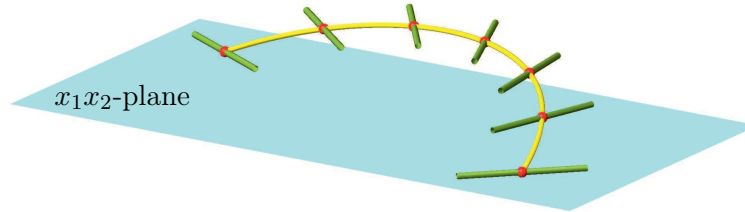


Figure 1: The ruled surface strip is conoidal as the generators are parallel to the x_1x_2 -plane.

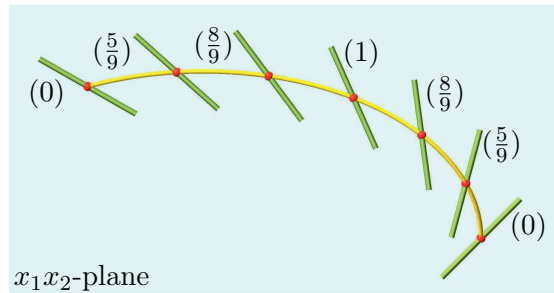


Figure 2: The image of the conoidal surface strip of Figure 1 under η is obtained by considering the top view. In addition we label the line-elements in the top view by the x_3 -coordinate. In German such a map is known as "kotierte Projektion".

Remark 4. By omitting the point of the line-element we get the lower-dimensional analogue for the situation discussed in Section 2. ◇

4. Straight lines

In the following we study the geometric objects, which are defined by straight lines in $P^5 \setminus L$ and $P^6 \setminus G$, respectively.

4.1. Straight lines in $P^5 \setminus L$

Given are two distinct lines $l_1, l_2 \in \mathcal{M}$ with quaternionic representation

$$(l_i, \mathbf{m}_i)\mathbb{R} \quad \text{with} \quad i = 1, 2. \tag{21}$$

Moreover we consider the line g spanned by the corresponding points in P^5 , which is given by

$$[t(l_1, \mathbf{m}_1) + (1 - t)(l_2, \mathbf{m}_2)]\mathbb{R} \quad \text{with} \quad t \in \mathbb{R}. \tag{22}$$

Now we can distinguish the following cases:

1. l_1 and l_2 are located within a plane. Then it can easily be seen that \mathfrak{g} corresponds to the pencil of lines spanned by l_1 and l_2 .
2. l_1 and l_2 are skew; i.e. they span a 3-space. The surface $\Gamma \in E^4$ which corresponds to \mathfrak{g} is studied next.

4.1.1. Properties of Γ

Clearly, Γ is a 2-surface in E^4 as it is generated by a 1-parametric set of lines. As all these lines belong to \mathcal{M} , we can say that Γ is a conoidal 2-surface with respect to the director hyperplane $x_0 = 0$. For the study of further properties of Γ we use the following coordinatisation:

Without loss of generality we can assume that

- l_1 is located in E^3 (given by $x_0 = 0$),
- l_2 belongs to the hyperplane $x_0 = h$ with $h \in \mathbb{R}$,
- l_1 coincides with the x_2 -axis,
- $\pi(l_2)$ is parallel to the x_2x_3 -plane,
- x_1 -axis is the common normal of l_1 and $\pi(l_2)$.

This choice of the coordinate system, which is illustrated in Figure 3, yields:

$$l_1 = \mathbf{j}, \quad l_2 = \cos \alpha \mathbf{j} + \sin \alpha \mathbf{k}, \quad (23)$$

$$\mathfrak{F}_1 = 0, \quad \mathfrak{F}_2 = h + n \mathbf{i}, \quad (24)$$

implying $\mathfrak{m}_1 = 0$ and

$$\mathfrak{m}_2 = (h \cos \alpha - n \sin \alpha) + (h \sin \alpha + n \cos \alpha) \mathbf{i},$$

which can be inserted into Eq. (22).

According to Eq. (10) the pedal point of $l \in \Gamma$ (in dependency of t) reads as

$$\mathbf{s} := \begin{pmatrix} \frac{(t-1)(th \cos \alpha - tn \sin \alpha - th + h)}{2(t \cos \alpha - t)(t-1) - 1} \\ \frac{(t-1)(th \sin \alpha + tn \cos \alpha - tn + n)}{2(t \cos \alpha - t)(t-1) - 1} \\ 0 \\ 0 \end{pmatrix} \quad (25)$$

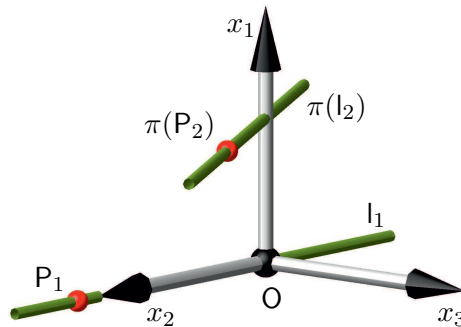


Figure 3: $\pi(l_2)$ intersects the x_1 -axis at height n and encloses with the x_2 -axis the angle $\alpha \in]0^\circ, 180^\circ[$.

and the direction \mathbf{r} of the ruling $l \in \Gamma$ (in dependency of t) equals

$$\mathbf{r} := \begin{pmatrix} 0 \\ 0 \\ \frac{t + (1 - t) \cos \alpha}{2(t \cos \alpha - t)(t - 1) - 1} \\ \frac{(1 - t) \sin \alpha}{2(t \cos \alpha - t)(t - 1) - 1} \end{pmatrix}. \tag{26}$$

Therefore $\Gamma \in E^4$ can be rationally parametrized as follows with respect to the parameters t and u :

$$\Gamma : \mathbf{g} = \mathbf{s} + u\mathbf{r} \quad \text{with} \quad u \in \mathbb{R}. \tag{27}$$

Theorem 3. Γ possesses a rational quadratic parametrization and it is a so-called LN-surface.

Proof. LN-surfaces in E^4 are discussed in [18] and can be characterized as follows: For any three-space of E^4 there exists a unique tangent plane of Γ parallel to it. According to [18, Eq. (12)] this is equivalent with the existence of a rational solution of the two equations

$$\langle \mathbf{w}, \frac{\partial}{\partial t} \mathbf{g} \rangle = 0 \quad \text{and} \quad \langle \mathbf{w}, \frac{\partial}{\partial s} \mathbf{g} \rangle = 0 \tag{28}$$

for t and u in dependence of $\mathbf{w} \in \mathbb{R}^4$. It can easily be checked that this criterion is fulfilled. \square

Theorem 4. Γ is a cubic conoidal 2-surface.

Proof. We consider Eq. (27) coordinate-wise and eliminate from these four equations the parameters t and u by resultant method. This yields the following set of equations:

$$\begin{aligned} (a) \quad & \sin \alpha (g_1 g_2^2 + g_1 g_3^2 - h g_3 g_2 - n g_3^2) - g_3 \cos \alpha (n g_2 - h g_3) = 0, \\ (b) \quad & \sin \alpha (h g_3^2 - n g_3 g_2 - g_0 g_3^2 - g_0 g_2^2) + g_3 \cos \alpha (h g_2 + n g_3) = 0, \\ (c) \quad & \sin \alpha (g_0^2 - n g_1 + g_1^2 - h g_0) - \cos \alpha (n g_0 - g_1 h) = 0. \end{aligned} \tag{29}$$

In order to obtain the projective closure P^4 of E^4 , we homogenize the coordinates by $g_i = \frac{v_i}{v}$ for $i = 0, \dots, 3$, whereby v denotes the homogenizing variable. The degree of $\Gamma \in P^4$ corresponds to the number of intersection points (counted by multiplicity) of this 2-surface with a plane. We choose this plane as follows:

$$v_0 = v_2 + v_3 + v, \quad v_1 = v_2 - v_3 - v. \tag{30}$$

Under consideration of these relations the three equations (a,b,c) of Eq. (29) are homogenous equations in v, v_2, v_3 . We eliminate from equation (i) and (j) the unknown v by resultant method for pairwise distinct $i, j \in \{a, b, c\}$, which yields $\text{Res}(i, j)$. The greatest common divisor of $\text{Res}(a, b)$, $\text{Res}(a, c)$ and $\text{Res}(b, c)$ equals $\sin \alpha (v_2^2 + v_3^2) V$ with

$$V := \sin \alpha (2v_2 + h v_3 + n v_3) + v_3 \cos \alpha (n - h).$$

This yields one real and two conjugate complex intersection points (with multiplicity one), thus we can conclude that Γ is cubic. \square

Finally, it should be noted (cf. [3, Sec. 2]) that the image of Γ under π is the *Plücker conoid*, which is also known as *cylindroid*.

4.2. Straight lines in $P^6 \setminus G$

Now we consider two distinct line-elements $(l_i, P_i) \in \mathcal{N}$ with quaternionic representation

$$(l_i, l_i + \mathbf{m}_i)\mathbb{R} \quad \text{with } i = 1, 2 \quad (31)$$

and the straight line \mathbf{q} in $P^6 \setminus G$ spanned by the corresponding points in P^6 ; i.e.

$$[t(l_1, l_1 + \mathbf{m}_1) + (1 - t)(l_2, l_2 + \mathbf{m}_2)]\mathbb{R}. \quad (32)$$

If the underlying lines l_1 and l_2 are

1. **coplanar**, then we can distinguish two cases:

(a) $l_1 \parallel l_2$: In this case \mathbf{q} corresponds to a ruled surface strip which consists of a parallel line pencil (spanned by l_1 and l_2) with a line on it (cf. Figure 4, right).

For the special case $l_1 = l_2$ the line \mathbf{q} corresponds to the set of line-elements which have the same carrier line $l_1 = l_2$.

(b) $l_1 \not\parallel l_2$: In this case \mathbf{q} corresponds to a ruled surface strip consisting of a line pencil, where the vertex \mathbf{V} is the intersection point of l_1 and l_2 , and a circle on it, which is determined by \mathbf{V}, P_1, P_2 (cf. Figure 4, left).

The statements given in item (1a) are trivial (proofs are left to the reader) and those of item (1b) follow from [13, Cor. 2].

2. **skew**, then \mathbf{g} corresponds to a ruled surface strip (Γ, \mathbf{k}) with the underlying ruled surface Γ of Section 4.1. Therefore we are only left with the question for the curve \mathbf{k} on Γ , which is answered next.

Theorem 5. *\mathbf{k} is a circle, which implies that Γ carries a 2-parametric set of circles.*

Proof. The curve \mathbf{k} on Γ is given by

$$\mathbf{k} = \mathbf{s} + (tl_1 + (1 - t)l_2)\mathbf{r}. \quad (33)$$

As this is a rational quadratic parametrization in t , the curve \mathbf{k} has to be a conic section in E^4 . By introducing homogenous coordinates $k_i = \frac{q_i}{q}$ for $i = 0, \dots, 3$, we are able to intersect the curve \mathbf{k} with the hyperplane at infinity determined by $q = 0$. The corresponding equation

$$2(t \cos \alpha - t)(t - 1) - 1 = 0 \quad (34)$$

has the solutions $t = \frac{1 - \cos \alpha \pm I \sin \alpha}{2 - 2 \cos \alpha}$ where I denotes the complex unit. Direct computation shows that the corresponding two intersection points are located on the absolute sphere $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 0$ ($\Rightarrow \mathbf{k}$ is a circle).

Γ carries a 2-parametric set of circles as l_1 and l_2 can take arbitrary real values. \square

Theorem 6. *The striction curve \mathbf{s} of Γ is a circle. Moreover \mathbf{s} is a geodesic curve of Γ .*

Proof. According to [22, p. 364] the *striction point* of a ruling is the point of contact of the tangent plane, which is orthogonal to the asymptotic plane (= tangent plane in the ruling's ideal point). Moreover it should be noted (cf. [20, p. 11]) that the tangent planes along a ruling of a 2-surface in E^4 are located within a hyperplane.

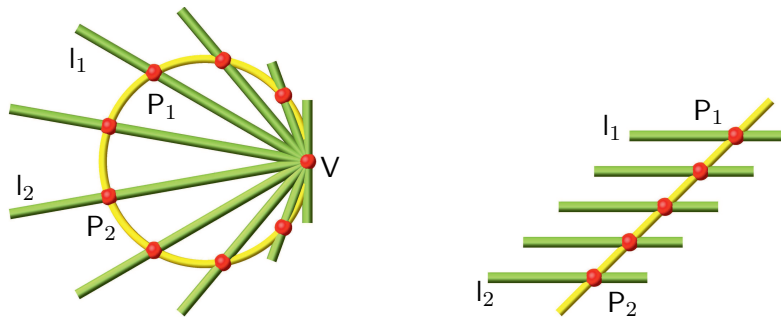


Figure 4: Right/Left: The special ruled surface strips of item (1a)/(1b).

The asymptotic plane is spanned by the vectors \mathbf{r} and $\frac{\partial}{\partial t} \mathbf{r}$. It can easily be checked that the tangent plane along \mathbf{s} of Eq. (25), which is spanned by \mathbf{r} and $\frac{\partial}{\partial t} \mathbf{s}$, is orthogonal to the asymptotic plane. Therefore the striction curve \mathbf{s} is given by Eq. (25). Moreover \mathbf{s} is a circle, which follows from Theorem 5 by setting $l_1 = l_2 = 0$ in Eq. (33).

According to [20, p. 14] the following statement holds: If a curve \mathbf{s} on a ruled 2-surface in E^4 possesses two of the following properties, it possesses the third:

1. \mathbf{s} is the striction curve,
2. \mathbf{s} is a geodesic,
3. \mathbf{s} cuts the rulings with a fixed angle.

It can easily be verified that \mathbf{s} intersects all rulings orthogonal. Therefore \mathbf{s} is a geodesic. \square

Finally it should be noted that $\pi(\mathbf{s})$ coincides with the common normal of l_1 and $\pi(l_2)$. All other circles \mathbf{k} on Γ are mapped to ellipses $\pi(\mathbf{k})$.

4.3. Kinematic relevance of Γ

Now we want to study the one-parametric motion in E^4 , which is generated by reflecting the coordinate frame in the one-parametric set of Γ 's rulings. Such a motion is called *line-symmetric* and Γ is the corresponding *basic surface* (cf. [1, Chap. 9,§7]).

Without loss of generality (cf. Section 4.1) we can assume that the rulings $l \in \Gamma$ with quaternionic representation $(\mathbf{l}, \mathbf{m})\mathbb{R}$ have the direction $(0, 0, \cos \beta, \sin \beta)^T$ for $\beta \in [0^\circ, 180^\circ]$. Moreover we consider the orthogonal vectors

$$\mathbf{h}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{h}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{h}_3 = \begin{pmatrix} 0 \\ 0 \\ -\sin \beta \\ \cos \beta \end{pmatrix}. \tag{35}$$

Then the reflection κ in l can be replaced by the composition of three reflections ρ_i in the hyperplanes H_i through l , which are orthogonal to \mathbf{h}_i of Eq. (35) for $i = 1, 2, 3$. Therefore it is clear that κ is a orientation-reversing isometry.

But we want to describe the obtained one-parametric motion in E^4 by orientation-preserving isometries (= displacements). Therefore we first reflect the coordinate frame in the hyperplane $x_0 = 0$ (reflexion ρ_0) and then we apply κ . In total a point \mathbf{P} of E^4 is transformed by

$$\kappa(\rho_0(\mathbf{P})) = \rho_3(\rho_2(\rho_1(\rho_0(\mathbf{P}))). \tag{36}$$

As $\rho_1(\rho_0(\mathbf{P}))$ is a composition of two reflexions in parallel hyperplanes it equals a pure translation orthogonal to these hyperplanes. We denote this translation along the x_0 -axis by τ . Therefore we only have to clarify the action of the composition of ρ_3 and ρ_2 on $\tau(\mathbf{P})$. As the hyperplanes H_2 and H_3 intersect orthogonally along a plane T through \mathbf{l} , which is parallel to the x_0 -axis, the composition of ρ_3 and ρ_2 equals the rotation δ about T by 180° (= reflexion in T). This yields

$$\kappa(\rho_0(\mathbf{P})) = \delta(\tau(\mathbf{P})). \quad (37)$$

It can easily be verified by direct computations that this displacement can be written in quaternionic notation as follows:

$$\mathfrak{P} \mapsto \mathbf{l} \circ \mathfrak{P} \circ \tilde{\mathbf{l}} - 2\mathbf{l} \circ \tilde{\mathbf{m}} = \mathbf{l} \circ \mathfrak{P} \circ \tilde{\mathbf{l}} + 2\mathfrak{C} \quad (38)$$

assuming that \mathbf{l} is a unit-quaternion.

Under consideration of [12, Thm. 1] this representation implies that the one-parametric displacement can be represented by a straight line in the ambient space of the Study quadric. From [12, Thm. 3] we can conclude immediately the following theorem:

Theorem 7. *The line-symmetric motion in E^4 with basic surface Γ is a circular Darboux 2-motion³, which is neither spherical nor a pure translation, and vice versa.*

5. Application

As the Plücker quadric Ψ is a point-model of lines, one can use well-known methods for curves (freeform techniques, interpolation, approximation, ...) for the design of ruled surfaces. The challenge for applying this standard technique is that one has to deal with the side condition that the curve in P^5 has to be located on Ψ . For the task of interpolating given lines by a ruled surface one can apply for example the

- rational interpolation of points on hyperquadrics [4],
- interpolation with rational quadratic spline curves (biarc construction) [26], [7, Sec. 2.12].

Another possibility is to project the Plücker quadric stereographically onto an affine 4-space \mathbb{A}^4 [22, p. 212]. Based on this approach a G^1 -Hermite interpolation of ruled surfaces with low degree rational ruled surfaces is given in [17].

Within this paper we do not focus on the interpolation problem but want to discuss the modification of the well-known algorithm of De Casteljau for the design of rational ruled surfaces. Applications in this context are e.g. wire cut EDM (electric discharge machining), laser beam machining, cylindrical milling, generation of line-symmetric motions, ...

For the understanding of the following review on this topic we have to repeat a point-model for the set of oriented lines of E^3 .

5.1. Study sphere

The two vectors \mathbf{l} and $\widehat{\mathbf{l}}$ of Section 1.1 can be combined by the so-called dual unit ε with the property $\varepsilon^2 = 0$ to a dual vector $\underline{\mathbf{l}} := \mathbf{l} + \varepsilon\widehat{\mathbf{l}}$ with $\langle \underline{\mathbf{l}}, \underline{\mathbf{l}} \rangle \in \mathbb{R} \setminus \{0\}$; i.e.

$$\langle \underline{\mathbf{l}}, \underline{\mathbf{l}} \rangle = \langle \mathbf{l} + \varepsilon\widehat{\mathbf{l}}, \mathbf{l} + \varepsilon\widehat{\mathbf{l}} \rangle = \langle \mathbf{l}, \mathbf{l} \rangle + 2\varepsilon\langle \mathbf{l}, \widehat{\mathbf{l}} \rangle = \langle \mathbf{l}, \mathbf{l} \rangle. \quad (39)$$

Note that $\underline{\mathbf{l}}$ is an element of \mathbb{D}^3 , where \mathbb{D} denotes the ring of dual numbers $a + \varepsilon b$ with $a, b \in \mathbb{R}$.

³All points have circular trajectories.

As $\mathbf{l} \neq \mathbf{o}$ holds one can additionally assume that \mathbf{l} is a unit-vector; i.e. $\langle \mathbf{l}, \mathbf{l} \rangle = 1$. As a consequence $\underline{\mathbf{l}} := \mathbf{l} + \varepsilon \widehat{\mathbf{l}}$ is a so-called dual unit-vector representing a spear (oriented line). Therefore there is a bijection between the points of the dual unit-sphere $S_{\mathbb{D}}^2 \in \mathbb{D}^3$ with

$$S_{\mathbb{D}}^2 := \{ \underline{\mathbf{l}} \in \mathbb{D}^3 \quad \text{with} \quad \langle \mathbf{l}, \mathbf{l} \rangle = 1 \} \quad (40)$$

and the set of spears of E^3 . Note that antipodal points of this so-called Study sphere correspond to oppositely oriented lines.

5.2. Review

One can think of the following possibilities for adapting De Casteljau's algorithm for the design of ruled surfaces using:

5.2.1. Oriented Lines

★ According to ODEHNAL [15, Sec. 2] the quartic manifold $M^4 \in \mathbb{R}^6$ given by

$$M^4 : \quad \langle \mathbf{l}, \mathbf{l} \rangle = 1, \quad \langle \mathbf{l}, \widehat{\mathbf{l}} \rangle = 0 \quad (41)$$

can be used as point-model for the set of oriented lines. Then one can perform the algorithm of De Casteljau in \mathbb{R}^6 and project the resulting curve $(\mathbf{c}, \widehat{\mathbf{c}}) \in \mathbb{R}^6$ back onto M^4 . This back projection θ is the composition of the mapping $(\mathbf{c}, \widehat{\mathbf{c}}) \mapsto (\mathbf{a}, \widehat{\mathbf{a}})$ according to Eq. (13) and the normalization

$$(\mathbf{a}, \widehat{\mathbf{a}}) \mapsto \left(\frac{\mathbf{a}}{\|\mathbf{a}\|}, \frac{\widehat{\mathbf{a}}}{\|\mathbf{a}\|} \right).$$

This approach can be modified by projecting the points of \mathbb{R}^6 obtained after each iteration step of De Casteljau's algorithm back on M^4 .

Remark 5. Analogue algorithms for oriented line-elements are given in [11, Sec. 5.2.1]. \diamond

★ The above described algorithm can also be performed within another setting; namely one can use as point-model the dual unit-sphere $S_{\mathbb{D}}^2$. Then De Casteljau's algorithm can be executed in \mathbb{D}^3 and the resulting curve is projected back onto $S_{\mathbb{D}}^2$ by the normalization to dual unit-vectors. This normalization is equivalent to the mapping θ . This was in fact done by LI and GE [8] for the generation of rational Bézier line-symmetric motions.

★ Instead of these two projection algorithms one can think of a so-called geodesic one. It is based on the idea to replace the straight lines of the control polygon in the ambient space of the used point-model by their analog on the point-model, which are geodesics. SPROTT and RAVANI [25] used as point-model the dual unit-sphere $S_{\mathbb{D}}^2$ where the geodesics (dual great circles) on $S_{\mathbb{D}}^2$ correspond to right helicoids⁴. Their resulting geodesic algorithm of De Casteljau on $S_{\mathbb{D}}^2$ was further studied in the context of light-weight concrete elements [5].

In contrast to the projection algorithms the geodesic one has the disadvantage that the resulting ruled surfaces are not rational, which is an important feature for the interactive design of ruled surfaces in CAGD.

Remark 6. Geodesic algorithms for oriented line-elements are given in [11, Sec. 5.2.2]. \diamond

⁴They are the only ruled minimal surfaces in E^3 (and E^4 ; cf. [20, p. 17]). A right helicoid is generated during a helical motion by a line intersecting the axis orthogonally.

5.2.2. Un-oriented Lines

- ★ One can use the already mentioned stereographic projection of the Plücker quadric Ψ onto an affine 4-space \mathbb{A}^4 and perform De Casteljau's algorithm in \mathbb{A}^4 . Finally the obtained curve is stereographically projected back to Ψ . The obtained ruled surface is rational but it depends on the chosen stereographic projection, which is less satisfactory from the geometric point of view.
- ★ A very sophisticated method was presented by GE and RAVANI [3]. They identified antipodal points of $S_{\mathbb{D}}^2$ by considering the set of lines spanned by antipodal points, which is equivalent to the projective plane over \mathbb{D} . Within this projective dual plane they applied a *projective De Casteljau algorithm*.

They studied the resulting rational ruled surfaces in detail and also determined surface patches by using a second De Casteljau algorithm, which provides the distances of the boundary points of the patch along the ruling from the striction point. Clearly this approach can easily be modified for the generation of rational ruled surface strips.

In a projective De Casteljau algorithm (cf. [21, Sec. 3]) the control points come with weights, which are dual numbers in the case of [3]. But dual weights cannot be handled very intuitively by designers⁵. GE and RAVANI also mentioned that one can get rid of them by using so-called Farin points (which is also known as frame points or weight points), but they have not outlined a user-friendly method for interactive design. Based on our alternative interpretation such a method is presented in the next subsection.

Moreover the straight forward extension to line-elements allows an analogue handling of ruled surface strips, which therefore can be generated by one projective De Casteljau algorithm instead of the combination of two algorithms as proposed in [3].

Remark 7. Note that our approach can easily be modified for Bézier ruled surface patches by replacing Eq. (5) by

$$(l_{01} : l_{02} : l_{03} : l_{23} : l_{31} : l_{12} : l_1 : l_2) \quad (42)$$

with $l_i := \langle \mathbf{p}_i, \mathbf{l} \rangle$ for $i = 1, 2$, where \mathbf{p}_i are the two boundary points of the patch along the ruling. The non-homogenous version of this representation goes back to RAVANI and WANG [23] and was furthered in [2]. \diamond

5.3. Concluding algorithm

In principle, the following algorithm is identical with the one of GE and RAVANI [3] but instead of working in the projective dual plane we use P^5 (for ruled surface strips/patches we work in P^6 and P^7 , respectively). In this way we gain a better geometric understanding, which results in a user-friendly control. The procedure is as follows:

We perform a projective De Casteljau algorithm in the projective space of dimension 5 (6 and 7, respectively), which uses Farin points instead of weights. The resulting curve can be interpreted as a conoidal ruled 2-surface (strip/patch) in E^4 with respect to the director hyperplane $x_0 = 0$ (cf. Section 2 and 3). By applying the orthogonal projection π in x_0 -direction we obtain the desired ruled surface (strip/patch) in E^3 . Moreover we label the projected lines (line-elements/line-segments) by the x_0 -coordinate ("*kotierte Projektion*", cf. Figure 2). In this way the user can modify very intuitively the control structure; i.e. the

⁵Moreover the weights are not invariant under projective transformations.

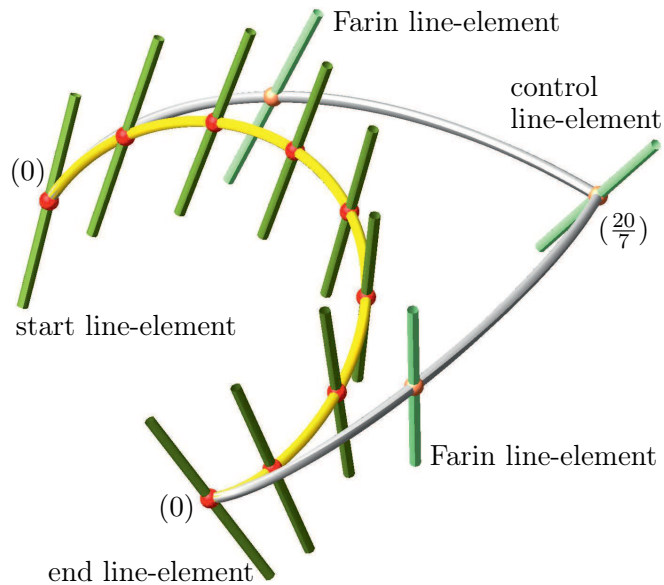


Figure 5: Example of the concluding algorithm: The design of a rational ruled surface strip is based on a very intuitively and user-friendly control structure.

Farin and control lines (line-elements/line-segments) can be changed by *mouse action* and their x_0 -heights by the *scroll wheel*.

In Figure 5 we illustrated a quartic rational ruled surface strip, which corresponds to a quadratic Bezier curve in P^6 . Each *Farin line-element* can only be modified within the ruled surface strip (composed of a Plücker conoid and an ellipse on it; cf. Section 4) determined by the *control line-element* and *start/end line-element*, respectively. In contrast the *control line-element* has 6 degrees of freedom. The x_0 -values of the control, start and end line-element are given in parentheses.

Finally it should be noted that an analogue algorithm for the design of rational motions in E^3 is given in [12].

Remark 8. The rational surface obtained by our algorithm can be written in Bézier representation. A geometric interpretation of the corresponding Farin lines and control lines in terms of linear complexes is given in [16]. \diamond

Acknowledgments

The author is supported by Grant No. P 24927-N25 of the Austrian Science Fund FWF.

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Received August 4, 2018; final form November 19, 2018