Journal for Geometry and Graphics
Volume 22 (2018), No. 2, 245–251.

# Packing Three Cubes in 8-Dimensional Space

**Zuzana Sedliačková**

*Dept. of Applied Mathematics, Faculty of Mechanical Engineering, University of Žilina*
*Univerzitná 1, 010 26 Žilina, Slovakia*
*email:* zuzana.sedliackova@fstroj.uniza.sk

**Abstract.** Let $V_n(d)$ denote the least number such that every system of $n$ cubes with total volume 1 in the $d$-dimensional (Euclidean) space can be packed into some rectangular parallelepiped of volume $V_n(d)$. In this paper two new results can be found: $V_2(8)$ and $V_3(8)$.

*Key Words:* packing of cubes, extreme

*MSC 2010:* 52C17

## 1. Introduction

Let $V_n(d)$ denote the least number such that every system of $n$ cubes with total volume 1 in the $d$-dimensional (Euclidean) space can be packed into some rectangular parallelepiped of volume $V_n(d)$. All admitted cubes have their edges parallel to the coordinate axes. We want to determine $V_n(d)$ and also the maximum $V(d)$ of the set of all $V_n(d)$ for $n = 1, 2, 3, \ldots$

Some results are known for $n \leq 8$ in the 2-dimensional space. There are also some estimates for $V(2)$. See, for example, [4, 5, 7, 8, 9, 10]. $V_n(3)$ is known for $n = 2, 3, 4, 5$ (see [6, 11, 12, 13]).

There are results for two and for three cubes in the 4-dimensional space [2]: $V_2(4) \doteq 1.420319245$, $V_3(4) \doteq 1.63369662$, and in the 6-dimensional space [3]: $V_2(6) \doteq 1.534554558$, $V_3(6) \doteq 1.94449161$.

## 2. Main results

In this paper new results are provided for packings of two and three cubes in the 8-dimensional Euclidean space. We use the same method as [2] and [3].

**Theorem 2.1.** $V_2(8) \doteq 1.6074984$.

*Proof.* In the 8-dimensional space, let us take two cubes with edge lengths $x$, $y$ such that $1 \geq x \geq y \geq 0$, with the total volume $x^8 + y^8 = 1$. These two cubes can be packed into some rectangle of volume $f(x, y) = x^7(x + y)$. So, we are looking for the maximum of the function

ISSN 1433-8157/\$ 2.50 

$f(x,y) = x^7(x+y)$ under the condition $g(x,y) = x^8 + y^8 - 1 = 0$. We solve the system of equations

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = 0 \quad \text{and} \quad g(x,y) = 0,$$

which gives

$$8xy^7 + 7y^8 - x^8 = 0 \quad \text{and} \quad x^8 + y^8 - 1 = 0.$$

We express $y$ from the second equation and plug it into the first equation. The substitution $t := x^8$ yields $8^8 t(1-t)^7 - (8t-7)^8 = 0$. It is obvious that $x^8 \geq \frac{1}{2}$, and therefore the solution of the equation is $t \doteq 0.948777458$. Thus we obtain $x \doteq 0.993448926$ and $y \doteq 0.689735597$, and the proof is complete. $\square$

**Theorem 2.2.** $V_3(8) \doteq 2.14930609$.

*Proof.* Consider in the 8-dimensional Euclidean space three cubes with edge lengths $x$, $y$, $z$, where $1 \geq x \geq y \geq z \geq 0$ and the total volume is $x^8 + y^8 + z^8 = 1$. It is sufficient to consider only the two cases of packing these three cubes as shown in Figure 1.

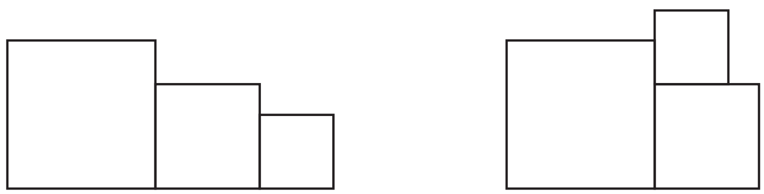

Figure 1: Two cases for packing three cubes

In the first case the volume $W_1 = x^7(x+y+z)$ is sufficient for packing, in the second case volume $W_2 = x^6\,(x+y)\,(y+z)$ is sufficient. We need to find max min $\{W_1, W_2\}$ under the conditions $x^8 + y^8 + z^8 = 1$ and $1 \geq x \geq y \geq z \geq 0$.

There are three cubes with edge lengths $x \doteq 0.986333649$, $y \doteq 0.704812693$, $z \doteq 0.675488876$, for which a volume $W_1 = W_2 \doteq 2.14930609$ is necessary. And so, $V_3(8) \geq 2.14930609$.

From $y^8 \leq z^8 + y^8 = 1 - x^8$ and therefore $y \leq \sqrt[8]{1-x^8}$ follows $y + z \leq 2y \leq 2\sqrt[8]{1-x^8}$. For $x \geq \dfrac{2}{\sqrt[8]{257}}$ we get $y + z \leq 2\sqrt[8]{1-x^8} \leq \dfrac{2}{\sqrt[8]{257}} \leq x$. If $y + z \leq x$ then we can pack the cubes as in the second case and the volume $V_2(8)$ is sufficient. It is obvious that $x^8 \geq \frac{1}{3}$, hence $x \geq \dfrac{1}{\sqrt[8]{3}}$. This implies that we can consider only $x \in \left\langle \dfrac{1}{\sqrt[8]{3}}, \dfrac{2}{\sqrt[8]{257}} \right\rangle$, i.e., $0.8716 \leq x \leq 0.9996$.

Equality $W_1 = W_2$ holds if $x^2 = y^2 + y\,z$. Then $z = \dfrac{x^2 - y^2}{y}$ and $W_1 = W_2 = x^8 + \dfrac{x^9}{y}$. When we substitute $z = \dfrac{x^2-y^2}{y}$ into $x^8 + y^8 + z^8 = 1$, we get the curve

$$C\colon\; x^8\,y^8 + y^{16} - y^8 + (x^2 - y^2)^8 = 0$$

(see Figure 2).

The interval for $x$ can be reduced. If we choose $x \in \langle a, b\rangle$ for $0 < a < b < 1$, then $1 - b^8 \leq 1 - x^8 \leq 1 - a^8$. If $y = z$, then $1 - x^8 = y^8 + z^8 = 2y^8$ and therefore $y = \sqrt[8]{\dfrac{1-x^8}{2}}$.

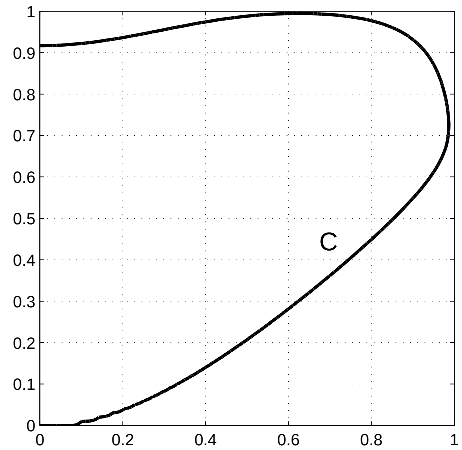


Figure 2: The curve $C$

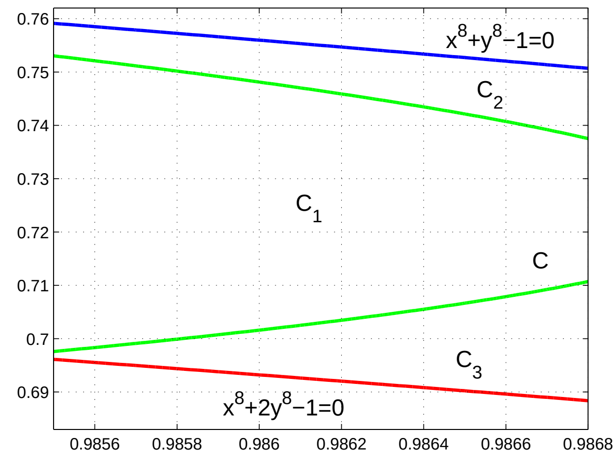


Figure 3: The regions $C_1$, $C_2$ and $C_3$

The function $W_1 = x^7(x+y+z)$ takes its greatest value if $y = z$, i.e., $y = \sqrt[8]{\dfrac{1-x^8}{2}}$. For $x \in \langle a,\, b\rangle$, we get

$$W_1 \;\le\; x^7(x+2y) \;\le\; W_1(a,b) := b^7\left(b + 2\sqrt[8]{\frac{1-a^8}{2}}\right).$$

For the following intervals holds $W_1(a,b) < 2.1493$:

$$\begin{array}{lll} x \in \langle 0.8716,\, 0.9673\rangle, & x \in \langle 0.9673,\, 0.9782\rangle, & x \in \langle 0.9782,\, 0.9819\rangle, \\ x \in \langle 0.9819,\, 0.9836\rangle, & x \in \langle 0.9836,\, 0.9845\rangle, & x \in \langle 0.9845,\, 0.9850\rangle, \\ x \in \langle 0.9850,\, 0.9853\rangle, & x \in \langle 0.9853,\, 0.9855\rangle\,. & \end{array}$$

For the asked maximum we have $x \ge 0.9855$.

Next we define the algorithm, which for a suitably chosen interval $x \in \langle a,\, b\rangle$ assigns step by step the numbers $y_0 = \sqrt[8]{1-a^8}$, $z_0 = \dfrac{2.1493}{b^6\,(b+y_0)} - y_0$, and if $z_i \le y_i$, then

$$y_{i+1} = \sqrt[8]{1-a^8-z_i^8}\,, \quad z_{i+1} = \frac{2.1493}{b^6\,(b+y_{i+1})} - y_{i+1}.$$

The clarification of the algorithm is as follows.

For every $x \in \langle a,\, b\rangle$ is $y \le \sqrt[8]{1-x^8} \le \sqrt[8]{1-a^8}$. If we denote $y_0 = \sqrt[8]{1-a^8}$, then $W_2 = x^6(x+y)(y+z) \le b^6(b+y_0)(y_0+z)$. For every $z \ge 0$ such that $z < \dfrac{2.1493}{b^6(b+y_0)} - y_0 = z_0$, we get $W_2 \le b^6(b+y_0)(y_0+z) < 2.1493$. So, for $z < z_0$ the asked maximum cannot be achieved.

Let $z \ge z_0$, then $y^8 = 1-x^8-z^8 \le 1-a^8-z_0^8$ and therefore $y \le \sqrt[8]{1-a^8-z_0^8}$. We set $y_1 := \sqrt[8]{1-a^8-z_0^8}$. Then $W_2 \le b^6(b+y_1)(y_1+z)$ and for every $z < z_1 := \dfrac{2.1493}{b^6(b+y_1)} - y_1$ is $W_2 < 2.1493$. So, for $z < z_1$ the asked maximum cannot be achieved.

We repeat this process until we reach $y_i < z_i$. If it succeeds we get $W_2 < 2.1493$ for $x \in \langle a,\, b\rangle$. And so, for $x \in \langle a,\, b\rangle$ the asked maximum cannot be achieved.

Take $x \in \langle 0.9950,\, 0.9996\rangle$. Then the algorithm generates the sequence $y_0 \doteq 0.667281$, $z_0 \doteq 0.625232$, $y_1 \doteq 0.596157$, $z_1 \doteq 0.753964$. So the asked maximum cannot be achieved for $x \in \langle 0.9950,\, 0.9996\rangle$.

If we take $x \in \langle 0.9920,\, 0.9950\rangle$, then the algorithm generates the sequence $y_0 \doteq 0.706733$, $z_0 \doteq 0.594835$, $y_1 \doteq 0.681561$, $z_1 \doteq 0.639550$, $y_2 \doteq 0.655885$, $z_2 \doteq 0.685773$. So the asked maximum cannot be achieved for $x \in \langle 0.9920,\, 0.9950\rangle$.

Similarly using the algorithm we can exclude these intervals:

$$\begin{array}{lll} x \in \langle 0.990,\, 0.992\rangle, & x \in \langle 0.989,\, 0.990\rangle, & x \in \langle 0.988,\, 0.989\rangle, \\ x \in \langle 0.9875,\, 0.9880\rangle, & x \in \langle 0.9873,\, 0.9875\rangle, & x \in \langle 0.9871,\, 0.9873\rangle, \\ x \in \langle 0.9870,\, 0.9871\rangle, & x \in \langle 0.9869,\, 0.9870\rangle, & x \in \langle 0.9868,\, 0.9869\rangle. \end{array}$$

So, we have shown that the asked maximum $\max\min\{W_1,\, W_2\}$ will be attained for $x \in \langle 0.9855,\, 0.9868\rangle$.

Consider the closed region $M$ determined by inequalities

$$0.9855 \le x \le 0.9868\,, \quad x^8 + y^8 \le 1\,, \quad x^8 + 2\,y^8 \ge 1.$$

The curve $C$ divides the region $M$ into three open regions $C_1$, $C_2$, $C_3$ (Figure 3).

We are looking for $\max\min\{W_1,\, W_2\}$, when $W_1 = x^7\,(x+y+z)$, $W_2 = x^6\,(x+y)\,(y+z)$. From the condition $x^8 + y^8 + z^8 = 1$ we get

$$W_1 = W_1(x,\, y) \,=\, x^7\left(x + y + \sqrt[8]{1 - x^8 - y^8}\,\right), \tag{2.1}$$

$$W_2 = W_2(x,\, y) \,=\, x^6\,(x+y)\left(y + \sqrt[8]{1 - x^8 - y^8}\,\right). \tag{2.2}$$

Let $\overline{C}_i$ denote the closure of the set $C_i$. The functions $W_1$, $W_2$ are continuous on $M$, and the equality $W_1 = W_2$ holds just in the point of the curve $C$.

Take the point $A_1 = (0.9862,\, 0.72) \in C_1$. Since $W_1(A_1) < W_2(A_1)$, the inequality $W_1(X) < W_2(X)$ holds in every point $X \in C_1$. So, for the asked maximum holds $\max\limits_{X\in\overline{C}_1}\min\{W_1(X),\, W_2(X)\} = \max\limits_{X\in\overline{C}_1}\{W_1(X)\}$.

Take the point $A_2 = (0.9862,\, 0.75) \in C_2$. Since $W_1(A_2) > W_2(A_2)$, the inequality $W_1(X) > W_2(X)$ holds in every point $X \in C_2$. So, for the asked maximum holds $\max\limits_{X\in\overline{C}_2}\min\{W_1(X),\, W_2(X)\} = \max\limits_{X\in\overline{C}_2}\{W_2(X)\}$.

Take the point $A_3 = (0.9862,\, 0.70) \in C_3$. Since $W_1(A_3) > W_2(A_3)$, the inequality $W_1(X) > W_2(X)$ holds in every point $X \in C_3$. So, for the asked maximum holds $\max\limits_{X\in\overline{C}_3}\min\{W_1(X),\, W_2(X)\} = \max\limits_{X\in\overline{C}_3}\{W_2(X)\}$.

**(1)** On the compact set $\overline{C}_1$ the function (2.1) achieves its maximum in some point $B$. It holds $\dfrac{\partial W_1}{\partial y} = x^7\left(1 - \dfrac{y^7}{\sqrt[8]{(1-x^8-y^8)^7}}\right)$. The equality $\dfrac{\partial W_1}{\partial y} = 0$ holds if $x^8 + 2\,y^8 - 1 = 0$. But the points of the curve $x^8 + 2\,y^8 - 1 = 0$ do not belong to the region $\overline{C}_1$. For every point $X \in C_1$ holds $\dfrac{\partial W_1}{\partial y} < 0$. And so, the point $B$ must lie on the curve $C$.

For every point $X = (x, y)$, $x \in \langle a,\, b\rangle$, $y \in \langle c,\, d\rangle$ the inequality $z \le \sqrt[8]{1 - a^8 - c^8}$ holds, and so

$$\begin{aligned} W_1 &= \quad x^7(x+y+z) \quad \le b^7\left(b + d + \sqrt[8]{1 - a^8 - c^8}\,\right), \\ W_2 &= x^6(x+y)(y+z) \le b^6\,(b+d)\left(d + \sqrt[8]{1 - a^8 - c^8}\,\right). \end{aligned}$$

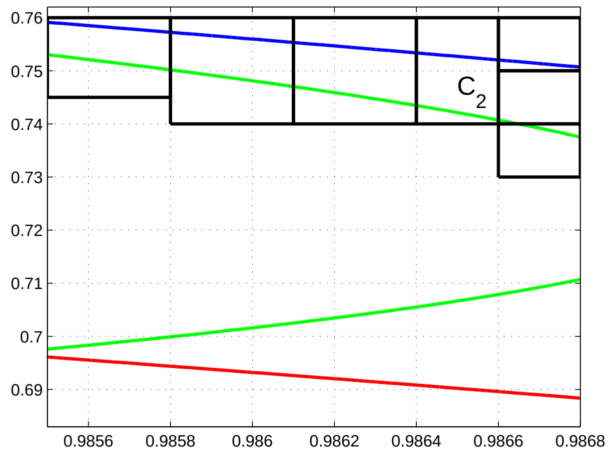


Figure 4: Exclusion of the region $C_2$

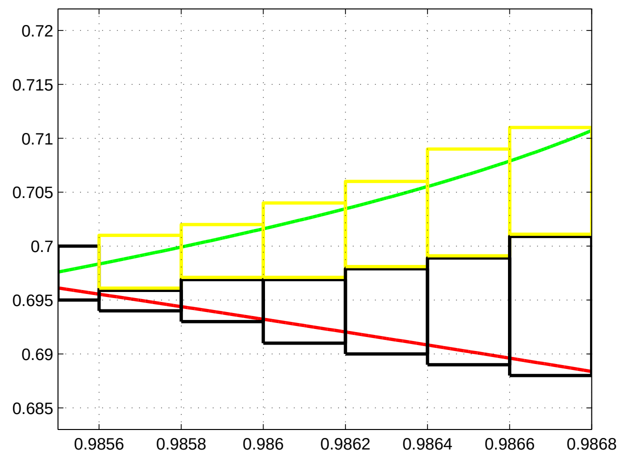


Figure 5: Exclusion of the region $C_3$

Denote

$$W_{11}(a,b,c,d) := b^7\left(b+d+\sqrt[8]{1-a^8-c^8}\right),$$
$$W_{22}(a,b,c,d) := b^6\,(b+d)\left(d+\sqrt[8]{1-a^8-c^8}\right).$$

**(2)** We examine the region $C_2$:

$$\text{For}\begin{cases} x\in\langle 0.9855,\,0.9858\rangle,\ y\in\langle 0.745,\,0.76\rangle & \text{is}\ \ W_{11}<2.1493.\\ x\in\langle 0.9858,\,0.9861\rangle,\ y\in\langle 0.74,\,0.76\rangle & \text{is}\ \ W_{11}<2.1493.\\ x\in\langle 0.9861,\,0.9864\rangle,\ y\in\langle 0.74,\,0.76\rangle & \text{is}\ \ W_{11}<2.1493.\\ x\in\langle 0.9864,\,0.9866\rangle,\ y\in\langle 0.74,\,0.76\rangle & \text{is}\ \ W_{11}<2.1493.\\ x\in\langle 0.9866,\,0.9868\rangle,\ y\in\langle 0.73,\,0.74\rangle & \text{is}\ \ W_{11}<2.1493.\\ x\in\langle 0.9866,\,0.9868\rangle,\ y\in\langle 0.74,\,0.75\rangle & \text{is}\ \ W_{11}<2.1493,\ \text{also}\ W_{22}<2.1493.\\ x\in\langle 0.9866,\,0.9868\rangle,\ y\in\langle 0.75,\,0.76\rangle & \text{is}\ \ W_{11}<2.1493,\ \text{also}\ W_{22}<2.1493.\end{cases}$$

This implies that the asked maximum cannot be achieved on the region $\overline{C}_2$ (Figure 4).

**(3)** We examine the region $C_3$:

For $x\in\langle 0.9855, 0.9856\rangle$ and, step by step, for $y\in\langle 0.695, 0.696\rangle$, $\langle 0.696, 0.697\rangle$, $\langle 0.697, 0.698\rangle$, $\langle 0.698, 0.699\rangle$, or $\langle 0.699, 0.700\rangle$ is always $W_{11}<2.1493$.

For $x\in\langle 0.9856, 0.9858\rangle$ and, step by step, for $y\in\langle 0.694, 0.695\rangle$ or $\langle 0.695, 0.696\rangle$ is always $W_{22}<2.1493$.

We do not exclude the area $x\in\langle 0.9856, 0.9858\rangle$, $y\in\langle 0.696, 0.701\rangle$ by this way. Maybe, we should have to divide the intervals into smaller intervals and it is not effective.

For $x\in\langle 0.9858, 0.9860\rangle$ and, step by step, for $y\in\langle 0.693, 0.694\rangle$, $\langle 0.694, 0.695\rangle$, $\langle 0.695, 0.696\rangle$, or $\langle 0.696, 0.697\rangle$ is always $W_{22}<2.1493$.

We do not exclude the area $x\in\langle 0.9858, 0.9860\rangle$, $y\in\langle 0.697, 0.702\rangle$ by this way.

For $x\in\langle 0.9860, 0.9862\rangle$ and, step by step, for $y\in\langle 0.691, 0.692\rangle$, $\langle 0.692, 0.693\rangle$, $\langle 0.693, 0.694\rangle$, $\langle 0.694, 0.695\rangle$, $\langle 0.695, 0.696\rangle$, or $\langle 0.696, 0.697\rangle$ is always $W_{22}<2.1493$.

We do not exclude the area $x\in\langle 0.9860, 0.9862\rangle$, $y\in\langle 0.697, 0.704\rangle$ by this way.

For $x\in\langle 0.9862, 0.9864\rangle$ and, step by step, for $y\in\langle 0.690, 0.691\rangle$, $\langle 0.691, 0.692\rangle$, $\langle 0.692, 0.693\rangle$, $\langle 0.693, 0.694\rangle$, $\langle 0.694, 0.695\rangle$, $\langle 0.695, 0.696\rangle$, $\langle 0.696, 0.697\rangle$, or $\langle 0.697, 0.698\rangle$ is always $W_{22}<2.1493$.

We do not exclude the area $x\in\langle 0.9862, 0.9864\rangle$, $y\in\langle 0.698, 0.706\rangle$ by this way.

For $x \in \langle 0.9864,\, 0.9866 \rangle$ and, step by step, for $y \in \langle 0.689,\, 0.690 \rangle$, $\langle 0.690,\, 0.691 \rangle$, $\langle 0.691,\, 0.692 \rangle$, $\langle 0.692,\, 0.693 \rangle$, $\langle 0.693,\, 0.694 \rangle$, $\langle 0.694,\, 0.695 \rangle$, $\langle 0.695,\, 0.696 \rangle$, $\langle 0.696,\, 0.697 \rangle$, $\langle 0.697,\, 0.698 \rangle$, or $\langle 0.698,\, 0.699 \rangle$ is always $W_{22} < 2.1493$.

We do not exclude the area $x \in \langle 0.9864,\, 0.9866 \rangle$, $y \in \langle 0.699,\, 0.709 \rangle$ by this way.

For $x \in \langle 0.9866,\, 0.9868 \rangle$ and, step by step, for $y \in \langle 0.688,\, 0.689 \rangle$, $\langle 0.689,\, 0.690 \rangle$, $\langle 0.690,\, 0.691 \rangle$, $\langle 0.691,\, 0.692 \rangle$, $\langle 0.692,\, 0.693 \rangle$, $\langle 0.693,\, 0.694 \rangle$, $\langle 0.694,\, 0.695 \rangle$, $\langle 0.695,\, 0.696 \rangle$, $\langle 0.696,\, 0.697 \rangle$, $\langle 0.697,\, 0.698 \rangle$, $\langle 0.698,\, 0.699 \rangle$, $\langle 0.699,\, 0.700 \rangle$, or $\langle 0.700,\, 0.701 \rangle$ is always $W_{22} < 2.1493$.

We do not exclude the area $x \in \langle 0.9866,\, 0.9868 \rangle$, $y \in \langle 0.701,\, 0.711 \rangle$ by this way.

Now let us look at the areas which we could not exclude by using the previous method. From (2.2) we have

$$\frac{\partial W_2}{\partial x} = \frac{x^5}{\sqrt[8]{(1-x^8-y^8)^7}} \left[(7x+6y)\left(y\sqrt[8]{(1-x^8-y^8)^7}+1-y^8\right) - 8x^9 - 7x^8\,y\right]$$

and $\dfrac{x^5}{\sqrt[8]{(1-x^8-y^8)^7}} > 0\,.$

For every point $X = (x,y)$, $x \in \langle a,\, b \rangle$, $y \in \langle c,\, d \rangle$ the inequality holds

$$\begin{aligned}&(7x+6y)\left(y\sqrt[8]{(1-x^8-y^8)^7}+1-y^8\right) - x^8(8x+7y)\\ &\le (7b+6d)\left(d\sqrt[8]{(1-a^8-c^8)^7}+1-c^8\right) - a^8(8a+7c).\end{aligned}$$

Let us denote $DW2(a,b,c,d) := (7b+6d)\left(d\sqrt[8]{(1-a^8-c^8)^7}+1-c^8\right) - a^8(8a+7c)$. Then for the following areas

$$\begin{array}{ll} x \in \langle 0.9856,\, 0.9858 \rangle,\ y \in \langle 0.696,\, 0.701 \rangle, & x \in \langle 0.9858,\, 0.9860 \rangle,\ y \in \langle 0.697,\, 0.702 \rangle,\\ x \in \langle 0.9860,\, 0.9862 \rangle,\ y \in \langle 0.697,\, 0.704 \rangle, & x \in \langle 0.9862,\, 0.9864 \rangle,\ y \in \langle 0.698,\, 0.706 \rangle,\\ x \in \langle 0.9864,\, 0.9866 \rangle,\ y \in \langle 0.699,\, 0.709 \rangle, & x \in \langle 0.9866,\, 0.9868 \rangle,\ y \in \langle 0.701,\, 0.711 \rangle \end{array}$$

is $DW2(a,b,c,d) < 0$ and therefore $\dfrac{\partial W_2}{\partial x} < 0$. This implies that the asked maximum cannot be achieved on the region $C_3$ (Figure 5.)

Now we determine the constrained maximum of the function

$$W(x,y) = x^8 + \frac{x^9}{y} \tag{2.3}$$

on the curve

$$C(x,y) = x^8y^8 + y^{16} - y^8 + (x^2-y^2)^8 = 0 \tag{2.4}$$

for $x \in \langle 0.9855,\, 0.9868 \rangle$. The system of equations $\dfrac{\partial W}{\partial x}\dfrac{\partial C}{\partial y} - \dfrac{\partial W}{\partial y}\dfrac{\partial C}{\partial x} = 0$ and $C(x,y) = 0$ has the form

$$\begin{gathered}10x^9y^8 + 18xy^{16} - 9xy^8 + 8x^8y^9 + 16y^{17} - 8y^9 + (x^2-y^2)^7(2x^3 - 16y^3 - 18xy^2) = 0\,,\\ x^8y^8 + y^{16} - y^8 + (x^2-y^2)^8 = 0.\end{gathered}$$

The solution is $x \doteq 0.986333649$, $y \doteq 0.704812693$, and $z \doteq 0.675488876$. The proof is complete. $\square$

## 3. Conclusions

**Conjecture 3.1.** $V_4(d) = V(d)$ *for* $d = 2, 3, 4$.

Our conjecture indicates that the case $n = 4$ is crucial for $d \leq 4$. The complexity of the problem greatly increases with increasing $n$ and $d$. The conjecture is not true for higher dimensions because $\lim\limits_{d\to\infty} V_n(d) = n$ for $n = 2, 3, 4, \ldots$ The statement $\lim\limits_{d\to\infty} V_n(d) = n$ has been proved for $n = 5, 6, 7, \ldots$ in [1], but this theorem is true for all $n \geq 2$.

## References


[1] P. Adamko, V. Bálint: *Universal asymptotical results on packing of cubes.* Studies of the University of Žilina, Mathematical Series, vol. **28**, 1–4 (2016).

[2] V. Bálint, P. Adamko: *Minimalizácia objemu kvádra pre uloženie troch kociek v dimenzii 4* [in Slovak]. G: slovenský časopis pre geometriu a grafiku **12**, 5–16 (2015).

[3] V. Bálint, P. Adamko: *Minimization of the parallelepiped for packing of three cubes in dimension 6.* In Aplimat: 15th Conference on Applied Mathematics 2016, pp. 44–55.

[4] S. Hougardy: *On packing squares into a rectangle.* Comput. Geom. **44**, 456–463 (2011).

[5] D.J. Kleitman, M.M. Krieger: *An optimal bound for two dimensional bin packing.* In 16th Annual Symposium on Foundations of Computer Science, 1975, pp. 163–168.

[6] A. Meir, L. Moser: *On packing of squares and cubes.* J. Combin. Theory **5**, 126–134 (1968).

[7] P. Novotný: *A note on a packing of squares.* Studies of the University of Transport and Communications in Žilina, Math.-Phys. Series **10**, 35–39 (1995).

[8] P. Novotný: *On packing of squares into a rectangle.* Arch. Math. (Brno) **32**, 75–83 (1996).

[9] P. Novotný: *On packing of four and five squares into a rectangle.* Note Mat. **19**, 199–206 (1999).

[10] P. Novotný: *Využitie počítača pri riešení ukladacieho problému* [in Slovak]. In Proceedings of Symposium on Computational Geometry 2002, pp. 60–62.

[11] P. Novotný: *Pakovanie troch kociek* [in Slovak]. In Proceedings of Symposium on Computational Geometry 2006, pp. 117–119.

[12] P. Novotný: *Najhoršie pakovateľné štyri kocky* [in Slovak]. In Proceedings of Symposium on Computational Geometry 2007, pp. 78–81.

[13] P. Novotný: *Ukladanie kociek do kvádra* [in Slovak]. In Proceedings of Symposium on Computational Geometry 2011, pp. 100–103.


Received February 16, 2018; final form August 3, 2018