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A Triangle "Broken" into Four Triangles – the Special Status of the Central Triangle

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Abstract. The cevians drawn through a point inside a given triangle intersect the opposite sides at three points. They form a triangle which subdivides the given triangle into four parts. The paper focuses on properties of the central triangle with respect to the other subtriangles.

Key Words: Cevians, central subtriangle

MSC: 51M04, 51M16, 51M25

1. Introduction

In an arbitrary triangle three cevians were drawn, which meet at a single point in the triangle's interior. Their points of intersection with the sides divide the original triangle into a *central triangle* containing the common point of the cevians and three *corner triangles*, each of which includes a vertex of the original triangle. Thus, the area of the original triangle "is broken" into four subtriangles (Figure 2). The paper focuses on special and surprising properties of the central triangle with respect to the other three triangles.

Description of the task. Let AD, BE, CF be cevians passing trough the point O in the interior of the triangle $\triangle$ABC, as shown in Figure 1. We denote

$$\alpha := \frac{\mathrm{BD}}{\mathrm{DC}}, \quad \beta := \frac{\mathrm{CE}}{\mathrm{EA}}, \quad \gamma := \frac{\mathrm{AC}}{\mathrm{FB}} \quad (\alpha\beta\gamma = 1).$$

The points $\mathrm{D}, \mathrm{E}, \mathrm{F}$ form four triangles, whose areas are

$$S_{\mathrm{A}} = S_{\triangle\mathrm{AEF}}, \quad S_{\mathrm{B}} = S_{\triangle\mathrm{BDF}}, \quad S_{\mathrm{C}} = S_{\triangle\mathrm{DEC}}, \quad S_{\mathrm{O}} = S_{\triangle\mathrm{DEF}}.$$

We look for properties of the central triangle $\triangle$DEF with respecto to the three corner triangles.

Lemma 1. *There holds*

$$(1) \quad \frac{S_{\mathrm{A}}}{S_{\triangle\mathrm{ABC}}} = \frac{\gamma}{(\gamma+1)(\beta+1}, \qquad (3) \quad \frac{S_{\mathrm{C}}}{S_{\triangle\mathrm{ABC}}} = \frac{\beta}{(\beta+1)(\alpha+1)},$$

$$(2) \quad \frac{S_{\mathrm{B}}}{S_{\triangle\mathrm{ABC}}} = \frac{\alpha}{(\alpha+1)(\gamma+1)}, \qquad (*) \quad \frac{S_{\mathrm{O}}}{S_{\triangle\mathrm{ABC}}} = \frac{\alpha\beta\gamma+1}{(\alpha+1)(\beta+1)(\gamma+1)}.$$

Statement (*) is Routh's Theorem [2, 5, 6].

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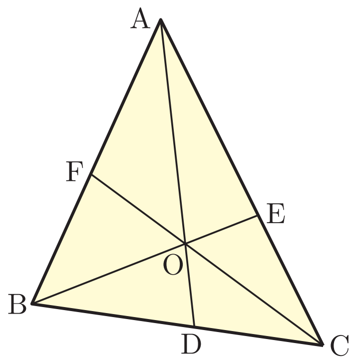

Figure 1: The cevians through O

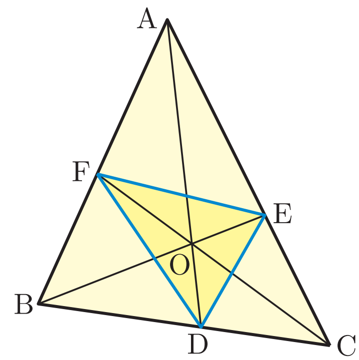

Figure 2: The four subtriangles

Proof. $\dfrac{S_{\mathrm{A}}}{S_{\triangle \mathrm{ABC}}} = \dfrac{\mathrm{AF}}{\mathrm{AB}} \cdot \dfrac{\mathrm{AE}}{\mathrm{AC}} \cdot \dfrac{\sin \mathrm{A}}{\sin \mathrm{A}} = \dfrac{\gamma}{(\gamma+1)(\beta+1)}$, and the proof is similar for the other equalities. Routh's theorem results from the identity $\dfrac{S_{\triangle \mathrm{ABC}} - S_{\mathrm{A}} - S_{\mathrm{B}} - S_{\mathrm{C}}}{S_{\triangle \mathrm{ABC}}} = \dfrac{S_{\mathrm{O}}}{S_{\triangle \mathrm{ABC}}}$ and the proof of the equalities (1), (2) and (3). □

A first inequality. From (*) follows in the case $\alpha\beta\gamma = 1$

$$\frac{S_{\mathrm{O}}}{S_{\triangle \mathrm{ABC}}} = \frac{2}{(\alpha+1)(\beta+1)(\gamma+1)} \leq \frac{2}{8\sqrt{\alpha\beta\gamma}} = \frac{1}{4}, \text{ hence } S_{\mathrm{O}} \leq \frac{1}{4} S_{\triangle \mathrm{ABC}}, \tag{4}$$

and equality holds only if $\alpha = \beta = \gamma = 1$.

2. The theorems

There are three different means of the areas S_{A}, S_{B} and S_{C} to distinguish,

- the arithmetic mean $M_{\mathrm{a}} = \dfrac{S_{\mathrm{A}} + S_{\mathrm{B}} + S_{\mathrm{C}}}{3}$,
- the geometric mean $M_{\mathrm{g}} = \sqrt[3]{S_{\mathrm{A}} \cdot S_{\mathrm{B}} \cdot S_{\mathrm{C}}}$, and
- the harmonic mean $M_{\mathrm{h}} = \dfrac{3}{\frac{1}{S_{\mathrm{A}}} + \frac{1}{S_{\mathrm{B}}} + \frac{1}{S_{\mathrm{C}}}}$.

Theorem 1. $\quad M_{\mathrm{a}} \geq M_{\mathrm{g}} \geq S_{\mathrm{O}} \geq M_{\mathrm{h}}$. (5)

Proof. a) We first prove that $M_{\mathrm{a}} \geq S_{\mathrm{O}}$. It was proven that $S_{\mathrm{O}} \leq \frac{1}{4} S_{\triangle \mathrm{ABC}}$, therefore $S_{\mathrm{A}} + S_{\mathrm{B}} + S_{\mathrm{C}} \geq \frac{3}{4} S_{\triangle \mathrm{ABC}}$, hence

$$\frac{S_{\mathrm{A}} + S_{\mathrm{B}} + S_{\mathrm{C}}}{3} \geq \frac{1}{4} S_{\triangle \mathrm{ABC}} \geq S_{\mathrm{O}}.$$

b) We prove that $M_{\mathrm{g}} \geq S_{\mathrm{O}}$.

$$M_{\mathrm{g}} = \sqrt[3]{S_{\mathrm{A}} \cdot S_{\mathrm{B}} \cdot S_{\mathrm{C}}} = S_{\triangle \mathrm{ABC}} \cdot \sqrt[3]{\frac{\alpha\beta\gamma + 1}{(\alpha+1)^2(\beta+1)^2(\gamma+1)^2}} = S_{\triangle \mathrm{ABC}} \cdot \sqrt[3]{\frac{1}{(\alpha+1)^2(\beta+1)^2(\gamma+1)^2}}.$$

From $S_{\mathrm{O}} = S_{\triangle \mathrm{ABC}} \cdot \dfrac{2}{(\alpha+1)(\beta+1)(\gamma+1)}$ follows

$$\frac{S_{\mathrm{O}}}{M_{\mathrm{g}}} = 2\left((\alpha+1)(\beta+1)(\gamma+1)\right)^{-\frac{1}{3}} = \frac{2}{\sqrt[3]{(\alpha+1)(\beta+1)(\gamma+1)}} \leq \frac{2}{\sqrt[3]{8}} = 1. \tag{6}$$

c) Let us prove that $S_{\mathrm{O}} \geq \dfrac{3}{\frac{1}{S_{\mathrm{A}}} + \frac{1}{S_{\mathrm{B}}} + \frac{1}{S_{\mathrm{C}}}}$, in other words, $\dfrac{1}{S_{\mathrm{A}}} + \dfrac{1}{S_{\mathrm{B}}} + \dfrac{1}{S_{\mathrm{C}}} \geq \dfrac{3}{S_{\mathrm{O}}}$.

Using the formulas (1)–(3), one has to prove that

$$\frac{(\gamma+1)(\beta+1)}{\gamma}+\frac{(\alpha+1)(\gamma+1)}{\alpha}+\frac{(\beta+1)(\alpha+1)}{\beta}\geq\frac{3(\alpha+1)(\beta+1)(\gamma+1)}{2}.$$

After dividing by $(\alpha+1)(\beta+1)(\gamma+1)$, one has to prove the inequality

$$\frac{1}{\gamma(\alpha+1)}+\frac{1}{\alpha(\beta+1)}+\frac{1}{\beta(\gamma+1)}\ \geq\ 1.5\,. \tag{7}$$

Here we proceed as follow: From Menelaus' Theorem [1, 3, 4] in the triangle ABD, cut by COF (see Figure 2), there holds $\frac{\alpha+1}{\alpha}\cdot\frac{\mathrm{DO}}{\mathrm{OA}}\cdot\frac{\gamma}{1}=1$, therefore $\frac{\mathrm{OD}}{\mathrm{OA}}=\frac{1}{\gamma(\alpha+1)}$, and similarly $\frac{\mathrm{OE}}{\mathrm{OB}}=\frac{1}{\alpha(\beta+1)}$ and $\frac{\mathrm{OF}}{\mathrm{OC}}=\frac{1}{\beta(\gamma+1)}$. We denote $x_3:=S_{\triangle\mathrm{AOB}}$, $x_2:=S_{\triangle\mathrm{AOC}}$, $x_1:=S_{\triangle\mathrm{BOC}}$. Then

$$\frac{\mathrm{OD}}{\mathrm{OA}}=\frac{x_1}{x_2+x_3}\,,\qquad\frac{\mathrm{OE}}{\mathrm{OB}}=\frac{x_2}{x_1+x_3}\,,\qquad\frac{\mathrm{OF}}{\mathrm{OC}}=\frac{x_3}{x_1+x_2}\,.$$

Therefore, one has to prove that

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_1+x_3}+\frac{x_3}{x_1+x_2}\geq 1.5\,.$$

This is a well-known inequality (Nesbitt's inequality [7, 8]), but we give a short proof. If $y_1:=x_1+x_2$, $y_2:=x_1+x_3$ and $y_3:=x_2+x_3$ then

$$x_1=\frac{y_1+y_2-y_3}{2}\,,\qquad x_2=\frac{y_3+y_1-y_2}{2}\,,\qquad x_3=\frac{y_2+y_3-y_1}{2}\,,$$

and hence remains to prove that

$$\frac{y_1+y_2-y_3}{2y_3}+\frac{y_3+y_1-y_2}{2y_2}+\frac{y_2+y_3-y_1}{2y_1}\geq 1.5\,.$$

We obtain that

$$\frac{1}{2}\Big(\underbrace{\frac{y_1}{y_2}+\frac{y_2}{y_1}}_{\geq 2}+\underbrace{\frac{y_2}{y_3}+\frac{y_3}{y_2}}_{\geq 2}+\underbrace{\frac{y_1}{y_3}+\frac{y_3}{y_1}}_{\geq 2}\Big)-\frac{3}{2}\geq 1.5\,.$$

We have thus proven (7) and Theorem (1). □

Important conclusion. $S_{\mathrm{O}}\geq\min\{S_{\mathrm{A}},S_{\mathrm{B}},S_{\mathrm{C}}\}$.

Proof. S_{O} is not smaller (usually larger) than the harmonic mean of S_{A}, S_{B} and S_{C}. Therefore it must be larger than or equal to (usually larger than) one of the areas S_{A}, S_{B}, S_{C}. Since S_{O} is smaller than or equal to the geometric mean of S_{A}, S_{B}, S_{C}, it must be smaller than or equal to one of S_{A}, S_{B}, S_{C}. □

An interesting probability question. If the triangle ABC is broken into four parts by the points $\mathrm{D},\mathrm{E},\mathrm{F}$, so that AD, BE and CF are cevians, what is the probability that S_{O} is the smallest? Surprisingly, the probability is 0. Similarly, what is the probability that S_{O} is the largest? The answer is: 0 again.

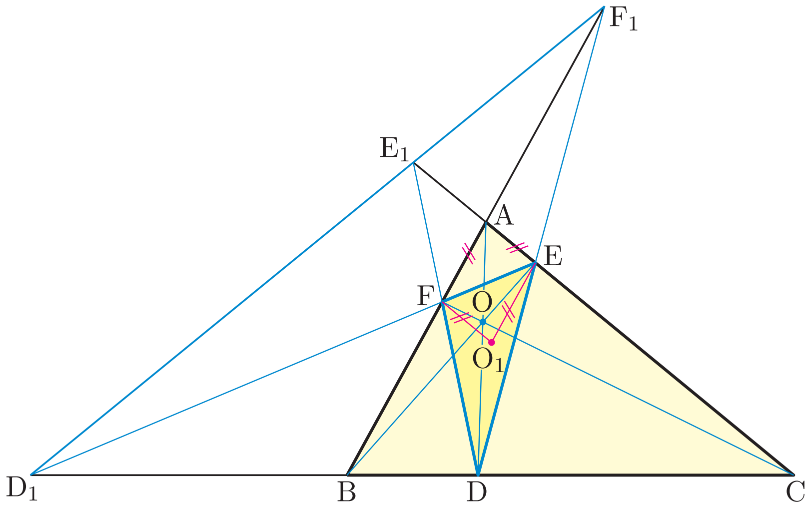

Figure 3: Comparing the perimeters of the subtriangles

Theorem 2. *Let* D, E, F *be points on* BC, CA *and* AB*, respectively, so that* BE*,* AD *and* CF *meet at the point* O*. We denote* $\alpha := \frac{\mathrm{BD}}{\mathrm{DC}}$, $\beta := \frac{\mathrm{CE}}{\mathrm{EA}}$ *and* $\gamma := \frac{\mathrm{AC}}{\mathrm{FB}}$, *where* $\alpha\beta\gamma = 1$.
We also denote the perimeters of the triangles $\triangle$AFE*,* $\triangle$CDE*,* $\triangle$FBD *by* $\mathrm{P_A}$*,* $\mathrm{P_B}$*,* $\mathrm{P_C}$*, respectively, and the perimeter of* $\triangle$DEF *by* $\mathrm{P_O}$*. Then there holds*

$$\mathrm{P_O} \geq \min\left(\mathrm{P_A},\, \mathrm{P_B},\, \mathrm{P_C}\right).$$

Proof. Without detracting from the generality, we can assume $\alpha < 1$, $\beta > 1$, $\gamma < 1$; the cases $\alpha = 1$, $\beta = 1$ or $\gamma = 1$ are easy to prove in a similar manner.
In our case the harmonic points of D, E, F are the points $\mathrm{D_1}$, E1 and F1 respectively, which are positioned, as shown in Figure 3, on a single straight line, the polar line of O. We draw from E a parallel to AB, and from F a parallel to AC. These parallels intersect at the point $\mathrm{O_1}$ (inside $\triangle$DEF). It is now clear that $\mathrm{P}_{\triangle \mathrm{FO_1E}} = \mathrm{P_A}$ and $\mathrm{P}_{\triangle \mathrm{FO_1E}} < \mathrm{P_O}$, therefore $\mathrm{P_O} > \mathrm{P_A}$. ☐

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