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On the Number of Points at Distance at Least 1 in the Unit Four-Dimensional Cube

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Abstract. We give an elementary proof of the following theorem: The maximum number of points which one can choose in the unit four-dimensional cube so that all mutual distances are at least one is 17.

Key Words: extremum, packing of cubes

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1. Introduction

Denote by $f(n)$ the maximum number of points which one can choose in the unit n-dimensional cube so that all mutual distances are at least 1. Such placing of points we call permissible placing. Trivially, $f(n) = 2^n$ for $n = 1, 2, 3$. A first reliable proof of the result $f(4) = 17$ can be found in the paper [1] (see [5]). In higher dimensions there are only estimates, see, e.g., [2, 3, 4, 7, 10].

This problem appears in [8] as problem 41, and was repeated besides others in [6, p. 244], [9] and some others.

2. The theorem

Theorem. $f(4) = 17$.

Proof. Let us consider the 4-dimensional unit cube $C = \langle 0, 1\rangle^4$. So, point $(x_1, x_2, x_3, x_4) \in C$ if and only if $x_i \in \langle 0, 1\rangle$ for $i = 1, 2, 3, 4$. The coordinates of 2^4 vertices of C are all ordered quadruples consisting of zeros and ones. The centre of C has the coordinates (h, h, h, h), where $h = \frac{1}{2}$. If we consider the 16 vertices and the centre of C, we get $f(4) \geq 17$.

We dissect the unit cube C into 16 congruent small cubes c_i. Each c_i contains exactly one vertex of the cube C, so the small cube c_i is uniquely determined by this vertex. The ordered quadruple (x_1, x_2, x_3, x_4) is a vertex of some small cube if and only if $x_i \in \{0, h, 1\}$ for $i = 1, 2, 3, 4$. So, our small cubes have exactly $3^4 = 81$ vertices. From every small cube

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we remove all its vertices and then we add exactly the specified vertices. So, we get the undermentioned set of 16 *adapted* small cubes.

$$
\begin{aligned}
M_1&\colon (0,0,0,0), (0,0,0,h), (0,0,h,0), (0,0,h,h), (0,h,0,0);\\
M_2&\colon (0,0,0,1), (0,0,h,1), (0,h,0,h), (0,h,0,1), (0,h,h,h);\\
M_3&\colon (0,0,1,0), (0,0,1,h), (0,h,h,0), (0,h,1,0), (0,h,1,h);\\
M_4&\colon (0,0,1,1), (0,h,h,1), (0,h,1,1), (h,0,h,1), (h,0,1,1);\\
M_5&\colon (0,1,0,0), (0,1,0,h), (0,1,h,0), (0,1,h,h), (h,1,0,0);\\
M_6&\colon (0,1,0,1), (0,1,h,1), (h,h,0,1), (h,h,h,1), (h,1,0,h);\\
M_7&\colon (0,1,1,0), (0,1,1,h), (h,h,1,0), (h,h,1,h), (h,1,h,0);\\
M_8&\colon (0,1,1,1), (h,h,1,1), (h,1,h,h), (h,1,h,1), (h,1,1,h);\\
M_9&\colon (1,0,0,0), (h,0,0,0), (h,0,0,h), (h,0,h,0), (h,0,h,h);\\
M_{10}&\colon (1,0,0,1), (h,0,0,1), (h,h,0,h), (1,0,0,h), (1,0,h,h);\\
M_{11}&\colon (1,0,1,0), (h,0,1,0), (h,0,1,h), (h,h,h,0), (1,0,h,0);\\
M_{12}&\colon (1,0,1,1), (1,0,h,1), (1,0,1,h), (1,h,h,h), (1,h,h,1);\\
M_{13}&\colon (1,1,0,0), (h,h,0,0), (1,h,0,0), (1,h,0,h), (1,h,h,0);\\
M_{14}&\colon (1,1,0,1), (h,1,0,1), (1,h,0,1), (1,1,0,h), (1,1,h,h);\\
M_{15}&\colon (1,1,1,0), (h,1,1,0), (1,h,1,0), (1,h,1,h), (1,1,h,0);\\
M_{16}&\colon (1,1,1,1), (h,1,1,1), (1,h,1,1), (1,1,h,1), (1,1,1,h).
\end{aligned}
$$

E.g., the adapted small cube M_3 containing the vertex $(0,0,1,0)$ contains from its 15 other vertices only four vertices $(0,0,1,h)$, $(0,h,h,0)$, $(0,h,1,0)$ and $(0,h,1,h)$.

Let us denote $M_{17} = (h,h,h,h)$. It is not hard to ascertain that $\bigcup_{i=1}^{17} = C$. Further, one can permissible place maximally 1 point into each of the *adapted* small cubes.

Now, the pigeon-hole principle gives $f(4) \le 17$. □

Remark. Let us note that the authors in [1] proved not only $f(4) = 17$, but — besides others — also the fact that the only configuration with $f(4) = 17$ consists of 16 vertices of the cube C and its centre.

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