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Shell Structures vs. Tessellation Patterns, a Didactic Experiment Between Architecture and Mathematics

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Abstract. Architecture has always relied on mathematics to achieve proportioned aesthetics, geometrical consistency, structural performance and reasonable construction. However, since computational tools have given architects the means to design and build complex spatial concepts that would have been inconceivable even twenty years ago, the discussion on how to integrate computation into the architectural curriculum is still ongoing. Against this background the academic project "Shell Structures vs. Tessellation Pattern" focused on the early integration of mathematical strategies regarding geometric description, structural performance, physical properties, and material specification as well as aspects of fabrication to inform the architectural design. The emphasis of the curriculum, developed jointly by the Department of Architecture and the Department of Mathematics, was put on research-based design-strategies that aimed to unfold hidden complexities of rather simple geometric definitions. The paper presents the didactic methodology and discusses three selected case studies that were carried out within this interdisciplinary Master course at the Politecnico di Milano.

Key Words: Didactics, Architecture curriculum, architecture and mathematics, parametric modelling, tessellation, shell structures

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1. Introduction

The emphasis of the course was put on research-based design-strategies that aimed to unfold hidden complexities of rather simple geometric definitions. 3D-geometries are in first place mathematical objects and, as such, they are a sequence of mathematical functions and relations used to describe a set of volumes and surfaces that constitute their separating boundaries. CAD applications are built using these mathematical concepts but generally, do

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not unveil them. In order to apply these principles in a comprehensive way, they have to be taught and understood by using the language of Mathematics. Moreover, the constraints that a geometrical shape must satisfy in order to be fabricated conveniently are described as well by mathematical equations. Hence, Mathematics plays a major role for the operability within the design and building process.

The curriculum had been conceived and carried out jointly by the Department of Architecture and the Department of Mathematics at the Milan Polytechnic University. The research focused on an in-depth understanding of geometric principles and their mathematical definitions as a starting point for the development of individual architectural projects. Next to the interdisciplinary exchange between architecture and mathematics, the intercultural set-up of the teams (students originating from ten different nationalities) proved to be both challenging and inspiring throughout the process.

1.1. Methodology

The academic project not only dissolves the boundaries between research, teaching and practice but also creates a change of perspective from teaching to learning. The following didactic models support such an approach:

- *DesignThinking* [1] serves as the basis for a creative process and interdisci-plinary exchange. It is based on the assumption that problems can be solved better when people of different disciplines work together in a creativity-promoting environment, jointly develop a research question, consider the needs and motivations of people and then develop concepts that are tested multiple times.
- *Research-based Design* [7] describes a research-led development process. The design method is aimed at the construction of prototypes and involves the exploration of various design concepts as well as the process-accompanying evaluation of the results and successive optimization of the proposed solutions.
- *Design-Build Projects* [16] combine practice and teaching. The realization of a building from the conceptual idea, through design and planning to execution is carried out together with students. If one understands the structural realization as a goal of the creative activity, the confrontation with the constructing and building offers an enormous learning potential.

Against this background the teaching approach combined the definition and discussion of mathematical principles for the generation of spatial geometry and their conceptual, structural, and functional potential from an architectural point of view as well as the potential and constraints for the fabrication of prototypes throughout the process. The considerations exposed above lead to the awareness of the opportunity for architects to build a familiarity with the basic concepts of Computational Geometry, in particular, with the representation and approximation of curves, surfaces, and volumes, and showed the necessity to build up a firm knowledge of the language appropriate for the discussion of such concepts.

Starting from the historic context of architecture and mathematics [15] and the theoretical background presented at the beginning of the course [10] — introducing the geometric principles of tessellation patterns (2D) and shell structures (single- and doubly-curved) — the course was conceived as an interconnected cycle of computation, design and fabrication. Hence, the process was driven by non-linear, but alternating methods and diverse application of tools. As a common ground for the operative part the software application Rhinoceros

(3D-Modelling) was chosen with the extensions of Grasshopper (Visual Programming) and Python (Programming Language) in order to connect architectural design and mathematics in a manageable framework for the students. After understanding the underlying principles of tessellations and shells, the findings were used to develop architectural case studies that profit from the synergy of both.

The presentation of mathematical tools to support the process started with the introduction of linear (frieze) and planar (tile) symmetry groups [12, 13]. Students were guided to develop the ability of recognizing the symmetry 'signature' of a tessellation of the plane and to design symmetric tiling by working with unit cells and affine isometries. Exercises with special tiling classes such as *Truchet tiling* [14, 2] were conducted as an example. To move out of the plane into space, tiled shell structures were presented as the mapping of planar patterns on 2D manifolds embedded in 3D space, smooth invertible parametrizations of 3D surfaces and the local distortion properties of different parametrizations were discussed [17]. Mappings of spherical surfaces were used as an example of particular interest and the distortion properties of different world map projections were discussed.

A second guided design process was that leading to the construction of 3D geometries via (un-)folding. To introduce the required mathematical concepts, first the definition of smooth parametrizations of surfaces in 3D was introduced together with the definition of their main differential properties [17]. In particular the definition of Gaussian curvature and its relation to unfoldability was discussed, then examples of common geometrical shapes with zero (cones, cylinders, unfoldable ruled surfaces) and non-zero (sphere) Gaussian curvature were given.

For the final design exercise the concept of Constructive Solid Geometry [4] was introduced and students were guided through the creation of 3D geometrical models by applying Boolean operations to solids with unfoldable boundaries.

1.2. Tessellations and shells

Tessellations consist of, or are generated from, simple geometric forms like a square, a hexagon or a circle. Through algorithmic operations — such as rotate, copy, array, mirror, offset and scale — intricate and fascinating patterns can be developed from these basic geometries. Patterns have an underlying mathematical structure; indeed, mathematics can be seen as the search for regularities, and the output of any function is a mathematical pattern. Tile patterns, mosaics, stained-glass windows, inlaid woodworks and other high skilled and time-consuming types of surface decorations, belonging to the world of ornamentation, are part of our cultural conceptions of mathematical beauty [8]. As such, the aesthetic and intellectual appreciation of tessellations has always demanded a certain degree of challenge of the creating process by developing coherent figures without gaps or overlaps of the chosen base units. Platonic and Archimedean tessellation patterns are representative for this principle.

Next to its geometric and aesthetic identity a base unit can express a load-paths model distributing internal and external forces throughout the whole surface it occupies. Intriguing examples of this integral approach between single unit (part) and overall shape (whole) can be seen in Pier Luigi Nervi's work on large-span structures, where he explored the relationship between material patterning, function and structural performance [3]. The fascinating masonry shell structures by Rafael Guastavino, the so called *Catalan vaults* [6, 11] as well as the work of Eladio Dieste [9] are testimonials for the synergetic potential of tessellated shells that goes beyond pure decoration.

Beyond this integral and performative approach, obviously architectural spaces are made from singular elements or building components that need to be organized according to their

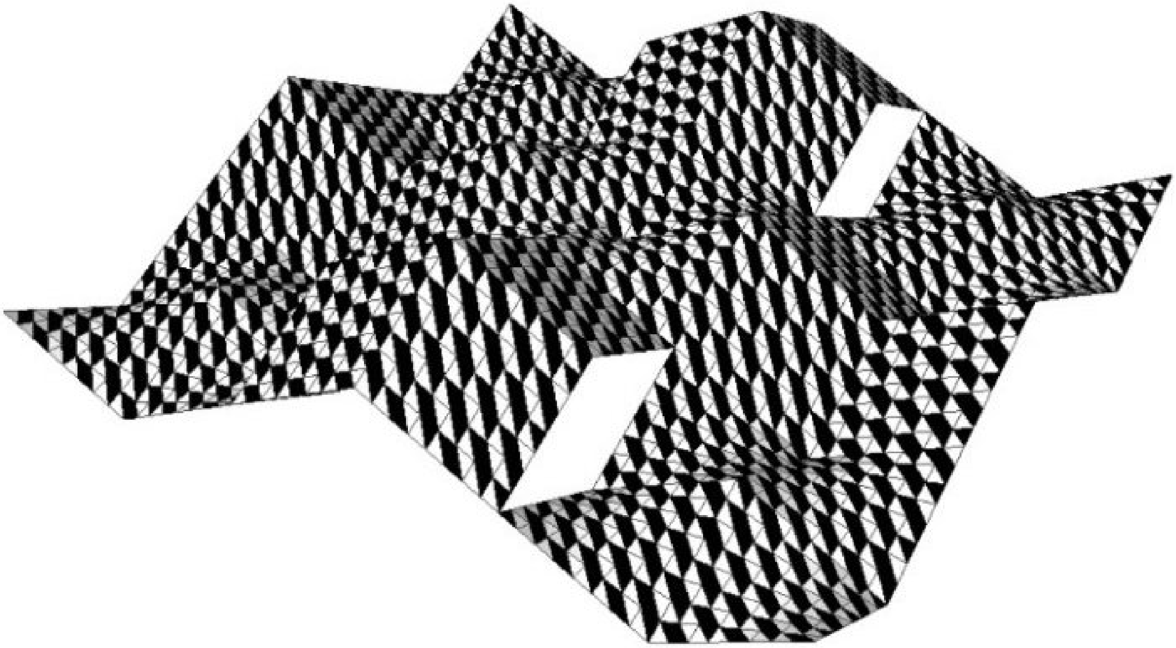

Figure 1: Digital model of the tessellated surfaces.

geometric constraints, constructability and assembly parameters. Hence, there is a mutual necessity to define the configuration of the parts to form the whole. Hence, the course tried to explore the synergetic potential of tessellations and shells to develop an informed architecture. Thus, the students were asked to develop strong and fresh architectural projects from fairly simple methods of space-defining surface-geometry. And while doing so ideally unfold a conceptual, historical, theoretical or technical framework around the project that will take it beyond the mere development and application of computational tools (Figure 1).

2. Case studies

Three representative projects serve in the following chapter as examples of the before described researched-based design approach for tessellation shells.

In the *"Wafer Saddle"* design a grid rectilinear pattern dictate the structure of the vault suggesting a waving movement due to the repetition of the inclined linear beams. The resulting saddle structure is potentially infinite and could be fabricated to create, for example, a woodwork roof structure. In the *"Reciprocal Tubes"* project, 'Hook's law of inversion' [5] is explored and applied in scale 1:1 with the use of simple cardboard tubes and plastic ties as joints. Finally, in the *"Zebra Pattern"* design, the black and white pattern was reduced to a squared portion in order to generate a repetitive pattern which could be applied to the designed shell structure and becoming part of the structural and acoustic performance.

2.1. Wafer Saddle

The geometric concept of this project is based on a diversified surface, which consists of joined rectilinear and curved parts that form a continuity. Next to quite simple flat surfaces the project consists of hyperbolic paraboloids (HP). These doubly ruled surfaces contain two families of mutual skew lines; which are parallel to a common plane, but not to each other. It

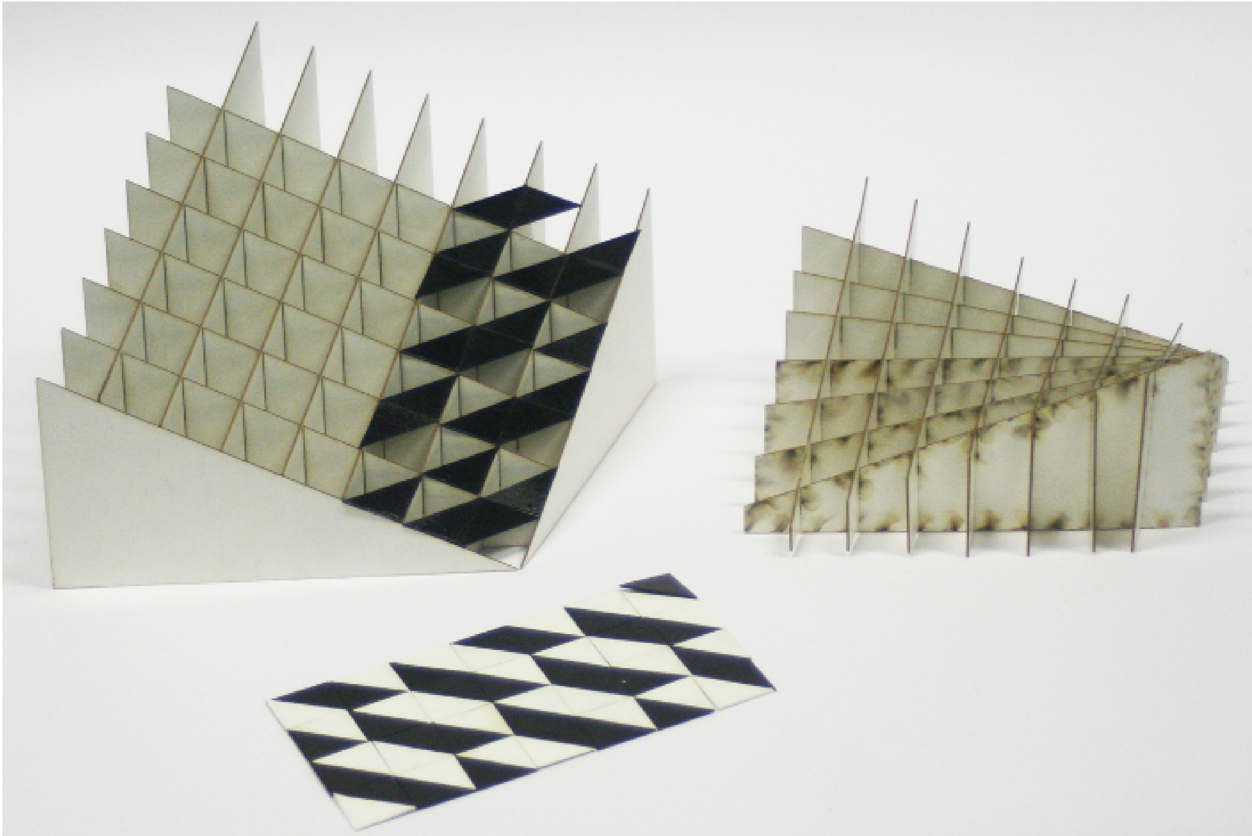

Figure 2: Physical models of the flat and HP-surface with tessellation pattern.

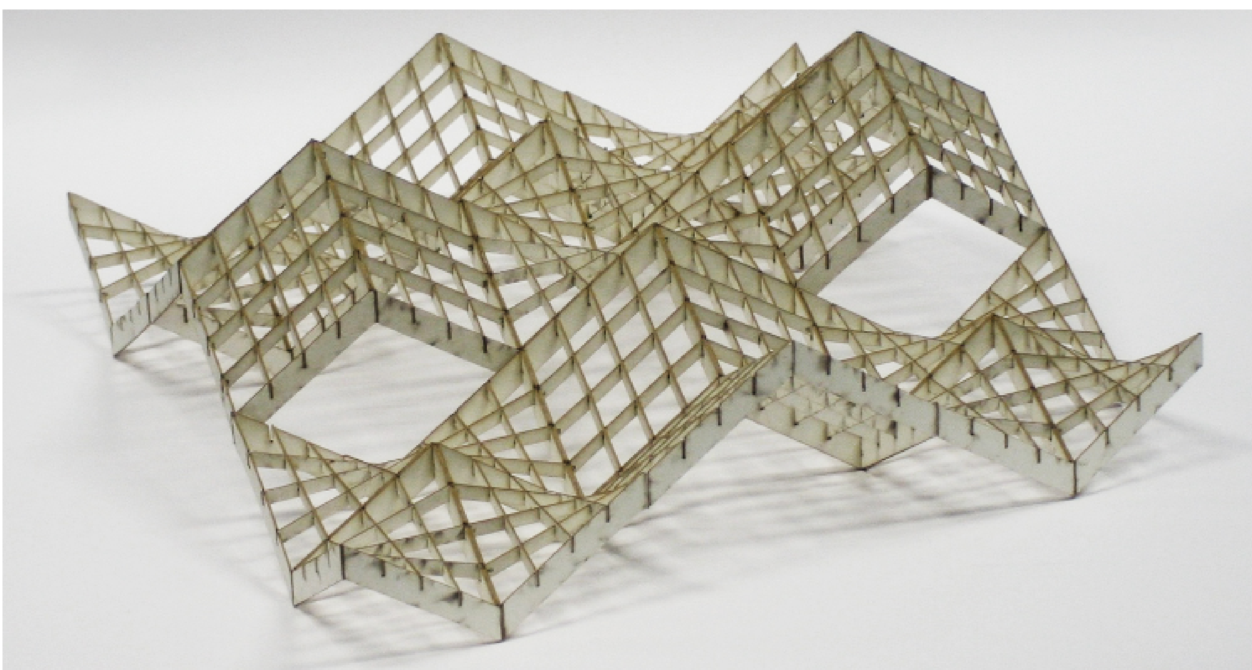

Figure 3: Physical models of the flat and HP-surface with tessellation pattern. (Project by Karyna Kalchenko, Karolina Rachwał, Dániel Simoncsics)

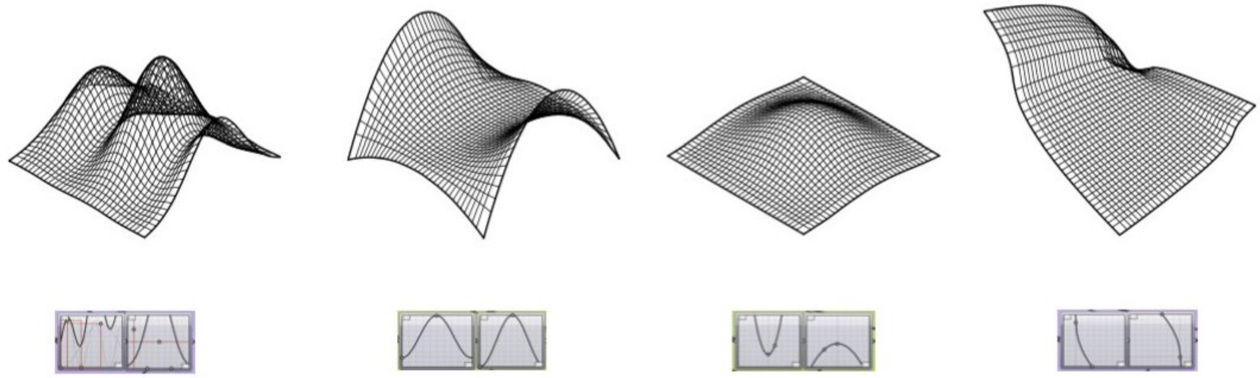

Figure 4: Parametrically generated variations of the shell surface by defining the outline curves, using Rhinoceros/Grasshopper/Kangaroo.

can be described from straight lines, which are rotated along the axis, forming a saddle surface. The possibility of constructing it from straight beams is one of the pretexts in formulating the further research on how a doubly curved surface would interplay with a rectilinear one. Moreover, the continuity was enhanced by the application of a common tessellation pattern that is based on the isocurves of the different surface typologies. As a result the project shows seamingless transitions as well as articulated differences in the overall shape.

As a general rule for the surface generation two simple units were created that could be further rotated, duplicated and organized into different components. The HP surface cannot only be connected with a non-curved surface but also be connected from all sides of the surface. Hence, it creates multiple variations of closed and open parts. The final surface consists of 4 HP surfaces and 4 rectilinear surfaces put together either to create a concave or convex module. The alternating modules generate a geometric system, which could be further developed as a roof structure. As a proof of concept this apparently multisided surface was finally translated into a physical model, fabricated from intersecting, lasercut cardboard beams (Figures 2–4).

2.2. Reciprocal Tubes

This project is based on a self-supporting, reciprocal system, which is derived from the isocurve mesh of a tessellated shell. The idea of an expanded node generates the possibility of fairly simple connections of linear elements. As such the node is defined as a number of bar-elements meeting at or near one point within the system. This system allowed for experimenting with relatively simple materials such as cardboard tubes and cable binders. Hence, physical models of this principle served as an early proof for the later construction of the surface

However, before fabricating the final physical model, simulations and variations of the structural performance were tested on the digital model, using Rhinoceros/Grasshopper/Kangaroo for the form-finding. The design variations of the shell structure are generated by a translational surface based on two mathematical graphs. Using this principle, variations of two parabolic functions were used to generate an elliptic paraboloid for the large-scale model and a hyperbolic paraboloid surface for a smaller scale model.

The reciprocal structure was generated on the given surface out of lines, using the "reciprocal" component from the Kangaroo 1 plugin. The geometry can be generated on any geometrical mesh shape — triangular, square or polygonal. A square shaped mesh was chosen, as it uses less constructive elements and is easier to assemble in a physical model. The

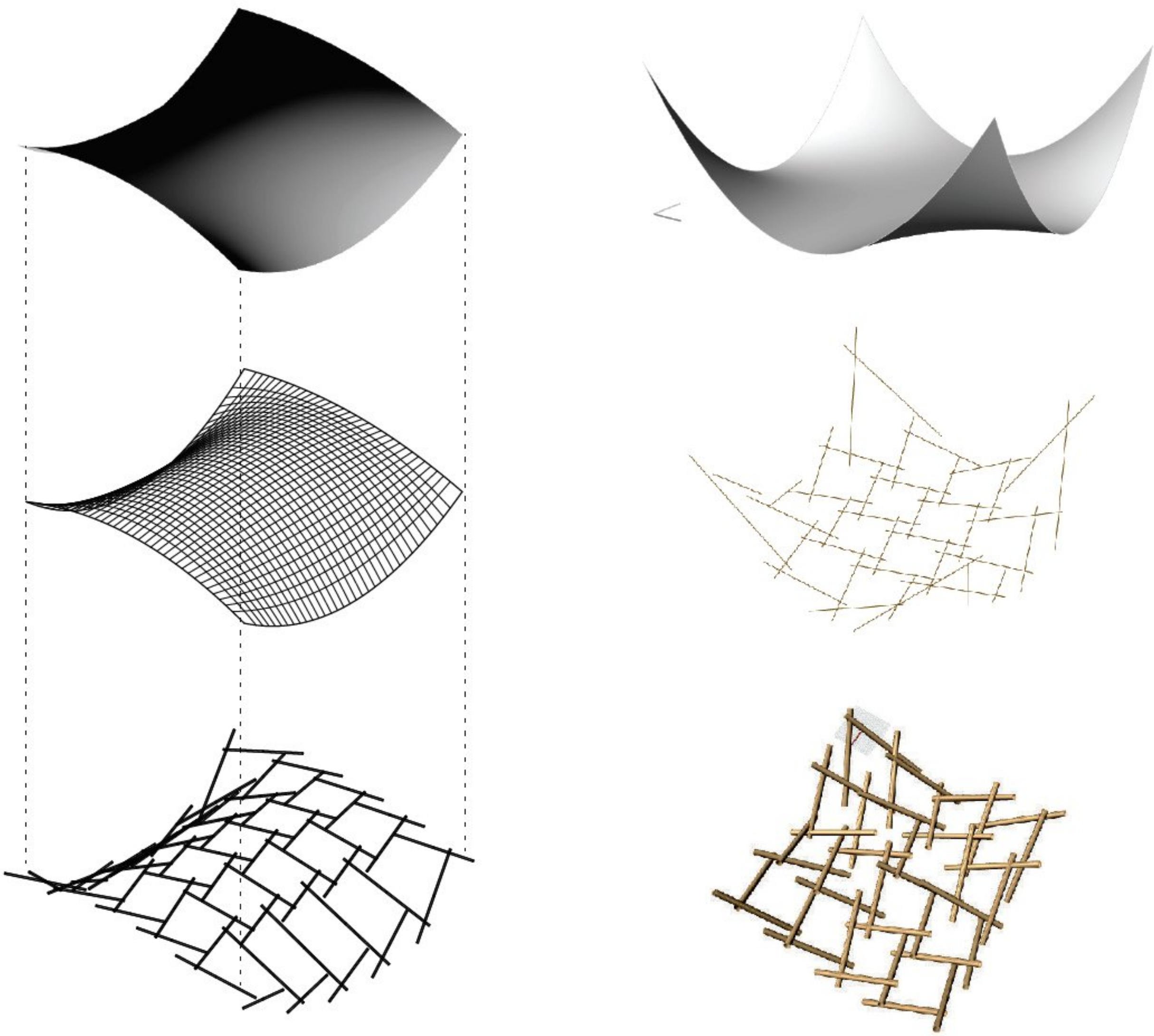

Figure 5: Application of the reciprocal system to the different surfaces.

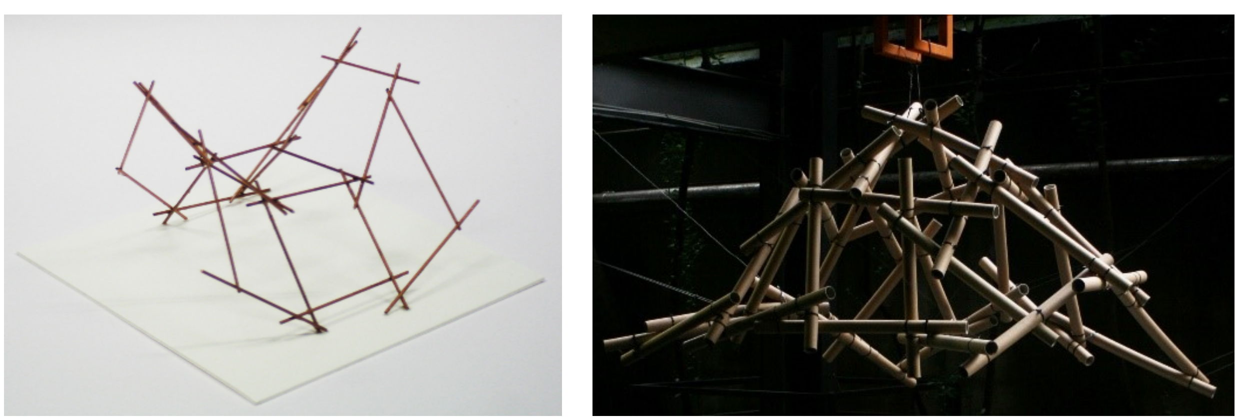

Figure 6: Physical prototypes of the reciprocal structure in different scales and materials. (Project by Gleb Mikhnovets, Alexandre Marguerie, Elizaveta Kunina)

digitally generated structure was tested using the Kangaroo plugin in a physical simulation, to see how steady the physical model would potentially be. To test the digital model, four anchor points were given, along with gravitation and the diameter of the tubes. From the optimized digital model simulation, data was extracted, such as the quantity of the tubes and the connection points. Moreover, each tube was numbered and the connection points for the assembly were marked on the tubular elements, by defining the distance of the connection from the edge of each tube. Using these instructions two physical models were assembled a largescale model from cardboard tubes connected with cable binders and a smaller scale model out of welded metallic rods (Figures 5–6).

2.3. Zebra Pattern

Of particular interest for this study was a specific group of self-organizing, inherent tessellation patterns — the one of zebra's coat (Figure 7). The alternating mechanism of its pigment cells was studied through a cellular automata model in 2D. With reference to Felix Candela's shell structures, the group developed a spatial design, which initially resulted in a doubly curved vault-like geometrical unit as a target surface for the tessellation. The zebra pattern was applied over the shell through the use of algorithmic procedures developed with grasshopper (from 2D to 3D), evaluating black and white areas and assigning positive or negative vertex translation respectively, creating a relief-like acoustically performative surface. Considering that fabrication of doubly curved shells is still difficult in its execution, further shell developments aimed at obtaining its design through developable surfaces that could be more easily fabricated in unrolled configuration and afterwards bent in their position. In parallel to the computational design process the group developed physical prototypes using 3D-printing and CNC-milling for the production of casting panels for both surface typologies (single- and doubly-curved).

The design for the shell structure was inspired by Felix Candela's well known *Cascarones*. In this project Candela tried to prove the structural efficiency of reinforced concrete in a dome like shape, where the shape tends to eliminate completely the tensile forces in the concrete. In his approach, he tended to rely on the geometric properties of the shell for analysis, instead of complex mathematical means. Considering that fabrication and construction of doubly curved shells requires more complexities and difficulties in its execution, the group aimed at obtaining its design through developable surfaces that could be more easily fabricated in unrolled configuration and afterwards bent into their position. Therefore, the final design resulted in a sixsided Candela-like vaulted structure. Furthermore, the six vaults are joined and supported with additional six vaults in opposite direction and complementary curvature, creating a sinusoidal wave-like shell edges. The characteristics of modularity and repetitiveness remained in this design solution as well.

The pattern application along the shell followed the same procedure as in previous iteration. However, in this design solution, the external side of the shell remained smooth without pattern, while the internal shell side resulted in a relief-like acoustic surfaces derived from the pattern. Two main reasons led to the final solution: first, the potential ease of fabrication where the developable surface would be unrolled and bent back in its position after the pattern form is applied upon it. Secondly, the need of acoustically performative surfaces only inside of the shell.

Finally, considering the overall process in creation and development of this project, two fabrication methods and two different materials of the final shell were considered and evaluated: CNC milled wooden structure and concrete structure casted with additively manu-

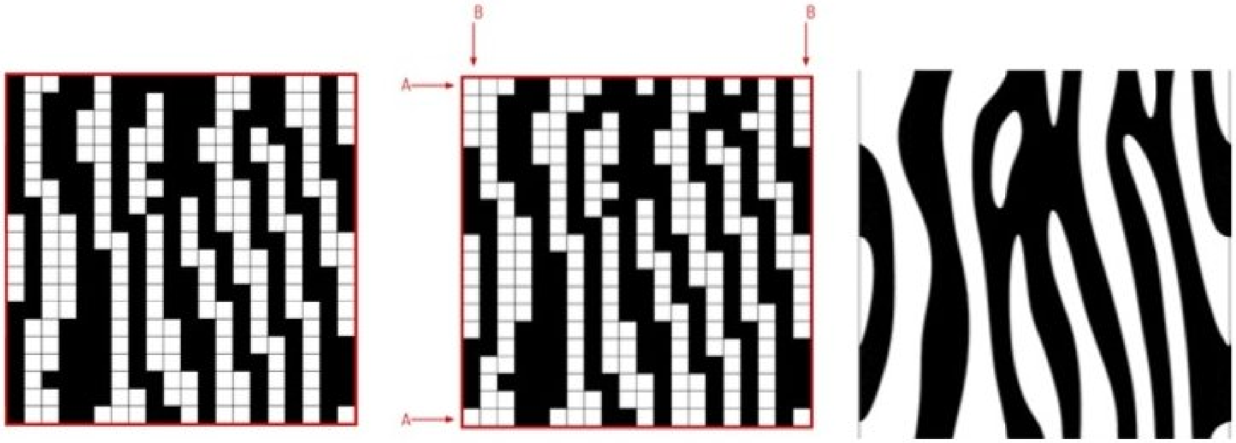

Figure 7: 2D-Tessellation of the Zebra pattern.

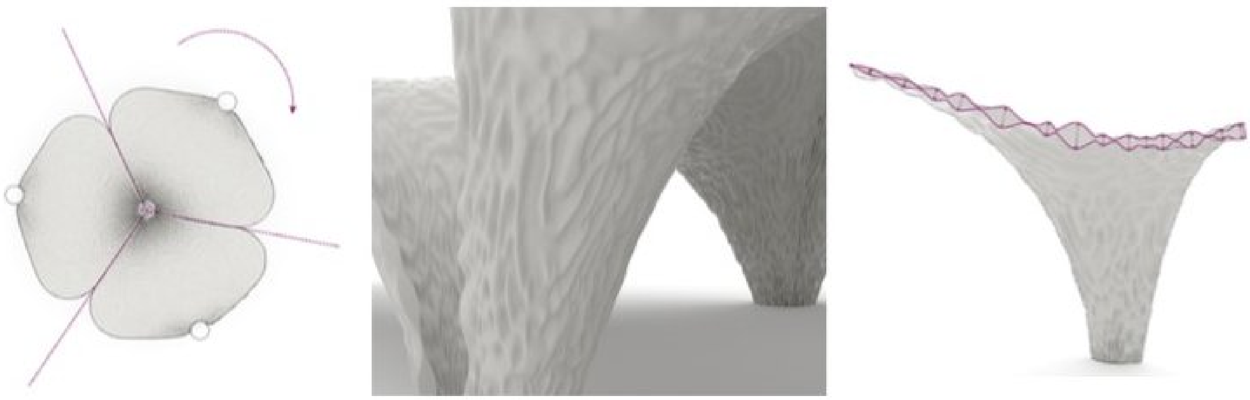

Figure 8: Digital 3D-Model. Tessellation applied to a trefoil shell structure.

Figure 9: 3D-printed parts of the tessellation shell.
(Project by Anja Kunic, Aleksandra Rochala, Mateusz Rejniak)

factured moulds. In the first case the pattern surface milling is performed over the planar unrolled shell surface, after which the so-called wood kerfing is applied in order to allow for the bending of the flat wooden form. In the second case the mould is 3D printed in its actual form and concrete casted directly, after which the plastic cast is removed. The two methods are further compared and evaluated in terms of the material performance (structural, acoustic and formability), fabrication time and cost, feasibility and flexibility to adapt to various geometrical and formal requirements.

3. Conclusion

The Master course connected complexity deliberately with a bottom-up design methodology to constitute an attainable learning base for the students. As a result, almost all of the projects reached a high level of complexity that started from a fairly simple analysis of a phenomenon or mathematical problem — in this case the interconnection of tessellation principles and space defining shell structures. In order to generate an accessibility to the chosen topic, the abstraction of the principle was an important first step (and a threshold for most of the teams as it appeared to be quite different from a standard design process). The definition of a mathematical model that represented the basic idea served as an essential starting point. Hence, the more profound the model was elaborated, the more potential it inherited for the further design process. Furthermore, the approach allowed for the successive development of a resilient system that is able to generate variations (as a base for design decision making) and to increase the level of complexity throughout the design process gradually.

Yet, the case studies also showed that the integration of parameters and requirements of functionality, efficiency, and aesthetics as well as structural performance shifts the focus from a purely formal design practice to an optimization process that relies on a systematically coherent approach. The understanding of the mathematical definition laid the solid foundation for the generation of a computational design concept and allowed for an accessible handling of complexity throughout the whole design to build process.

As such, the development of a physical product from the digital model was an important step within the design process. Most of the resulting prototypes can be seen as process or research models rather than final solutions. The structural performance, assembly logic, connection details, feasibility or scalability of the designs reached in many cases only in-between stages or partially convincing results. Therefore, the further improvement of the concepts should take into account an evaluation of the physical prototypes and use the findings for the enhancement of the digital design model and the development of a consistent materialization strategy. Against this background it might be necessary to extend the semester course to an annual academic module, which could enable also a realization of the concepts in scale 1:1. In such a way the previously described benefits of design-build projects could be integrated in the academic context (Figures 8–9).

In order to achieve a more effective learning success the course should be organized in intensive blocks of 2 – 3 days or an entire week, rather than weekly lectures and seminars. Such a focused working process (e.g., studying and applying mathematical principles as well as developing and testing physical prototypes) would increase the generation of knowledge in a short time. An eLearning platform (e.g., flipped classroom, online tutoring and lectures, chatrooms) would support this didactic concept and provide a beneficial environment to document and share the findings more efficiently within the peer group.

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