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Characterization of an Isosceles Tetrahedron

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Abstract. A tetrahedron in which each edge is equal to its opposite is an *isosceles* tetrahedron. We will use vectors to prove the following statement: A tetrahedron $OABC$ is isosceles if, and only if the centroid of the parallelepiped defined by the three edges OA, OB, and OC is an ex-center of the tetrahedron $OABC$.

Key Words: Isosceles tetrahedron, in-center, ex-center, centroid, circum-center.

MSC 2010: 52B10, 51M04, 51N20

1. Characterization of an isosceles tetrahedron

A tetrahedron in which each edge is equal to its opposite is an *isosceles (equifacial)* tetrahedron. There are many characterizations of an isosceles tetrahedron. Recently, Mazur [3] has published two theorems characterizing an isosceles tetrahedron. Many other characterizations are listed on pages 94–102 of [1]. Among them, the most striking theorem to us is the following.

Theorem 1 (See Theorem 307 of [1], and [2])**.** *A tetrahedron is isosceles if, and only if the four faces of a tetrahedron have the same area.*

In order to introduce other ideas, we need following definitions.

Definition 1. An outside sphere of the tetrahedron $OABC$ tangent to all the planes OAB, OAC, OBC, and to the triangle ABC simultaneously is called an *ex-center*. A sphere that circumscribes (inscribed in) a tetrahedron is called a *circum-sphere (in-sphere)* and its center a *circum-center (in-center)*.

Note: On pages 74–75 in [1], "truncs" and "escribed" spheres of a tetrahedron are defined. Please note that escribed spheres in truncs are our definition of ex-spheres. There are exactly four ex-spheres (see Theorem 250 of [1]), while there is only one in-sphere to a tetrahedron.

Some known equivalent statements to being an isosceles tetrahedron related to this paper are listed below.

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Theorem 2. *The following statements are equivalent.*

(1) *A tetrahedron is isosceles.*

(2) *The circum-center and the in-center of a tetrahedron are identical* (*see* [1, Theorem 304]; *also, see* [2]).

(3) *The circum-center and the centroid of a tetrahedron are identical* (*see* [1, Theorem 298]. *Also, see the statement at the end of the proof of Theorem 3 in* [3]).

It is our purpose to expand Theorem 2 using Theorem 1 in terms of ex-centers and centroids. We use vectors to prove our results. For this, let us introduce notations.

Notations: Let $OABC$ be a tetrahedron. Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$. We denote the parallelepiped defined by the three edges, OA, OB, and OC by Γ. Let O' be the diagonally opposite vertex of O in Γ. Thus $\vec{OO'} = \vec{a} + \vec{b} + \vec{c}$. We denote the volume of Γ by γ. Then the volume of the tetrahedron $OABC$ is $\frac{1}{6}\gamma$. Let M be the point defined by $\vec{OM} = \frac{1}{2}\left(\vec{OA} + \vec{OB} + \vec{OC}\right)$. The point M is the *centroid* (the center of gravity) *of the parallelepiped* Γ. Let G be the point defined by $\vec{OG} = \frac{1}{4}\left(\vec{OA} + \vec{OB} + \vec{OC}\right)$. Then G is the *centroid of the tetrahedron* $OABC$. (For your information, if the point S is defined by the vector equation $\vec{OS} = \frac{1}{3}(\vec{a} + \vec{b} + \vec{c})$. Then S is the centroid of the triangle ABC.)

Theorem 3. *The following statements are equivalent.*

(1) *A tetrahedron $OABC$ is isosceles.*

(2) *The centroid G of the tetrahedron $OABC$ is the in-center of the tetrahedron.*

(3) *The centroid M of the parallelepiped Γ is an ex-center of the tetrahedron $OABC$.*

Remark. The equivalence (1) $\Longleftrightarrow$ (2) in Theorem 3 can be obtained from Theorem 2, or see Theorems 300 and 303 of [1]. In view of Consequence 305 in [1], the ex-center and the centroid of Γ could have been a part of this statement, but it is not. So it became the motivation of this paper. We listed three equivalent statements in Theorem 3 because the proofs of these equivalences are almost identical using the next lemma.

Lemma 1. *Let O'' be a point defined by $\vec{OO''} = t(\vec{a} + \vec{b} + \vec{c})$ for some $0 \le t \le 1$. Let $\vec{AB} = \vec{b} - \vec{a} = \vec{u}$ and $\vec{AC} = \vec{c} - \vec{a} = \vec{v}$. Then the distance from O'' to the planes OAB, OAC, OBC, and ABC are $\dfrac{t\gamma}{\|\vec{a} \times \vec{b}\|}$, $\dfrac{t\gamma}{\|\vec{a} \times \vec{c}\|}$, $\dfrac{t\gamma}{\|\vec{b} \times \vec{c}\|}$, $\dfrac{|3t-1|\gamma}{\|\vec{u} \times \vec{v}\|}$, respectively.*

Proof. The tetrahedron $OABO''$ has the volume

$$\frac{1}{6}\,|\vec{OO''} \cdot \vec{a} \times \vec{b}\,| = \frac{1}{6}\,t\,|(\vec{a} + \vec{b} + \vec{c}\,) \cdot \vec{a} \times \vec{b}\,| = \frac{1}{6}\,t\,|\vec{c} \cdot \vec{a} \times \vec{b}\,| = \frac{1}{6}\,t\gamma.$$

Since the area of the triangle OAB is $\frac{1}{2}\|\vec{a} \times \vec{b}\|$, the distance from O'' to the plane OAB is given by

$$3 \cdot \frac{\frac{1}{6}\,t\gamma}{\frac{1}{2}\|\vec{a} \times \vec{b}\,\|} = \frac{t\gamma}{\|\vec{a} \times \vec{b}\,\|}.$$

Similarly, we can show that the distances from O'' to the planes OAC and OBC to be

$$\frac{t\gamma}{\|\vec{a} \times \vec{c}\,\|} \quad \text{and} \quad \frac{t\gamma}{\|\vec{b} \times \vec{c}\,\|},$$

respectively. Next, let us find the distance from O'' to the plane ABC. Note that $\vec{u} \times \vec{v} = \vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c}$. Since $\vec{AO''} = t(\vec{a} + \vec{b} + \vec{c}) - \vec{a} = t\vec{b} + t\vec{c} - (1-t)\vec{a}$, the volume of the tetrahedron $O''ABC$ is given by

$$\begin{aligned}\frac{1}{6}\left|\vec{AO''} \cdot (\vec{u} \times \vec{v})\right| &= \frac{1}{6}\left|\left(t\vec{b} + t\vec{c} - (1-t)\vec{a}\right) \cdot \left(\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c}\right)\right| \\ &= \frac{1}{6}\left| - t\vec{b} \cdot \vec{a} \times \vec{c} - t\vec{c} \cdot \vec{b} \times \vec{a} - (1-t)\vec{a} \cdot \vec{b} \times \vec{c}\right|.\end{aligned}$$

Since $\vec{b} \cdot \vec{a} \times \vec{c} = -\vec{a} \cdot \vec{b} \times \vec{c}$ and $\vec{c} \cdot \vec{b} \times \vec{a} = -\vec{a} \cdot \vec{b} \times \vec{c}$, we have the volume of the tetrahedron $O''ABC$ is

$$\begin{aligned}\frac{1}{6}\left|\vec{AO''} \cdot (\vec{u} \times \vec{v})\right| &= \frac{1}{6}\left|t\vec{a} \cdot \vec{b} \times \vec{c} + t\vec{a} \cdot \vec{b} \times \vec{c} - (1-t)\vec{a} \cdot \vec{b} \times \vec{c}\right| \\ &= \frac{1}{6}\left|3t-1\right| \left|\vec{a} \cdot \vec{b} \times \vec{c}\right| = \frac{1}{6}\left|3t-1\right|\gamma.\end{aligned}$$

Hence the distance from O'' to the plane ABC is given by

$$3\,\frac{\frac{1}{6}\left|3t-1\right|\gamma}{\frac{1}{2}\|\vec{u} \times \vec{v}\|} = \frac{\left|3t-1\right|\gamma}{\|\vec{u} \times \vec{v}\|}. \qquad \square$$

Proof of Theorem 3. Suppose the tetrahedron $OABC$ is isosceles. Then we have

$$\|\vec{u} \times \vec{v}\| = \|\vec{a} \times \vec{b}\| = \|\vec{c} \times \vec{c}\| = \|\vec{a} \times \vec{c}\|.$$

Let O'' defined by $\vec{OO''} = t(\vec{a} + \vec{b} + \vec{c})$ as in Lemma 1. Suppose O'' is equidistant from the planes OAB, OAC, OBC, and ABC. Then we have

$$\frac{t\gamma}{\|\vec{a} \times \vec{b}\|} = \frac{t\gamma}{\|\vec{a} \times \vec{c}\|} = \frac{t\gamma}{\|\vec{b} \times \vec{c}\|} = \frac{\left|3t-1\right|\gamma}{\|\vec{u} \times \vec{v}\|}.$$

This implies that $t = |3t-1|$. Then $t = \frac{1}{2}$ or $t = \frac{1}{4}$.

Suppose $t = \frac{1}{2}$. Then $\vec{OO''} = \frac{1}{2}(\vec{a} + \vec{b} + \vec{c}) = \vec{OM}$ so that M is an ex-center of the tetrahedron $OABC$.

Suppose $t = \frac{1}{4}$. Then $\vec{OO''} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c}) = \vec{OG}$ so that G is the in-center of the tetrahedron $OABC$.

Therefore (1) implies (2) and (3).

Conversely, suppose M, given by $\vec{OM} = \frac{1}{2}(\vec{a} + \vec{b} + \vec{c})$, is an ex-center of the tetrahedron $OABC$. By Lemma 1, we have

$$\frac{\gamma}{2\|\vec{u} \times \vec{v}\|} = \frac{\gamma}{2\|\vec{a} \times \vec{v}\|} = \frac{\gamma}{2\|\vec{b} \times \vec{c}\|} = \frac{\gamma}{2\|\vec{a} \times \vec{c}\|}$$

so that $\|\vec{u} \times \vec{v}\| = \|\vec{a} \times \vec{b}\| = \|\vec{b} \times \vec{c}\| = \|\vec{a} \times \vec{c}\|$. Similarly, if G is given by $\vec{OG} = \frac{1}{4}(\vec{a} + \vec{b} + \vec{c})$ is the in-center of the tetrahedron $OABC$, then we have

$$\frac{\gamma}{4\|\vec{u} \times \vec{v}\|} = \frac{\gamma}{4\|\vec{a} \times \vec{v}\|} = \frac{\gamma}{4\|\vec{b} \times \vec{c}\|} = \frac{\gamma}{4\|\vec{a} \times \vec{c}\|}$$

by Lemma 1. Again, we have that $\|\vec{u} \times \vec{v}\| = \|\vec{a} \times \vec{b}\| = \|\vec{b} \times \vec{c}\| = \|\vec{a} \times \vec{c}\|$. By Theorem 1, we know that "(3) implies (1)" and "(2) implies (1)".

This proves Theorem 3. $\square$

Corollary 3.1. *Let α be the volume of an isosceles tetrahedron. Let β be the area of a face of the isosceles tetrahedron.*

(1) *An ex-sphere of the isosceles tetrahedron has the radius* $\dfrac{3\alpha}{\beta}$.

(2) *The in-sphere of the tetrahedron $OABC$ has the radius* $\dfrac{3\alpha}{2\beta}$.

Proof. From the proof of the theorem, the ex-radius and inradius of the isosceles tetrahedron $OABC$ are $\dfrac{\gamma}{2\|\vec{a}\times\vec{b}\|} = \dfrac{6\alpha}{2\beta} = \dfrac{3\alpha}{\beta}$ and $\dfrac{\gamma}{4\|\vec{a}\times\vec{b}\|} = \dfrac{6\alpha}{4\beta} = \dfrac{3\alpha}{2\beta}$, respectively. □

Lemma 2. *Recall that O' is the diagonally opposite point of O of the parallelepiped Γ. Let Ω be the parallelepiped having $O'A$, $O'B$, and $O'C$ as edges. Then O is the centroid of Ω.*

Proof. Let T be the centroid of the parallelepiped Ω. Note that $\vec{O'A} = -(\vec{b}+\vec{c})$, $\vec{O'B} = -(\vec{a}+\vec{c})$, $\vec{O'C} = -(\vec{b}+\vec{a})$. Hence, $\vec{OT} = \vec{OO'} + \frac{1}{2}\lfloor \vec{O'A} + \vec{O'B} + \vec{O'C} \rfloor = \vec{OO'} - (\vec{a}+\vec{b}+\vec{c}) = \vec{0}$. So $T = O$, i.e., O is the centroid of Ω. □

Corollary 3.2. *The point O is an ex-center of the tetrahedron $ABCO'$ if, and only if the parallelepiped Γ is a rectangular box. (If Γ is a cube, then the tetrahedron $ABCO'$ is a regular tetrahedron.)*

Proof. Suppose O is an ex-center of the tetrahedron $ABCO'$. Hence, the tetrahedron $ABCO'$ is isosceles by Lemma 2 and Theorem 3. So $O'A = BC$, $O'B = AC$, and $O'C = AB$. If we let A', B', C' be the diagonally opposite points of A, B, C in Γ respectively. Then $O'A = OA'$, $O'B = OB'$, and $O'C = OC'$. So we have $OA' = BC$, $OB' = AC$, and $OC' = AB$. But the quadrilateral $OBA'C$ is a parallelogram, and the diagonals $OA' = BC$ implies that $OBA'C$ is a rectangle. Similarly, $OAC'B$, and $OCB'A$ are rectangles. Therefore, Γ is a rectangular box.

Conversely, suppose Γ is a rectangular box. Then the tetrahedron $ABCO'$ is isosceles. By Lemma 2, O is the centroid of Ω. By Theorem 3, O is an ex-center of the tetrahedron $ABCO'$. □

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