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Addendum to Pohlke's Theorem, a Proof of Pohlke-Schwarz's Theorem

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Abstract. This is a short proof of Pohlke-Schwarz's theorem, based on the results of the previous paper on Pohlke's theorem. Furthermore, explicit formulae for the solution are provided.

Key Words: Pohlke-Schwarz's theorem, oblique axonometry

MSC 2010: 51N10, 51N05

1. Introduction

In 1864 H.A. Schwarz published the proof of the following generalisation of Pohlke's fundamental theorem on oblique axonometry:

> *Three arbitrary straight line segments OP_1, OP_2, OP_3 in a plane, originating from a point O and not contained in a line, can be considered as the parallel projections of three edges OQ_1, OQ_2, OQ_3 of a tetrahedron that is similar to a given tetrahedron.*

Several purely geometric proofs and, in a few instances, analytic proofs were given. See, among the others, the references in [1]. Here, we give a straightforward proof of the above statement together with explicit formulae for the edges OQ_1, OQ_2, OQ_3 of the reference tetrahedron and the direction of the parallel projection onto the image plane.

As we did in [1] for Pohlke's theorem, we reformulate the Pohlke-Schwarz theorem as a result of linear algebra for square matrices. For this reason, and to avoid repetitions, all the matrices that we consider from now on are real 3×3 matrices. If A is such a matrix, we denote with A^i for $1 \leq i \leq 3$ the column vectors of A, and with A_i the row vectors.

1.1. Reformulation of the problem

Throughout the paper we use a fixed coordinate frame for both, the preimages in space as well as for the planar images. More precisely, we introduce a cartesian system of coordinate

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axes x, y, z such that

$$O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad P_i = \begin{pmatrix} x_i \\ y_i \\ 0 \end{pmatrix} \quad \text{for } 1 \le i \le 3 \tag{1}$$

are points of the image plane $\{z = 0\}$. Furthermore, we define the matrix

$$S = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & 0 \end{pmatrix} = \left(S^1,\, S^2,\, S^3\right). \tag{2}$$

In the same way, we represent a given tetrahedron with vertices O, Q_1, Q_2, Q_3 (i.e., with a vertex at the origin O) by an invertible matrix $T = (T^1,\, T^2,\, T^3)$ whose columns T^1, T^2, T^3 are the coordinates of Q_1, Q_2 and Q_3. After this, if the matrices T and $\widetilde{T}$ represent two given tetrahedrons, we say,

Definition 1.1. T and $\widetilde{T}$ are *geometrically similar* if $T = H\widetilde{T}$ with H being a nonzero multiple of an orthogonal matrix, i.e., $HH^t = \rho I$ for some $\rho > 0$.

Pohlke-Schwarz's theorem can now be stated as follows:

Theorem 1.2. *Assume that* $\operatorname{rank}(S) = 2$, *and let* $\widetilde{T}$ *be an invertible matrix. Then, there exists a matrix* T, *geometrically similar to* $\widetilde{T}$,

$$T = \begin{pmatrix} x'_1 & x'_2 & x'_3 \\ y'_1 & y'_2 & y'_3 \\ z'_1 & z'_2 & z'_3 \end{pmatrix} = \left(T^1,\, T^2,\, T^3\right), \tag{3}$$

and a parallel projection Π *onto the plane* $\{z = 0\}$ *such that* $\Pi\big(T^i\big) = S^i$ *for* $1 \le i \le 3$.

2. Proof of Theorem 1.2

Since $\widetilde{T}$ is invertible, we can define the matrix

$$A = S\,\widetilde{T}^{-1} = \begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \left(A^1,\, A^2,\, A^3\right). \tag{4}$$

Noting that $\operatorname{rank}(A) = 2$ and $A_3 = (0,0,0)$, we can apply to the matrix A Pohlke's theorem in the form stated in [1, Theorem 1.1]. This means that *there exist a matrix* B, *with orthogonal columns of equal norm, and a parallel projection* Π *onto the image plane* $\{z = 0\}$ *such that*

$$\Pi(B^i) = A^i, \quad 1 \le i \le 3. \tag{5}$$

Hence, there exists a column vector U, where $U \nparallel \{z = 0\}$, and $\mu_i \in \mathbb{R}$ such that

$$B^i - A^i = \mu_i\, U, \quad 1 \le i \le 3. \tag{6}$$

In other words, introducing the row vector $\mu = (\mu_1, \mu_2, \mu_3)$, we have

$$B - A = U\mu. \tag{7}$$

Then, setting
$$T = B\,\widetilde{T}\,, \tag{8}$$
from (7) we immediately find
$$T - S = (B - A)\,\widetilde{T} = U\,\mu\,\widetilde{T} = \big(\,(\mu\widetilde{T}^1)\,U,\ (\mu\widetilde{T}^2)\,U,\ (\mu\widetilde{T}^3)\,U\,\big). \tag{9}$$
That is, $T^i - S^i = (\mu\widetilde{T}^i)\,U$ for $1 \le i \le 3$.

Thus, we may conclude that the projection Π verifies
$$\Pi(T^i) = S^i, \quad 1 \le i \le 3\,. \tag{10}$$
Besides, since B is a nonzero a multiple of an orthogonal matrix, $T = B\,\widetilde{T}$ is geometrically similar to $\widetilde{T}$. This concludes the proof of Theorem 1.2. □

As a by-product, we can state a simple generalization, for oblique system of coordinate-axes, of the Gauss' fundamental theorem of orthogonal axonometry. More precisely, denoting with $\Pi_{\perp}$ the orthogonal projection onto the image plane $\{z = 0\}$, we have:

Corollary 2.1. *Let* $\widetilde{T}$ *be invertible and let* S *and* A *be the matrices defined in* (2) *and* (4)*. Then, there exists a matrix* T*, geometrically similar to* $\widetilde{T}$*, such that* $\Pi_{\perp}(T^i) = S^i$ *for* $1 \le i \le 3$ *if and only if*
$$\|A_1\| = \|A_2\| \neq 0 \quad \textit{with} \quad A_1 \perp A_2\,. \tag{11}$$

Proof. Taking into account (8), (10), it is enough to apply [1, Proposition 1.2] to the rows of the matrix $A = S\,\widetilde{T}^{-1}$. □

2.1. Reference tetrahedron and direction of projection

Following the steps of the proof of Theorem 1.2, we can now determine the explicit expressions of T, geometrically similar to $\widetilde{T}$, and of Π such that $\Pi\big(T^i\big) = S^i$ for $1 \le i \le 3$.

We begin by setting:
$$A_1 = S_1\,\widetilde{T}^{-1}\,, \quad A_2 = S_2\,\widetilde{T}^{-1}. \tag{12}$$
That is $A_1 = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, $A_2 = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ as in (4). Besides, as in formulae (3.6), (3.10), (3.21), (3.22) of [1], we define the quantities:
$$\gamma = \arccos\left(\frac{A_1 \cdot A_2}{\|A_1\|\,\|A_2\|}\right), \quad \lambda = \frac{\|A_1\|}{\|A_2\|}\,, \tag{13}$$
$$\eta = \frac{\lambda^2 + 1 + \sqrt{(\lambda^2+1)^2 - 4\lambda^2\sin^2\!\gamma}}{2\lambda^2\sin^2\!\gamma}\,, \tag{14}$$
$$\nu \,=\, \pm\,\varrho \quad \text{with} \quad \varrho = \frac{\|A_1\|}{\lambda\sqrt{\eta}} = \frac{\|A_2\|}{\sqrt{\eta}}\,, \tag{15}$$
and, finally,
$$(\alpha, \beta) = \pm\Big(\sqrt{\eta\,\lambda^2 - 1}\ ,\ \operatorname{sgn}(\cos\gamma)\sqrt{\eta - 1}\ \Big) \tag{16}$$
with the "signum" function:
$$\operatorname{sgn}(t) := \begin{cases} 1 & \text{if} \quad t \ge 0\,, \\ -1 & \text{if} \quad t < 0\,. \end{cases} \tag{17}$$

Then, by the results of [1, Chapter 4], the matrix B and the direction U of the projection Π satisfying (5) are given by the relations:

$$B = \frac{1}{1+\alpha^2+\beta^2}\begin{pmatrix} 1+\beta^2 & -\alpha\beta & -\alpha \\ -\alpha\beta & 1+\alpha^2 & -\beta \\ \alpha & \beta & 1 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 \\ \frac{\tilde{x}_2\tilde{y}_3-\tilde{y}_2\tilde{x}_3}{\nu} & \frac{\tilde{y}_1\tilde{x}_3-\tilde{x}_1\tilde{y}_3}{\nu} & \frac{\tilde{x}_1\tilde{y}_2-\tilde{y}_1\tilde{x}_2}{\nu} \end{pmatrix}, \tag{18}$$

$$U = \begin{pmatrix} -\alpha \\ -\beta \\ 1 \end{pmatrix}. \tag{19}$$

Summarizing up eqs. (12)–(19) and taking into account (8) and (10), we conclude:

Corollary 2.2. *Under the conditions of Theorem 1.2, the matrix T and the parallel projection Π are given by*

$$T = B\widetilde{T} \quad \textit{and} \quad \Pi\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+\alpha z \\ y+\beta z \\ 0 \end{pmatrix}. \tag{20}$$

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References

[1] R. Manfrin: *A Proof of Pohlke's Theorem with an Analytic Determination of the Reference Trihedron.* J. Geometry Graphics **22**, 195–205 (2018).

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