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Common Tangents to Ellipse and Circles, the 13-Point Circle and Other Theorems

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Abstract. New developments of the author's research project [3, 4, 5, 6] on the geometry of conics are presented. Special attention is paid to the relationships among the ellipse and three circles — denoted by Φ_1, Φ_2 and Φ_3 — previously introduced by the author [6] and belonging to the elliptic and hyperbolic pencil of circles defined by the ellipse foci. Among the newly defined points, arising as intersections of the geometrical objects under examination, eight triplets of collinear points (Theorems 10 and 13) and as many quadruplets of concyclic points (Theorems 11 and 14; Figures 4 and 5) are recognized. Eight new special points are shown (Theorem 17) to be concyclic on the well known circle through P and the ellipse foci.

Key Words: ellipse, harmonic range, collinear points, concyclic points, symbiotic ellipse, Monge's circle, circle inversion

MSC 2010: 51M04, 51N20

1. Introduction

In an orthogonal reference frame (Figure 1), let H be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b, \tag{1.1}$$

whose foci are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c = \sqrt{a^2 - b^2}$. Throughout this paper, the ellipse general point — that is, any point different from the vertices — is denoted by $P(a\cos\varepsilon;\ b\sin\varepsilon)$ or simply P. Nevertheless, to avoid the exceeding verbal complexity, I will formulate any statement assuming that P lies in the first quadrant ($x > 0,\ y > 0$). For the reader's convenience, some geometrical objects frequently referred to throughout this paper are listed and previous results are summarized.

1. The ellipse diameters with slope $m_e = \tan\varepsilon$ and $m_{e'} = -\tan\varepsilon$, which have been given by the author [3] the names *eccentric line* (1.2) and *symm-eccentric line* e' (1.3), respectively,

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$$y = x\tan\varepsilon, \qquad (1.2) \qquad\qquad y = -x\tan\varepsilon. \qquad (1.3)$$

2. The tangent t (1.4) to the ellipse H at P and its respective x- and y-intercepts T_x and T_y,

$$y = -x\frac{b}{a}\cot\varepsilon + \frac{b}{\sin\varepsilon}, \ (1.4) \qquad T_x\left(\frac{a}{\cos\varepsilon}; 0\right), \qquad (1.5) \qquad T_y\left(0; \frac{b}{\sin\varepsilon}\right). \qquad (1.6)$$

3. The normal n (1.7) to the ellipse H at P and its respective x- and y-intercepts N_x and N_y,

$$y = x\frac{a}{b}\tan\varepsilon - \frac{c^2}{b}\sin\varepsilon, \ (1.7) \qquad N_x\left(\frac{c^2}{a}\cos\varepsilon; 0\right), \qquad (1.8) \qquad N_y\left(0; -\frac{c^2}{b}\sin\varepsilon\right). \qquad (1.9)$$

4. The following points E (1.10) and I (1.11), where the normal (1.7) meets the eccentric (1.2) and the symm-eccentric (1.3) line of P, respectively,

$$E\left((a+b)\cos\varepsilon; (a+b)\sin\varepsilon\right), \qquad (1.10) \qquad I\left((a-b)\cos\varepsilon; -(a-b)\sin\varepsilon\right). \qquad (1.11)$$

This paper deals with relationships among the ellipse H (1.1) and the following circles (Fig. 1), previously introduced by the author [6]:

- the circle Φ_1, whose center is the y-intercept T_y (1.6) of the tangent (1.4); it passes through the foci and the points E (1.10) and I (1.11) ([6, Theorem 2.2]),

$$x^2 + \left(y - \frac{b}{\sin\varepsilon}\right)^2 = c^2 + \frac{b^2}{\sin^2\varepsilon}; \qquad (1.12)$$

- the circle Φ_2, whose center is the y-intercept N_y (1.9) of the normal (1.7); it passes through the foci,

$$x^2 + \left(y + \frac{c^2}{b}\sin\varepsilon\right)^2 = c^2 + \left(\frac{c^2}{b}\sin\varepsilon\right)^2; \qquad (1.13)$$

- the circle Φ_3, whose center is the x-intercept T_x (1.5) of the tangent (1.4) to H at P; it passes through the points E (1.10) and I (1.11),

$$\left(x - \frac{a}{\cos\varepsilon}\right)^2 + y^2 = \frac{a^2\sin^2\varepsilon + b^2\cos^2\varepsilon}{\cos^2\varepsilon}. \qquad (1.14)$$

The circles Φ_1, Φ_2 and Φ_3 taken pairwise are mutually orthogonal ([6, Theorem 2.1]), and the point N_x (1.8) is their radical center.

The circles Φ_2 and Φ_3 share the points (the notation $(acbs) = a^2\cos^2\varepsilon + b^2\sin^2\varepsilon$ is used)

$$\Psi_1\left(\frac{(a - c\sin\varepsilon)\,c^2\cos\varepsilon}{(acbs)}; \frac{(a - c\sin\varepsilon)\,bc}{(acbs)}\right); \qquad (1.15)$$

$$\Psi_2\left(\frac{(a + c\sin\varepsilon)\,c^2\cos\varepsilon}{(acbs)}; \frac{-(a + c\sin\varepsilon)\,bc}{(acbs)}\right) \qquad (1.16)$$

In the present paper, the vertices of the ellipse H (1.1) are denoted by $V_1(a,0)$, $V_2(0,b)$, $V_3(-a,0)$, and $V_4(0,-b)$, or simply $V_1,\dots,V_4$. The points where the circle $\Phi_i (i=1,3)$ meets the tangent drawn at the ellipse vertex V_j, $j=1,4$, are denoted by $T_{ij\lambda}$, $\lambda=1,2$.

2. Results

The tangent t (1.4) (Figure 1) drawn to the ellipse H (1.1) at P meets:

(i) the tangents drawn to the ellipse H (1.1) at its vertices $V_3(-a,0)$ and $V_1(a,0)$ in the following points T_{131} and T_{112}, respectively:

$$T_{131}\left(-a;\ \frac{b(1+\cos\varepsilon)}{\sin\varepsilon}\right), \qquad (2.1) \qquad\qquad T_{112}\left(a;\ \frac{b(1-\cos\varepsilon)}{\sin\varepsilon}\right); \qquad (2.2)$$

(ii) the tangents drawn to the ellipse H (1.1) at its vertices $V_2(0,b)$ and $V_4(0,-b)$ in the following points T_{321} and T_{342}:

$$T_{321}\left(\frac{a(1-\sin\varepsilon)}{\cos\varepsilon};\ b\right), \qquad (2.3) \qquad\qquad T_{342}\left(\frac{a(1+\sin\varepsilon)}{\cos\varepsilon};\ -b\right). \qquad (2.4)$$

Replacing the coordinates of the points T_{131} and T_{112} [T_{321} and T_{342}] in the equation (1.12) representing the circle Φ_1 [in the equation (1.14) representing the circle Φ_3], one can see that such equations are fulfilled. Accordingly, we may state the following:

Theorem 1. [Figure 1] *The tangent* (1.4) *drawn to the ellipse* H (1.1) *at* P *meets*

- *the tangents drawn to the ellipse at its vertices* $V_1(a,0)$ *and* $V_3(-a,0)$ *in points belonging to the circle* Φ_1 (1.12) [such points are denoted by T_{112} (2.2) and T_{131} (2.1), respectively]*;*
- *the tangents drawn to the ellipse at its vertices* $V_2(0,b)$ *and* $V_4(0,-b)$ *in points belonging to the circle* Φ_3 (1.12) [such points are denoted by T_{321} (2.3) and T_{342} (2.4), respectively].

The present author has shown in [6, Theorem 2.2], that the points E (1.10) and I (1.11) — where the normal (1.7) to H at P meets the eccentric (1.2) and the symm-eccentric line (1.2) of P, respectively — belong to the circle Φ_1 (1.12). Accordingly, we may state:

Theorem 2. [Figure 1] *The ellipse foci, the points* E (1.10) *and* I (1.11) *and the points* T_{112} (2.2) *and* T_{131} (2.1) *are concyclic about the y-intercept* T_y (1.6) *of the tangent to the ellipse* H *at* P *on the circle* Φ_1 (1.12)*.*

Also the circle Φ_3 (1.12) passes through E (1.10) and I(1.11). Accordingly, we may state:

Theorem 3. [Figure 1]. *The points* E (1.10) *and* I (1.11)*,* T_{321} (2.3) *and* T_{342} (2.4) *are concyclic about the x-intercept* T_x (1.5) *of the tangent to the ellipse* H *at* P *on the circle* Φ_3 (1.14)*.*

Of course, there exist points symmetrical to the afore mentioned ones about the ellipse symmetry axes, I will not mention for the sake of brevity.

Remembering that any couple of points defines an *elliptic* and a *hyperbolic pencil of circles* ([2], Chapter 7), we may regard the circles Φ_1 and Φ_2 as elements of the elliptic pencil of circles defined by the ellipse foci. The general element of this set, denoted here by Φ_E, is represented by the following equation,

$$x^2+(y-y_o)^2=y_o^2+c^2, \qquad (2.5)$$

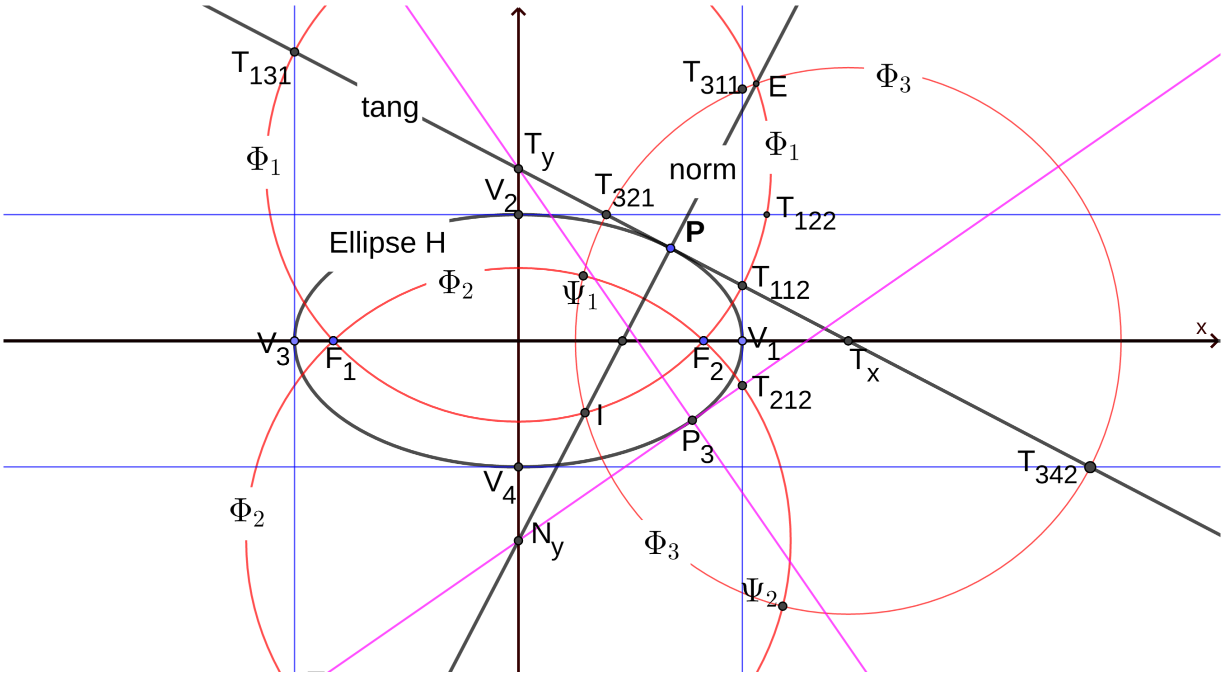

Figure 1: Illustrating the definition of circles Φ_1, Φ_2 and Φ_3 and the Theorems 2.1, 2.2 and 2.3. The points F_1 and F_2 are the foci of the ellipse H; the ellipse vertices are denoted by V_1 through V_4; the tangents to H at such vertices are drawn [blue]. The tangent to H at P meets the tangent at V_3, the y-axis x, the tangent at V_2, the tangent at V_1, the x-axis and the tangent at V_4 in the points T_{131}, T_y, T_{321}, T_{112}, T_x and T_{342}, respectively. Two circles [red] through the foci, denoted by Φ_1 and Φ_2, are constructed about T_y and N_y, respectively. The circle Φ_3 [red] is constructed about T_x; the circles Φ_1 and Φ_3 and the normal share the points E and I; the circles Φ_2 and Φ_3 share the points Ψ_1 and Ψ_2. The circle Φ_2 meets the tangent to H at V_1 in T_{212}; the Φ_2 diameter through T_{212} [magenta] touches the ellipse (Corollary 1) at P_3; the normal to H at P_3 [magenta] meets the minor axis at T_y (Theorem 8).

where the parameter y_o is the y-coordinate of the center. The circle Φ_3 belongs to the hyperbolic pencil of circles defined by the ellipse foci. The general element of this set, denoted by Φ_H, is represented by the following equation:

$$(x - x_o)^2 + y^2 = x_o^2 - c^2, \tag{2.6}$$

where the parameter x_o is the x-coordinate of the center. Any circle of the hyperbolic pencil passes through the ellipse imaginary foci $F_1^i\,(0, ic)$, $F_2^i\,(0, -ic)$. Any circle of either pencil is orthogonal to each circle of the other pencil.

Theorem 1 implies the following:

Corollary 1. [Figure 1] *Let Φ_E (2.5) be a circle through the ellipse foci which meets the line $x = a$ in real points. The Φ_E diameters through such points are tangent to the ellipse H.*

Let the tangents drawn to the ellipse H at its minor axis vertices (namely, the lines $y = b$ and $y = -b$) meet the circle Φ_1 (1.12) at the following points T_{122} and T_{142} (Figure 2):

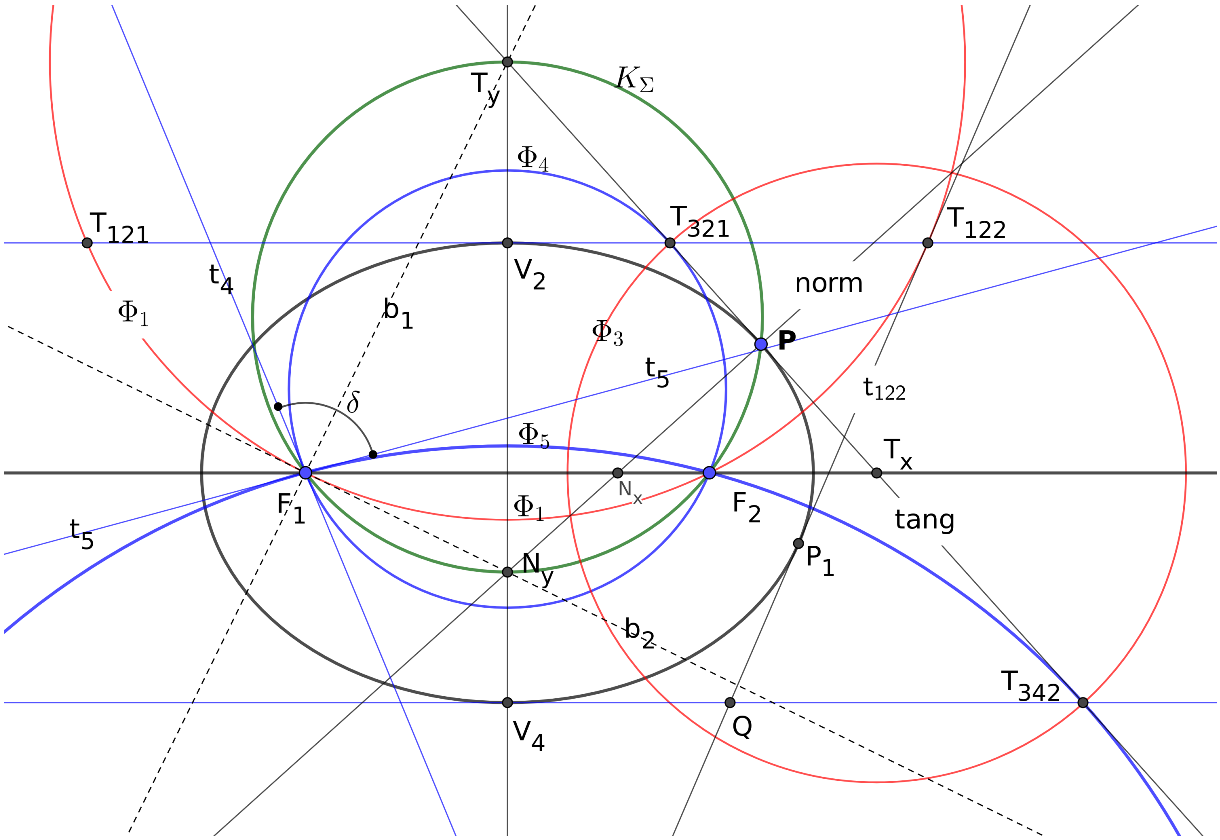

Figure 2: The circle Φ_1 [red] meets the line $y = b$ [blue] in T_{122}. The tangent t_{122} to the circle Φ_1 at T_{122} touches the ellipse (Lemma 1) at P_1. The tangent t to the ellipse at P meets the lines $y = b$ and $y = -b$ [blue] in the points T_{321} and T_{342}. The circles Φ_4 and Φ_5 [blue] symmetrically lie about the minor axis and touch the tangent t at T_{321} and T_{342}, respectively; they pass through the foci (Lemma 2). The tangents t_4 and t_5 [blue] to the circles Φ_4 and Φ_5 at the focus F_1 form the angle δ [Theorem 5, item (i)]. The bisectors b_1 and b_2 [black, dashed lines] of the angle δ and its supplementary meet the minor axis in T_y and N_y. The circle K_Σ [green] passes through T_y, N_y, P and the foci [Theorem 5, item (iii)].

$$T_{122}\left(\sqrt{a^2 + 2b^2\frac{1-\sin\varepsilon}{\sin\varepsilon}};\ b\right), \qquad (2.7) \qquad\qquad T_{142}\left(\sqrt{a^2 - 2b^2\frac{1+\sin\varepsilon}{\sin\varepsilon}};\ -b\right), \qquad (2.8)$$

respectively. The tangent t_{122} to the circle Φ_1 at T_{122} (2.7) has the equation

$$y = x\,\frac{\sqrt{(c^2-b^2)\sin^2\varepsilon + 2b^2\sin\varepsilon}}{b(1-\sin\varepsilon)} - \frac{c^2\sin\varepsilon + b^2}{b(1-\sin\varepsilon)}. \qquad (2.9)$$

Simultaneously solving the equations of the tangent t_{122} (2.9) to the circle Φ_1 at T_{122} and the ellipse H (1.1), we get the following, single unknown equation (the notation $(asbc) = a^2\sin^2\varepsilon + b^2\cos^2\varepsilon$ is used):

$$x^2(asbc) - 2xa^2(asbc)\sqrt{(c^2-b^2)\sin^2\varepsilon + 2b^2\sin\varepsilon} + a^4\left[\left(c^2-b^2\right)\sin^2\varepsilon + 2b^2\sin\varepsilon\right] = 0. \quad (2.10)$$

The vanishing of the discriminant of (2.10) shows that the tangent t_{122} (2.9) to the circle Φ_1 at T_{122} touches also the ellipse H. Moreover, observing that the circle Φ_1 (1.12) shares such property with the general circle through the foci, we may state the following:

Lemma 1. [Figure 2] *Let Φ_E (2.5) be a circle belonging to the elliptic pencil of circles defined by the ellipse foci, which meets the line $y = b$ in real points. The tangents drawn to Φ_E at such points are tangent to the ellipse, too.*

Now, let us consider a circle, denoted by Φ_y and represented by the following equation,

$$x^2 + (y - y_o)^2 = y_o^2 + x_o^2 \tag{2.11}$$

The center of the circle Φ_y (2.11) is a point belonging to the minor axis of the ellipse H (1.1), $|y_o|$ apart from the major axis. The circle meets the major axis in points $|x_o|$ apart from the minor axis. Let the line $y = b$ meet the circle Φ_y in the real point $T'_{122}\left(\sqrt{x_o^2 + 2by_o - b^2},\ b\right)$. The tangent drawn to Φ_y at such point is

$$y = x\frac{\sqrt{x_o^2 + 2by_o - b^2}}{y_o - b} - \frac{y_o b + x_o^2}{y_o - b}. \tag{2.12}$$

Let us assume that the line (2.12) is tangent to the ellipse (1.1), too. This hypothesis amounts to state that the following equation

$$\frac{x^2}{a^2} + \frac{1}{b^2}\left(x\frac{\sqrt{x_o^2 + 2by_o - b^2}}{y_o - b} - \frac{y_o b + x_o^2}{y_o - b}\right)^2 = 1, \tag{2.13}$$

written by replacing the r.h.s. of (2.12) in the ellipse equation (1.1), has a vanishing discriminant. Setting the discriminant of (2.13) to zero amounts, in turn, to write the following equation for the unknown x_o^2 (the conventional notation $R = x_o^2 + 2by_o - b^2$ is used):

$$a^4\left(y_o b + x_o^2\right)^2 R - \left[b^2(y_o - b)^2 + a^2 R\right] \times a^2\left[(y_o b + x_o^2)^2 - b^2\left(y_o - b\right)^2\right] = 0,$$

which may be written, by means of trivial manipulations, as follows:

$$x_o^4 + 2y_o b x_o^2 - a^2 x_o^2 - b^4 + 2b^3 y_o - 2a^2 b y_o + a^2 b^2 = 0$$

The unique acceptable solution is $x_o^2 = c^2$. This result means that the circle Φ_y passes through the foci. Accordingly, we may state:

Lemma 2. [Figure 2] *Let Φ_y (2.11) be a circle about a point lying on the ellipse minor axis, which meets the line $y = b$ in real points. If the tangents drawn to the circle at such points touch also the ellipse, then the circle Φ_y passes through the ellipse foci.*

The Lemmas 1 and 2 amount to the following:

Theorem 4. [Figure 2] *Let Φ_y (2.11) be a circle about a point lying on the ellipse minor axis, which meets the line $y = b$ in real points. A necessary and sufficient condition for the tangents to the circle at such points to touch also the ellipse is the membership of the circle in the elliptic pencil of circles defined by the ellipse foci.*

An obvious consequence of Theorem 4 is the following:

Corollary 2. [Figure 2] *Let the tangent t (1.4) to the ellipse H at P meet the lines $y = b$ and $y = -b$ in the real points T_{321} (2.3) and T_{342} (2.4), respectively. Let Φ_4 and Φ_5 be two circles whose centers lie on the minor axis, which touch the line t in the points T_{321} and T_{342}, respectively. Under such hypothesis, the circles Φ_4 and Φ_5 pass through the ellipse foci.*

The following equations (2.14) and (2.15) represent the circles Φ_4 and Φ_5, respectively (the notation $(asbc) = a^2\sin^2\varepsilon + b^2\cos^2\varepsilon$ is used):

$$x^2 + \left(y - \frac{(asbc) - a^2\sin\varepsilon}{b\cos^2\varepsilon}\right)^2 = \left(\frac{a(1-\sin\varepsilon)}{\cos\varepsilon}\right)^2 + \left(b - \frac{(asbc) - a^2\sin\varepsilon}{b\cos^2\varepsilon}\right)^2, \quad (2.14)$$

$$x^2 + \left(y + \frac{(asbc) + a^2\sin\varepsilon}{b\cos^2\varepsilon}\right)^2 = \left(\frac{a(1+\sin\varepsilon)}{\cos\varepsilon}\right)^2 + \left(-b + \frac{(asbc) + a^2\sin\varepsilon}{b\cos^2\varepsilon}\right)^2. \quad (2.15)$$

The tangents to the circles Φ_4 (2.14) and Φ_5 (2.15) at the focus $F_1(-c, 0)$ have the following slopes m_4 (2.16) and m_5 (2.17), respectively:

$$m_4\colon \ \frac{-bc\cos^2\varepsilon}{(asbc) - a^2\sin\varepsilon}, \qquad (2.16) \qquad\qquad m_5\colon \ \frac{bc\cos^2\varepsilon}{(asbc) + a^2\sin\varepsilon}. \qquad (2.17)$$

The tangents to the circles Φ_4 and Φ_5 at the focus F_1 form an angle δ,

$$\delta = \arctan\frac{m_4 - m_5}{1 + m_4 m_5} = \arctan\frac{-\dfrac{bc\cos^2\varepsilon}{(asbc) - a^2\sin\varepsilon} - \dfrac{bc\cos^2\varepsilon}{(asbc) + a^2\sin\varepsilon}}{1 - \dfrac{b^2c^2\cos^4\varepsilon}{(asbc)^2 - a^4\sin^2\varepsilon}} = \arctan\frac{2bc}{c^2 - b^2}.$$

This unexpected result means that — in spite of the dependence of any detail on the eccentric anomaly ε of P (namely, on the point P location on the ellipse) — *the measure of the angle δ does not depend on such variable but on the ellipse semiaxes only.*

A further, unexpected result arises from the bisectors of the angle δ and its supplementary. Indeed, observing that $\tan\frac{\delta}{2} = \frac{b}{c}$, the slopes of such bisectors — here denoted by m_{b_1} and m_{b_2} — can be written as follows:

$$m_{b_1} = \frac{m_5 + \tan\frac{\delta_b}{c}}{1 - m_5\tan\frac{b}{c}} = \frac{bc^2\cos^2\varepsilon + a^2b\sin^2\varepsilon + b^3\cos^2\varepsilon + a^2b\sin\varepsilon}{a^2c\sin^2\varepsilon + b^2c\cos^2\varepsilon + a^2c\sin\varepsilon - b^2c\cos^2\varepsilon} = \frac{b}{c\sin\varepsilon},$$

$$m_{b_2} = -\frac{1}{m_{b_1}} = -\frac{c\sin\varepsilon}{b}.$$

These slopes m_{b_1} and m_{b_2} equal the slopes of the tangents drawn to the circles Φ_2 and Φ_1 at the focus F_1, respectively. Because of the orthogonality of the circles Φ_1 and Φ_2, this finding means that the bisectors b_1 and b_2 of the angle δ and of its supplementary pass through the centers — T_y (1.6) and N_y (1.9) — of the circles Φ_1 and Φ_2, respectively. This result carries with itself a further consequence: *if a circle is constructed on the segment T_yN_y as diameter, such circle passes through the foci.*

We may summarize these results as follows:

Theorem 5. [Figure 2] *Drawn the tangent t to the ellipse H at P, let Φ_4 (2.14) and Φ_5 (2.15) be two circles whose centers lie on the minor axis, which touch the line t at the points T_{321} (2.3) and T_{342} (2.4), where such line meets the lines $y = b$ and $y = -b$, respectively. Let these circles* [which pass through the foci by virtue of the Corollary 2] *admit as tangent at the foci two lines denoted by t_4 and t_5. Then the following holds:*

(i) the lines t_4 and t_5 meet at constant angle $\delta = \arctan \dfrac{2bc}{c^2 - b^2}$*;*

(ii) the bisectors of the angles formed by the lines t_4 and t_5 are tangent to the circles Φ_1 (1.12) *and Φ_2* (1.13)*, respectively;*

(iii) the point P, the y-intercepts T_y and N_y of the tangent and normal to the ellipse at P and the foci are concyclic on the circle constructed on the segment T_yN_y as diameter.

The following equation represents the circle — here denoted by K_Σ — whose existence is stated by the last proposition of Theorem 5:

$$x^2 + \left(y - \frac{b^2 - c^2 \sin^2 \varepsilon}{2b \sin \varepsilon}\right)^2 = \left(\frac{b^2 + c^2 \sin^2 \varepsilon}{2b \sin \varepsilon}\right)^2. \tag{2.18}$$

The circle (2.18) is well known since a long time. The new proof I have given here of its existence adds to many others registered in the literature (e.g., see [7, p. 86], [1, Chapter VI, p. 215], and [6, subsect. 3.2, item 2]).

Remembering that the point T_{112} is common to (i) the circle Φ_1, (ii) the tangent $x = a$ drawn to the ellipse at $V_1(a, 0)$, and (iii) the tangent t (1.4) to the ellipse at P (Theorem 1), let us assume that a circle denoted Φ_x, symmetrically lying about the focal axis, passes through the point T_{112}. Remembering that the line t (1.4) is a diameter of Φ_1, it is obvious that, if such line t touches also the circle Φ_x at T_{112}, then the circle Φ_x is orthogonal to Φ_1. Since Φ_1 belongs to the elliptic pencil of circles defined by the ellipse foci, we conclude that the circle Φ_x belongs to the conjugated hyperbolic pencil of circles. On the other hand, if Φ_x orthogonally cuts the circle Φ_1 at the point T_{112} — namely: if Φ_x belongs to the hyperbolic pencil of circles defined by the ellipse foci — then the line t is tangent to Φ_x, too. Accordingly, we may state:

Theorem 6. [Figure 3] *Let Φ_x be a circle about a point lying on the ellipse focal axis, which meets the line $x = a$ in real points. A necessary and sufficient condition for the tangents to the circle Φ_x at such points to touch also the ellipse is the membership of the circle in the hyperbolic pencil of circles defined by the ellipse foci.*

Of course, what has been said for the point T_{112} holds also for the point T_{131}, where the tangent t (1.4) drawn to the ellipse at P meets the line $x = -a$. Therefore, we may construct two circles — let them be denoted by Φ_4' and Φ_5' — such that their centers lie on the focal axis and they touch the tangent t in the points T_{112} and T_{131}, respectively. The following statement is an obvious consequence of the Theorem 6:

Corollary 3. [Figure 3] *Let the tangent t* (1.4) *to the ellipse H at P meet the lines $x = a$ and $x = -a$ in the real points T_{112}* (2.2) *and T_{131}* (2.1)*, respectively. Let Φ_4' and Φ_5' be two circles whose centers lie on the major axis, which touch the line t in the points T_{112} and T_{131}, respectively.*
Under such hypothesis, the circles Φ_4' and Φ_5' belong to the hyperbolic pencil of circles defined by the ellipse foci.

A special circle belonging to the hyperbolic pencil defined by the foci is Φ_3 (1.14) (Figure 3). The line $x = a$ meets the circle Φ_3 (1.14) in two points; one of them is the following T_{311},

$$T_{311}\left(a,\ \sqrt{2\frac{a^2}{\cos \varepsilon} - a^2 - c^2}\right) \tag{2.19}$$

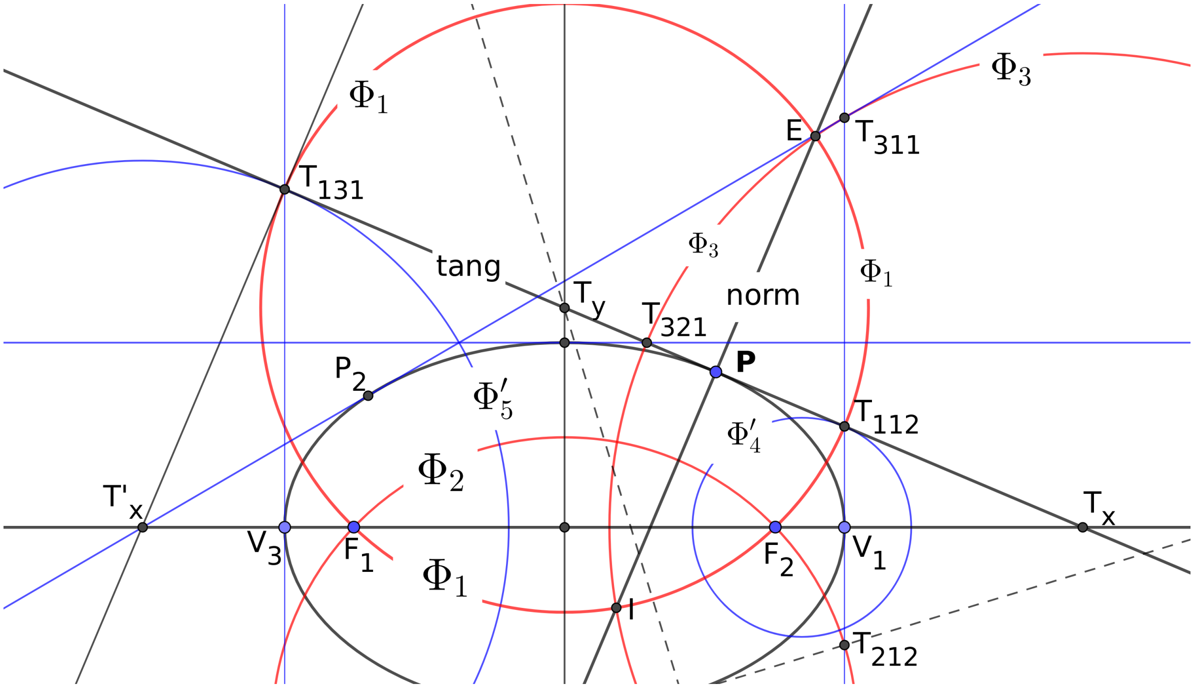

Figure 3: The circle Φ_3 [red] meets the line $x = a$ [blue] in T_{311}; the tangent to Φ_3 at T_{311} [blue] touches the ellipse (Theorem 6) at P_2. The tangent t to the ellipse at P meets the lines $x = a$ and $x = -a$ in the points T_{112} and T_{131}; the circles Φ_4' and Φ_5' [blue] touch the tangent t in T_{112} and T_{131}.

The following line t_{311} (2.20),

$$y = x\,\frac{a(1-\cos\varepsilon)}{\sqrt{2a^2\cos\varepsilon-(a^2+c^2)\cos^2\varepsilon}} + \frac{a^2-c^2\cos\varepsilon}{\sqrt{2a^2\cos\varepsilon-(a^2+c^2)\cos^2\varepsilon}}, \tag{2.20}$$

is tangent to the circle Φ_3 at T_{311} (2.19) and to the ellipse H at the following point P_2,

$$P_2\left(-\frac{a^3\,(1-\cos\varepsilon)}{a^2-c^2\cos\varepsilon};\ \frac{b^2\sqrt{2a^2\cos\varepsilon-(a^2+c^2)\cos^2\varepsilon}}{a^2-c^2\cos\varepsilon}\right). \tag{2.21}$$

We have recognized some special points on the tangent to the ellipse at P, which can be gathered into special quadruplets, as the following Theorem states:

Theorem 7. [Figure 1] *The following quadruplets of points form harmonic ranges.*

(i) T_{131} (2.1), T_{321} (2.3), T_{112} (2.2) and T_{342} (2.4),

(ii) T_y (1.6), T_{321} (2.3), P and T_{342} (2.4),

(iii) T_{131} (2.1), P, T_{112} (2.2) and T_x (1.5)

The orthogonality of the circles Φ_1 and Φ_3 accounts for the harmonic range formed by the first quadruplet. As regards the second and third quadruplet, replacing the points by their x- and y-coordinates, respectively, we may write down the following quadruplets:

$$\left(\frac{b}{\sin\varepsilon},\ b,\ b\sin\varepsilon,\ -b\right) \quad \text{and} \quad \left(-a,\ a\cos\varepsilon,\ a,\ \frac{a}{\cos\varepsilon}\right),$$

and check that, for both of them, the product of the 1st and 3rd term equals — apart from the sign — the product of the 2nd and 4th term.

In the paper [6], I have introduced the circle Φ_2 (1.13). It is constructed about the y-intercept N_y (1.9) of the normal to H at P, passes through the foci and forms, together with Φ_1 (1.12) and Φ_3 (1.14), a triplet of orthogonal circles ([6, Theorem 2.1]). The line $x = a$ meets the circle Φ_2 in the following points T_{211} and T_{212} (Figure 1):

$$T_{211}\left(a;\ \frac{-c^2\sin\varepsilon-\sqrt{c^4\sin^2\varepsilon-b^4}}{b}\right), \quad (2.22) \qquad T_{212}\left(a;\ \frac{-c^2\sin\varepsilon+\sqrt{c^4\sin^2\varepsilon-b^4}}{b}\right). \quad (2.23)$$

Let us restrict ourselves to consider the point T_{212} (2.23). Corollary 1 enables us to state that the following Φ_2 diameter through T_{212},

$$y = -\frac{c^2\sin\varepsilon}{b} + x\,\frac{\sqrt{c^4\sin^2\varepsilon-b^4}}{ab}, \tag{2.24}$$

is tangent to the ellipse H. Indeed, the tangency point is the following P_3:

$$P_3\left(\frac{a\sqrt{c^4\sin^2\varepsilon-b^4}}{c^2\sin\varepsilon};\ -\frac{b^3}{c^2\sin\varepsilon}\right). \tag{2.25}$$

The circle Φ_2 plays, w.r.t. the point P_3, the same role the circle Φ_1 plays w.r.t. the point P. Accordingly, as the normal to H at P is, by definition, a diameter of Φ_2, the normal to H at P_3 is a diameter of Φ_1 (Figure 1). Such normal passes, therefore, through the center T_y (1.6) of Φ_1, as the following Theorem states:

Theorem 8. [Figure 1] *The normal to the ellipse H at the point P_3* (2.25) *concurs with the tangent* (1.4) *to H at P in the minor axis point $T_y(0,\, b/\sin\varepsilon)$.*

If the procedure which has lead us from P to P_3 is followed starting from P_3, the circles Φ_1 and Φ_2 exchange their role with each other, as well as the points T_{112} and T_{212}, the points T_y and N_y and, accordingly, the points P and P_3, too. This accounts for the following

Theorem 9. *The correspondence associating the points P and P_3* (2.25) *is involutory.*

Observing that the y-intercepts of the normal and tangent to H at P_3 (2.25) coincide with the y-intercepts T_y (1.6) and N_y (1.9) of the tangent and normal to H at P, respectively, we conclude that a circle constructed on the segment T_yN_y as diameter passes through P and P_3. But we know (Theorem 5)that the segment T_yN_y is a diameter of the circle K_Σ (2.18), which passes through the point P and the foci. Moreover, also the following point C_Ψ (2.26), representing the midpoint between the points Ψ_1 (1.15) and Ψ_2 (1.16),

$$C_\Psi\left(\frac{ac^2\cos\varepsilon}{a^2\cos^2\varepsilon+b^2\sin^2\varepsilon};\ -\frac{bc^2\sin\varepsilon}{a^2\cos^2\varepsilon+b^2\sin^2\varepsilon}\right), \tag{2.26}$$

belongs to the circle K_Σ (2.18). Indeed, replacing the C_Ψ (2.26) coordinates in the l.h.s. of (2.18) and doing some simplifications, we get

$$a^2c^2\cos^2\varepsilon + b^2c^2\sin^2\varepsilon + \left(a^2\cos^2\varepsilon+b^2\sin^2\varepsilon\right)\left(b^2-c^2\sin^2\varepsilon\right) - \left(a^2\cos^2\varepsilon+b^2\sin^2\varepsilon\right)^2.$$

Few, trivial manipulations allow us to conclude that this expression vanishes. Accordingly, we may state the following

Lemma 3. *The points P_3 (2.25) and C_Ψ (2.26) are concyclic with the point P, the foci and the y-intercepts of the tangent [T_y (1.6)] and normal [N_y (1.9)] to the ellipse H at P in the circle K_Σ (2.18).*

3. The normal to the ellipse at the point P

The normal (1.7) to H at P meets the circle Φ_2 (1.13) (Figure 4) in points denoted by N_{22} and N_{21} (the farther and closer to P, respectively). Such points are

$$N_{22}\left(-c\cos\varepsilon;\ -\frac{c(a+c)\sin\varepsilon}{b}\right), \qquad (3.1) \qquad\qquad N_{21}\left(c\cos\varepsilon;\ \frac{c(a-c)\sin\varepsilon}{b}\right). \qquad (3.2)$$

Six lines link the points N_{21} (3.2), N_{22} (3.1), T_{131} (2.1) and T_{112} (2.2), two of them being the normal and the tangent to H at P. The remaining four lines are involved in relationships of collinearity and concyclicity with several other points.

Theorem 10. [Figure 4] *The focus $F_1(-c,0)$ is collinear*

1. *with the points T_{131} (2.1) and N_{22} (3.1) on the line*

$$T_{131}F_1N_{22}:\ y=-\frac{b(1+\cos\varepsilon)}{(a-c)\sin\varepsilon}(x+c), \tag{3.3}$$

2. *with the points T_{112} (2.2) and N_{21} (3.2) on the line*

$$T_{112}N_{21}F_1:\ y=\frac{(a-c)\sin\varepsilon}{b(1+\cos\varepsilon)}(x+c). \tag{3.4}$$

The lines $T_{131}F_1N_{22}$ (3.3) and $T_{112}N_{21}F_1$ (3.4) meet orthogonally at F_1.

The focus $F_2(c,0)$ is collinear

1. *with the points T_{131} (2.1) and N_{21} (3.2) on the line*

$$T_{131}N_{21}F_2:\ y=-\frac{b(1+\cos\varepsilon)}{(a+c)\sin\varepsilon}(x-c), \tag{3.5}$$

2. *with the points T_{112} (2.2) and N_{22} (3.1) on the line*

$$T_{112}F_2N_{22}:\ y=\frac{b(1-\cos\varepsilon)}{(a-c)\sin\varepsilon}(x-c). \tag{3.6}$$

The lines $T_{131}N_{21}F_2$ (3.5) and $T_{112}F_2N_{22}$ (3.6) meet orthogonally at F_2.

The collinearity of the mentioned triplets of points is demonstrated by checking that, for each of them, the points coordinates fulfill the equation of the corresponding line. A glance at the slope of the lines $T_{131}F_1N_{22}$ (3.3) and $T_{112}N_{21}F_1$ (3.4) reveals that such lines are orthogonal, as well as the lines $T_{131}N_{21}F_2$ (3.5) and $T_{112}F_2N_{22}$ (3.6) are.

In the light of these results, the following holds, too:

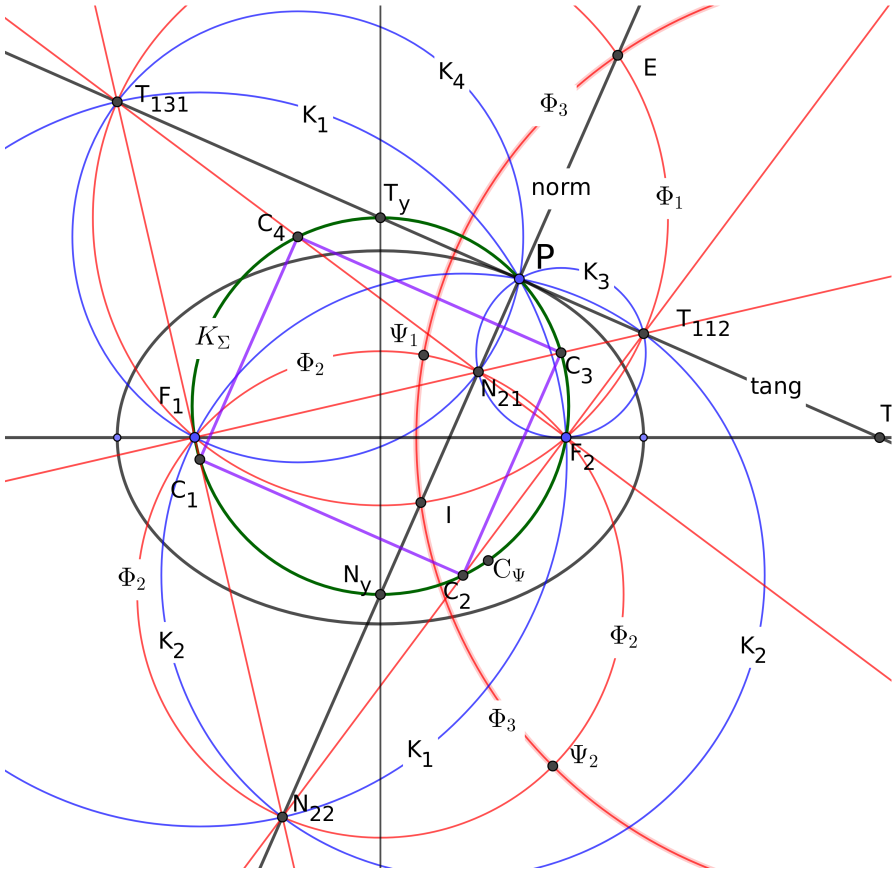

Figure 4: The normal to the ellipse at P meets the circle Φ_2 [red] in N_{21} and N_{22}. The points forming the triplets $T_{131}F_1N_{22}$, $T_{112}N_{21}F_1$, $T_{131}N_{21}F_2$ and $T_{112}F_2N_{22}$ are collinear [red lines] (Theorem 10). The points forming the quadruplets $T_{131}N_{22}F_2P$, $F_1N_{22}T_{112}P$, $N_{21}F_2T_{112}P$ and $T_{131}F_1N_{21}P$ are concyclic on the circles $K_i (i = 1, 4)$ [blue] (Theorem 11). The centers $C_i (i = 1, 4)$ of the circles K_i are the vertices of a rectangle, whose sides [violet] parallel the tangent and normal to the ellipse H at P (Theorem 12).

Corollary 4. [Figure 4] *The normal* (1.7) *to the ellipse H at P and the lines $T_{131}F_1$* (3.3) *and $T_{112}F_2$* (3.6) *concur in the point N_{22}* (3.1)*, which belongs to the circle Φ_2* (1.13)*.*

The normal (1.7) *to the ellipse H at P and the lines $T_{131}F_2$* (3.5) *and $T_{112}F_1$* (3.4) *concur in the point N_{21}* (3.2)*, which belongs to the circle Φ_2* (1.13)*.*

The tangents to the ellipse H at P (1.4) *and at the vertex $V_3(-a, 0)$ and the lines $N_{22}F_1$* (3.3) *and $N_{21}F_2$* (3.5) *concur in the point T_{131}* (2.1)*, which belongs to the circle Φ_1* (1.12)*.*

The tangents to the ellipse H at P (1.4) *and at the vertex $V_1(a, 0)$ and the lines $N_{21}F_1$* (3.4) *and $N_{22}F_2$* (3.6) *concur in the point T_{112}* (2.2)*, which belongs to the circle Φ_1* (1.12)*.*

Since the lines $T_{131}F_1N_{22}$ (3.3) and $T_{112}N_{21}F_1$ (3.4) meet at the focus F_1, both triangles $F_1N_{21}T_{131}$ and $F_1N_{22}T_{112}$ are right, the point F_1 being the common vertex of their right angles and the sides opposite to F_1 — namely, $N_{21}T_{131}$ and $N_{22}T_{112}$ — being their hypotenuses. By similar reasoning, we may achieve the conclusion that the segments $N_{21}T_{112}$ and $N_{22}T_{131}$ are the hypotenuses of as many right triangles sharing the vertex F_2. Moreover, the same segments $N_{21}T_{131}$, $N_{22}T_{112}$, $N_{21}T_{112}$ and $N_{22}T_{131}$ are the hypotenuses of a second set of four right triangles sharing the point P as the common vertex of their right angles, whilst their catheti are segments of the tangent and normal to H at P. Accordingly, if four circles K_i $(i = 1, 4)$ are constructed on the afore mentioned segments as diameters, each one of such circles passes through P and the focus non collinear with the diameter. Here, for each segment assumed as diameter, the center is given:

$$\text{midpoint of } T_{131}N_{22}\text{:}\quad C_1\left(-\frac{a + c\cos\varepsilon}{2};\ \frac{b^2(1 + \cos\varepsilon) - c(a + c)\sin^2\varepsilon}{2b\sin\varepsilon}\right), \tag{3.7}$$

$$\text{midpoint of } T_{112}N_{22}\text{:}\quad C_2\left(\frac{a - c\cos\varepsilon}{2};\ \frac{b^2(1 - \cos\varepsilon) - c(a + c)\sin^2\varepsilon}{2b\sin\varepsilon}\right), \tag{3.8}$$

$$\text{midpoint of } T_{112}N_{21}\text{:}\quad C_3\left(\frac{a + c\cos\varepsilon}{2};\ \frac{b^2(1 - \cos\varepsilon) + c(a - c)\sin^2\varepsilon}{2b\sin\varepsilon}\right), \tag{3.9}$$

$$\text{midpoint of } T_{131}N_{21}\text{:}\quad C_4\left(\frac{-a + c\cos\varepsilon}{2};\ \frac{b^2(1 + \cos\varepsilon) + c(a - c)\sin^2\varepsilon}{2b\sin\varepsilon}\right). \tag{3.10}$$

Which angles form the radii of the four circles $K_i (i = 1, 4)$ at their common point P? Here I give the respective slopes m_{C_1P} and m_{C_3P} of the radii through P of the circles K_1 and K_3:

$$m_{C_1P} = \frac{2b^2\sin^2\varepsilon - b^2(1 + \cos\varepsilon) + c(a + c)\sin^2\varepsilon}{(2a\cos\varepsilon + a + c\cos\varepsilon)b\sin\varepsilon},$$
$$m_{C_3P} = \frac{2b^2\sin^2\varepsilon - b^2(1 - \cos\varepsilon) - c(a - c)\sin^2\varepsilon}{(2a\cos\varepsilon - a - c\cos\varepsilon)b\sin\varepsilon}.$$

Few manipulations allow us to write the product of such slopes as

$$m_{C_1P}m_{C_3P} = \frac{-b^2\cos^2\varepsilon + 2a\cos\varepsilon(c - a\cos\varepsilon) + a^2\sin^2\varepsilon}{-a^2\sin^2\varepsilon + 2a\cos\varepsilon(-c + a\cos\varepsilon) + b^2\cos^2\varepsilon} = -1.$$

This means that the circles K_1 and K_3 meet orthogonally. In the same way, we can see that the circles K_2 and K_4 orthogonally meet, too. We may summarize these findings as follows:

Theorem 11. [Figure 4] *Let the circles K_1, K_2, K_3 and K_4 be constructed on the segments $T_{131}N_{22}$, $T_{112}N_{22}$, $T_{112}N_{21}$ and $T_{131}N_{21}$ as diameters, respectively. Each circle passes through the point P and through the focus non collinear with the segment taken as diameter. The circles K_1 and K_3, as well as the circles K_2 and K_4, are two couples of orthogonal circles.*

Moreover, as it is easy to check with a bit of algebra, the four midpoints C_1 (3.7) through C_4 (3.10) possess the properties described by the next statement:

Theorem 12. [Figure 4] *The midpoints C_1 (3.7), C_2 (3.8), C_3 (3.9) and C_4 (3.10) of the segments $T_{131}N_{22}$, $T_{112}N_{22}$, $T_{112}N_{21}$ and $T_{131}N_{21}$, respectively, are the vertices of a rectangle whose sides C_1C_2 and C_3C_4 are parallel to the tangent to the ellipse at P whilst the sides C_1C_4 and C_3C_2 are parallel to the normal.*
The same points are concyclic on the circle K_Σ (2.18), which is constructed on the segment of the ellipse minor axis intercepted by the tangent and normal to the ellipse at P.

The center of the circle K_Σ (2.18) is the common midpoint of the segments C_1C_3 and C_2C_4. These segments are, therefore, diameters of K_Σ. This result implies, in turn, that the triangles whose vertices are the points $C_i (i = 1, 4)$ taken three at a time are right, because each of them is inscribed in a semicircle. The lines C_1P and C_3P are orthogonal because they are inscribed in a semicircle, too. Since these lines are radii of the circles K_1 and K_2, respectively, such circles are orthogonal to each other. In this way we find again the orthogonality of the circles K_1 and K_3 and of K_2 and K_4.

In 2007, the author introduced in [3] the following circle K (Figure 5), taking as diameter the segment T_yT_x of tangent to the ellipse H at P intercepted by the axes,

$$\left(x-\frac{a}{2\cos\varepsilon}\right)^2+\left(y-\frac{b}{2\sin\varepsilon}\right)^2=\left(\frac{a}{2\cos\varepsilon}\right)^2+\left(\frac{b}{2\sin\varepsilon}\right)^2. \qquad (3.11)$$

The circle K (3.11) was shown to pass through many special points, two of them being the intersections E (1.10) and I (1.11) of the normal (1.7) to the ellipse H at P with the eccentric (1.2) and the symm-eccentric line (1.3) of P, respectively. A further paper [6] introduced the concept of *symbiotic conics*: taken a point P on the ellipse H (1.1), the symbiotic ellipse H_Σ of the ellipse H about P is an ellipse having P as center and passing through the center O of the ellipse H. The axes of H_Σ are the tangent and normal to H at P; the tangent and normal to H_Σ at O are the y- and x-axes of H, respectively. The equation of the ellipse H_Σ follows:

$$x^2\,\frac{a^2-b^2\cos^2\varepsilon}{a^2\cos^2\varepsilon}-2xy\,\frac{b}{a}\tan\varepsilon-2x\,\frac{c^2}{a\cos\varepsilon}+y^2=0. \qquad (3.12)$$

The foci of the ellipse H_Σ are the points E (1.10) and I (1.11) ([6, Theorem 3.1]). We can say that the point E, I and O are, for the ellipse H_Σ, objects *homologous* to the points F_1, F_2 and P. On the other hand, the symbiotic ellipse of H_Σ about O is the ellipse H.

Remembering that the circle K_Σ (2.18) is, by definition, *the circle constructed taking as diameter the segment T_yN_y of the y-axis intercepted by the tangent and normal to H at P*, the introduction of the symbiotic ellipse H_Σ allows to redefine such circle K_Σ (2.18) as *the circle constructed taking as diameter the segment T_yN_y of the tangent to the ellipse H_Σ intercepted by the axes of H_Σ*. Since this new definition faithfully traces the definition of the circle K, we can conclude that the circle K_Σ is homologous to the circle K (3.11). In [6], it was shown that the circles homologous to Φ_1, Φ_2 and Φ_3 are the circles Φ_1, Φ_3 and Φ_2, respectively. Objects which correspond to themselves, as the circle Φ_1, are said *auto homologous*. Now, we can determine objects homologous to the newly introduced points, lines and circles.

From Theorem 1, we know that the points T_{131} (2.1) and T_{112} (2.2) are the intersections of the tangent to H at P with the circle Φ_1, where they are diametrical opposite. More precisely, T_{112} lies on the same side as P, with respect to the ellipse H minor axis, at variance with T_{131}; the former will be said *homolateral* and the latter *contralateral*. Accordingly, as the circle Φ_1 is auto homologous, we may conclude that the points homologous to T_{131} and T_{112} are the intersections of the y-axis [namely, the tangent to H_Σ at O] with the circle Φ_1; such points are the following Φ_{1d} (3.13) and Φ_{1p} (3.14), respectively:

$$\Phi_{1d}\left(0;\ \frac{b+\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{\sin\varepsilon}\right), \quad (3.13) \qquad \Phi_{1p}\left(0;\ \frac{b-\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{\sin\varepsilon}\right). \quad (3.14)$$

The points Φ_{1d} (3.13) and Φ_{1p} (3.14) are the Φ_1 points lying at maximal and minimal distance to the H center O. They will be referred to as the Φ_1 *distal* and *proximal points*, respectively. Quite analogously, the points T_{131} and T_{112} are the Φ_1 points lying at maximal and minimal distance to the H_Σ center P.

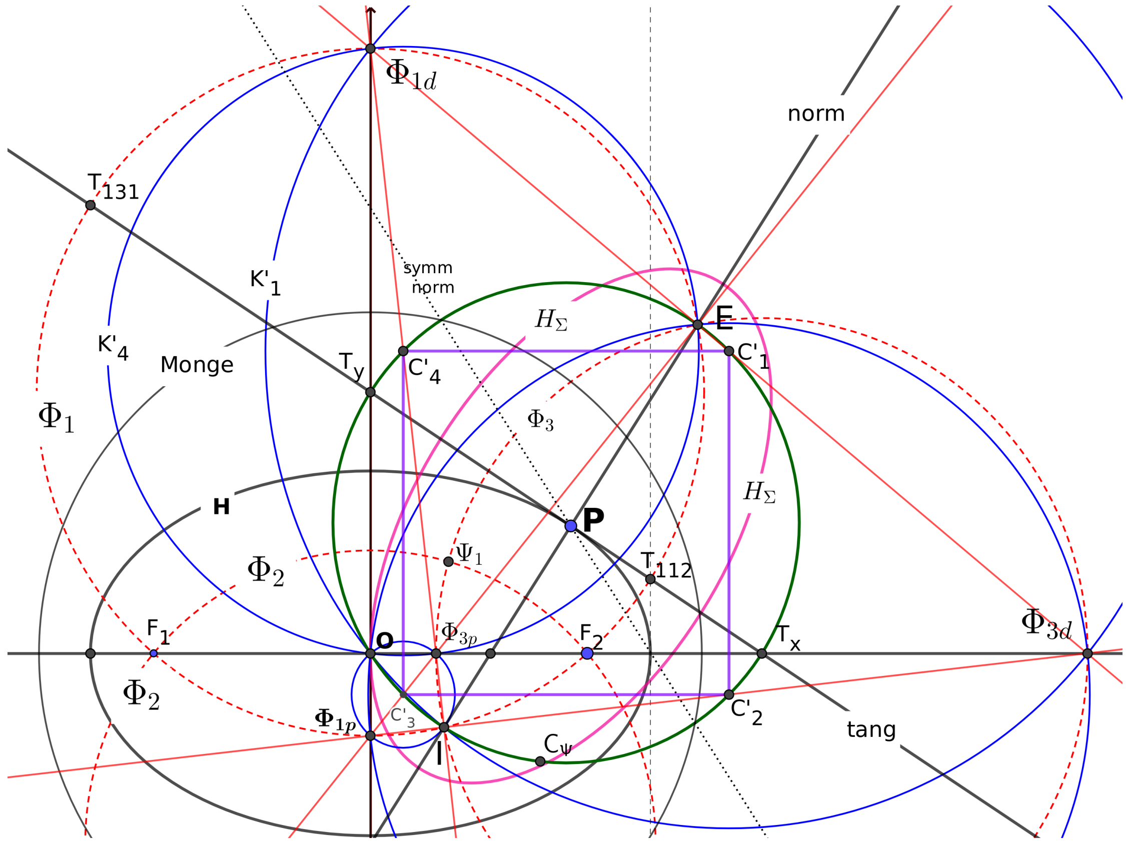

Figure 5: The circles $\Phi_i (i = 1, 3)$ are represented by red, dashed lines; their points lying at maximal and minimal distance to the origin are denoted by Φ_{id} and Φ_{ip}, respectively. The symbiotic ellipse H_Σ (3.12) [magenta] is constructed about the points E and I as foci. *The points forming the triplets* $\Phi_{1d}E\Phi_{3d}$, $\Phi_{1p}\Phi_{3p}E$, $\Phi_{1d}\Phi_{3p}I$ *and* $\Phi_{1p}I\Phi_{3d}$ *are collinear* [red lines] (Theorem 13). *The points forming the quadruplets* $\Phi_{1d}OI\Phi_{3d}$, $\Phi_{1p}\Phi_{3d}EO$, $\Phi_{1p}I\Phi_{3p}O$ *and* $\Phi_{1d}O\Phi_{3p}E$ *are concyclic on the circles* $K'_i (i = 1, 4)$ [blue] (Theorem 14). *The centers* $C'_i (i = 1, 4)$ *of the circles* K'_i *are the vertices of a rectangle whose sides* (violet) *parallel the* x- *and* y-*axes; the same points* C'_i *are concyclic with* E *and* I, T_y *and* T_x *and the midpoint* C_Ψ *of the chord* $\Psi_1\Psi_2$ *in the circle* K [green] (Theorem 15).

The points N_{22} (3.1) and N_{21} (3.2) are the intersections of the normal (1.7) to H at P with the circle Φ_2. Accordingly, they correspond to the intersections of the normal to H_Σ at O [namely, the x-axis] with the circle Φ_3 [the circles Φ_2 and Φ_3 are homologous to each other]. Precisely, as N_{22} and N_{21} are the farther and closer to P, their homologous points are the following Φ_{3d} and Φ_{3p}, namely the Φ_3 *distal* and *proximal points*, respectively:

$$\Phi_{3d}\left(\frac{a+\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{\cos\varepsilon};\ 0\right) \quad (3.15) \qquad \Phi_{3p}\left(\frac{a-\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{\cos\varepsilon};\ 0\right) \quad (3.16)$$

We may conclude by saying that the points E (1.10), I (1.11), Φ_{1d} (3.13), Φ_{1p} (3.14), Φ_{3d} (3.15) and Φ_{3p} (3.16) are homologous to the H foci F_1 and F_2, the points T_{131} (2.1), T_{112}

(2.2), N_{22} (3.1) and N_{21} (3.2) and *viceversa*, respectively. In other words, when dealing with the ellipse H and its point P, we regard the points E and I as the intersections of the normal to H at P with the eccentric and symm-eccentric line. If we change our perspective, dealing with the ellipse H_Σ and its point O, we regard the points E and I as the foci of H_Σ.

The fruit of this approach is that, from a given statement involving a special set of geometrical objects, we may generate a twin statement involving the set of the homologous objects. In many cases, the new statement, far from being a trivial duplicate of the previous statement, reveals new, worth mentioning facts.

The following Theorem 13 is an example of this approach; it is but the Theorem 10, invoked for the objects homologous to those mentioned in the original statement.

Theorem 13. [Figure 5] *The point E (1.10) is collinear*

1. *with the points Φ_{1d} (3.13) and Φ_{3d} (3.15) on the line*

$$\Phi_{1d}E\Phi_{3d}\colon\ y = \frac{b+\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{\sin\varepsilon} - x\,\frac{b+\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{a+\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}\cot\varepsilon, \tag{3.17}$$

2. *with the points Φ_{1p} (3.14) and Φ_{3p} (3.16) on the line*

$$\Phi_{1p}\Phi_{3p}E\colon\ y = \frac{b-\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{\sin\varepsilon} - x\,\frac{b-\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{a-\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}\cot\varepsilon. \tag{3.18}$$

The lines $\Phi_{1d}E\Phi_{3d}$ (3.17) and $\Phi_{1p}\Phi_{3p}E$ (3.18) meet orthogonally at E.

The point I (1.11) is collinear

1. *with the points Φ_{1d} (3.13) and Φ_{3p} (3.16) on the line*

$$\Phi_{1d}\Phi_{3p}I\colon\ y = \frac{b+\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{\sin\varepsilon} - x\,\frac{b+\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{a-\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}\cot\varepsilon, \tag{3.19}$$

2. *with the points Φ_{1p} (3.14) and Φ_{3d} (3.15) on the line*

$$\Phi_{1p}I\Phi_{3d}\colon\ y = \frac{b-\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{\sin\varepsilon} - x\,\frac{b-\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}{a+\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}}\cot\varepsilon. \tag{3.20}$$

The lines $\Phi_{1d}\Phi_{3p}I$ (3.19) and $\Phi_{1p}I\Phi_{3d}$ (3.20) meet orthogonally at I.

Now, if we construct four circles $K_i'(i=1,4)$ on the segments $\Phi_{1d}\Phi_{3d}$, $\Phi_{1p}\Phi_{3d}$, $\Phi_{1p}\Phi_{3p}$ and $\Phi_{1d}\Phi_{3p}$ as diameters, it suffices that we invoke Theorem 11 to state the following:

Theorem 14. [Figure 5] *Let the circles K_i' $(i=1,4)$ be constructed taking as diameters the segments $\Phi_{1d}\Phi_{3d}$, $\Phi_{1p}\Phi_{3d}$, $\Phi_{1p}\Phi_{3p}$ and $\Phi_{1d}\Phi_{3p}$, respectively.*

1. *The points Φ_{1d}, O, I and Φ_{3d} are concyclic on the following circle K_1' about the midpoint C_1' between Φ_{1d} and Φ_{3d},*

$$\left(x-\frac{a+\sqrt{(asbc)}}{2\cos\varepsilon}\right)^2+\left(y-\frac{b+\sqrt{(asbc)}}{2\sin\varepsilon}\right)^2=\left(\frac{a+\sqrt{(asbc)}}{2\cos\varepsilon}\right)^2+\left(\frac{b+\sqrt{(asbc)}}{2\sin\varepsilon}\right)^2. \tag{3.21}$$

2. *The points Φ_{1p}, Φ_{3d}, E and O are concyclic on the following circle K'_2 about the midpoint C'_2 between the points Φ_{1p} and Φ_{3d}:*

$$\left(x-\frac{a+\sqrt{(asbc)}}{2\cos\varepsilon}\right)^2+\left(y-\frac{b-\sqrt{(asbc)}}{2\sin\varepsilon}\right)^2=\left(\frac{a+\sqrt{(asbc)}}{2\cos\varepsilon}\right)^2+\left(\frac{b-\sqrt{(asbc)}}{2\sin\varepsilon}\right)^2. \quad (3.22)$$

3. *The points Φ_{1p}, O, Φ_{3p} and I are concyclic on the following circle K'_3 about the midpoint C'_3 between Φ_{1p} and Φ_{3p}:*

$$\left(x-\frac{a-\sqrt{(asbc)}}{2\cos\varepsilon}\right)^2+\left(y-\frac{b-\sqrt{(asbc)}}{2\sin\varepsilon}\right)^2=\left(\frac{a-\sqrt{(asbc)}}{2\cos\varepsilon}\right)^2+\left(\frac{b-\sqrt{(asbc)}}{2\sin\varepsilon}\right)^2. \quad (3.23)$$

4. *The points Φ_{1d}, O, Φ_{3p} and E are concyclic on the following circle K'_4 about the midpoint C'_4 between the points Φ_{1d} and Φ_{3p}:*

$$\left(x-\frac{a-\sqrt{(asbc)}}{2\cos\varepsilon}\right)^2+\left(y-\frac{b+\sqrt{(asbc)}}{2\sin\varepsilon}\right)^2=\left(\frac{a-\sqrt{(asbc)}}{2\cos\varepsilon}\right)^2+\left(\frac{b+\sqrt{(asbc)}}{2\sin\varepsilon}\right)^2. \quad (3.24)$$

The circles K'_1 (3.21) and K'_3 (3.23) are orthogonal, as well as the circles K'_2 (3.24) and K'_4 (3.22). The centers C'_i of the circles K'_i $(i=1,4)$ belong to the circle K (3.11).

Invoking Theorem 12 for the points homologous to C_i $(i=1,4)$, we may state the following:

Theorem 15. [Figure 5] *The midpoints C'_1, C'_2, C'_3, and C'_4 of the segments $\Phi_{1d}\Phi_{3d}$, $\Phi_{1p}\Phi_{3d}$, $\Phi_{1p}\Phi_{3p}$ and $\Phi_{1d}\Phi_{3p}$, respectively, are the vertices of a rectangle whose sides $C'_1C'_2$ and $C'_3C'_4$ are parallel to the y-axis* [that is the tangent to the ellipse H_Σ at O] *whilst the sides $C'_1C'_4$ and $C'_2C'_3$ are parallel to the x-axis* [that is the normal to the ellipse H_Σ at O].
The same points are concyclic on the circle K (3.11), which is constructed on the segment of the tangent to the ellipse H intercepted by the x- and y-axes.

The points T_y (1.6), N_y (1.9), Φ_{3p} (3.16) and Φ_{3d} (3.15) are linked by six lines, two of which are the x- and y-axes; the remaining four lines are the following:

$$T_y\Phi_{3d}\colon\ y=\frac{b}{\sin\varepsilon}-\frac{b\left(a-\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}\right)}{c^2\sin\varepsilon\cos\varepsilon}\,x\,, \quad (3.25)$$

$$N_y\Phi_{3p}\colon\ y=-\frac{c^2\sin\varepsilon}{b}+\frac{\sin\varepsilon\left(a+\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}\right)}{b\cos\varepsilon}\,x\,, \quad (3.26)$$

$$N_y\Phi_{3d}\colon\ y=-\frac{c^2\sin\varepsilon}{b}+\frac{c^2\sin\varepsilon\cos\varepsilon}{b\left(a+\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}\right)}\,x\,, \quad (3.27)$$

$$T_y\Phi_{3p}\colon\ y=\frac{b}{\sin\varepsilon}-\frac{b\cos\varepsilon}{\sin\varepsilon\left(a-\sqrt{a^2\sin^2\varepsilon+b^2\cos^2\varepsilon}\right)}\,x\,. \quad (3.28)$$

The lines $T_y\Phi_{3d}$ (3.25) and $N_y\Phi_{3p}$ (3.26) meet at the following point P_{12} (the notation $(asbc)=a^2\sin^2\varepsilon+b^2\cos^2\varepsilon$ is used):

$$P_{12}\left(\frac{c^2\cos\varepsilon\sqrt{(asbc)}}{a\sqrt{(asbc)}+c^2\sin^2\varepsilon-b^2};\ \frac{2bc^2\sin\varepsilon}{a\sqrt{(asbc)}+c^2\sin^2\varepsilon-b^2}\right). \quad (3.29)$$

The lines $N_y\Phi_{3d}$ (3.27) and $T_y\Phi_{3p}$ (3.28) meet at the following point P_{13},

$$P_{13}\left(\frac{c^2\cos\varepsilon\sqrt{(asbc)}}{a\sqrt{(asbc)}-c^2\sin^2\varepsilon+b^2};\ \frac{2b\sin\varepsilon\left(\sqrt{(asbc)}-a\right)}{(1+\sin^2\varepsilon)\sqrt{(asbc)}-2a\sin^2\varepsilon}\right). \tag{3.30}$$

The following holds:

Theorem 16. *The lines $T_y\Phi_{3d}$ (3.25) and $N_y\Phi_{3p}$ (3.26) are orthogonal, and their common point P_{12} (3.29) belongs to the circles Φ_3 (1.14) and K_Σ (2.18).*

The lines $N_y\Phi_{3d}$ (3.27) and $T_y\Phi_{3p}$ (3.28) are orthogonal, and their common point P_{13} (3.30) belongs to the circles Φ_3 (1.14) and K_Σ (2.18).

The orthogonality of the lines $T_y\Phi_{3d}$ (3.25) and $N_y\Phi_{3p}$ (3.26) clearly appears from the product of their slopes,

$$m_{T_y\Phi_{3d}}m_{N_y\Phi_{3p}}=-\frac{b\cos\varepsilon}{\sin\varepsilon\left(a+\sqrt{(asbc)}\right)}\frac{c^2\sin\varepsilon\cos\varepsilon}{b\left(a-\sqrt{(asbc)}\right)}=-\frac{c^2\cos^2\varepsilon}{a^2-a^2\sin^2\varepsilon-b^2\cos^2\varepsilon}=-1,$$

and the same holds for the lines $N_y\Phi_{3d}$ (3.27) and $T_y\Phi_{3p}$ (3.28),

$$m_{N_y\Phi_{3d}}m_{T_y\Phi_{3p}}=-\frac{c^2\sin\varepsilon\cos\varepsilon}{b\left(a+\sqrt{(asbc)}\right)}\frac{b\cos\varepsilon}{\sin\varepsilon\left(a-\sqrt{(asbc)}\right)}=-\frac{c^2\cos^2\varepsilon}{a^2-a^2\sin^2\varepsilon-b^2\cos^2\varepsilon}=-1.$$

To demonstrate that the point P_{12} (3.29) belongs to the circle K_Σ (2.18), we will replace the coordinates (3.29) in the l.h.s. of (2.18). Making some obvious manipulations, we get

$$c^2(asbc)\cos^2\varepsilon+4b^2c^2\sin^2\varepsilon-2\left(b^2-c^2\sin^2\varepsilon\right)\left(a\sqrt{(asbc)}+c^2\sin^2\varepsilon-b^2\right)$$
$$-\left(a\sqrt{(asbc)}+c^2\sin^2\varepsilon-b^2\right)^2.$$

Afterwards, further manipulations lead us to conclude that such expression identically vanishes and that P_{12} belongs, therefore, to K_Σ.

Quite similarly, to demonstrate that the point P_{12} (3.29) belongs to the circle Φ_3 (1.14), too, we will replace the coordinates (3.29) in the l.h.s. of (1.14), getting the following:

$$c^4(asbc)\cos^2\varepsilon-2ac^2\sqrt{(asbc)}\left(a\sqrt{(asbc)}+c^2\sin^2\varepsilon-b^2\right)+4b^2c^4\sin^2\varepsilon$$
$$+c^2\left(a\sqrt{(asbc)}+c^2\sin^2\varepsilon-b^2\right)^2.$$

A bit of manipulations is enough to show that this expression vanishes, too. This proves that P_{12} belongs to the circle Φ_3, too. Analogously, one proves that the point P_{13} (3.30) belongs to the circles K_Σ (2.18) and Φ_3 (1.14).

The new findings regarding the circle K_Σ — namely, the Lemma 3, the Theorem 12 and part of the Theorem 16 — may be summarized in the following statement:

Theorem 17. [The 13-Point Circle] *Let the following points be considered:*

(i) the point P_3 (2.25) [where the Φ_2 diameter through T_{212} (2.23) touches the ellipse H]*;*

(ii) the point C_Ψ (2.26) [the midpoint between the points Ψ_1 (1.15) and Ψ_2 (1.16), where the circles Φ_2 and Φ_2 meet]*;*

(iii) *the midpoints* C_1 (3.7), C_3 (3.9), C_4 (3.10) *and* C_2 (3.8) *of the segments* $T_{131}N_{22}$, $T_{112}N_{21}$, $T_{131}N_{21}$ *and* $T_{112}N_{22}$, *respectively;*

(iv) *the point* P_{12} (3.29) [where the lines $T_y\Phi_{3d}$ (3.25) and $N_y\Phi_{3p}$ (3.26) meet]*;*

(v) *the point* P_{13} (3.30) [where the lines $N_y\Phi_{3d}$ (3.27) and $T_y\Phi_{3p}$ (3.28) meet].

The afore mentioned eight points are concyclic with the ellipse foci, the point P and the y-intercepts of the normal and tangent to H at P in the circle $_\Sigma$ (2.18).

Invoking Theorem 16 for the homologous objects, the following results:

Theorem 18. *The lines T_yN_{22} and T_xN_{21} are orthogonal and their common point belongs to the circles* Φ_2 (1.13) *and* K (3.11). *The lines T_xN_{22} and T_yN_{21} are orthogonal and their common point belongs to the circles* Φ_2 (1.13) *and* K (3.11).

In [3, 4] the author has introduced the *symm-normal* line, whose equation is

$$y = b\sin\varepsilon - \frac{a}{b}\tan\varepsilon\,(x - a\cos\varepsilon)\,. \tag{3.31}$$

The symm-normal passes through P and shares two points with Monge's circle ($x^2 + y^2 = a^2 + b^2$). Each one of these points can be regarded as inverse of itself with respect to Monge's circle. Moreover, such points have been recognized [3] to belong to the circle K (3.11), too. The inversion of P with respect to Monge's circle yields the point

$$I_{P:M}\left(\frac{a^2+b^2}{a^2\cos^2\varepsilon + b^2\sin^2\varepsilon}\,a\cos\varepsilon;\ \frac{a^2+b^2}{a^2\cos^2\varepsilon + b^2\sin^2\varepsilon}\,b\sin\varepsilon;\right)$$

which belongs ([4, Theorem 1]) to the circle K (3.11), too. Having found three points which belong to the circle K and are inverse — w.r.t. Monge's circle — of as many points belonging to the symm-normal, we may state the following:

Theorem 19. *The circle* K (3.11) *is the inverse curve of the symm-normal* (3.31) *with respect to Monge's circle.*

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