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The Theorem of Gallucci Revisited

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Abstract. In this paper, we propose a well-justified synthetic approach to the projective space. We define the concepts of plane and space of incidence and also the statement of Gallucci as an axiom of our classical projective space. For this purpose, we deduce from our axioms the theorems of Desargues, Pappus, and the fundamental theorem of projectivities. Our approach doesn't use any information about analytical projective geometry like the concept of cross-ratios and homogeneous coordinates of points.

Key Words: Desargues theorem, Gallucci's axiom, Pappus axiom, projective space

MSC 2010: 51A05, 51A20, 51A30

1. Introduction

A very old and interesting question is the following: Is there a purely *geometric* reasoning of the fact that in the planes of a three-dimensional projective space the theorem of Pappus must be true? The classical investigations contain the following clear *algebraic* reasoning:

For the points in a Desarguesian plane of incidence, we can introduce projective homogeneous coordinates from a skew field. This skew field is commutative if and only if the theorem of Pappus is true. Consequently, the theorem of Pappus is necessary for defining cross-ratios of collinear points and concurrent lines and for developing the standard tools of "classical projective geometry". Hence, a projective plane can be defined as Desarguesian plane of incidence in which the theorem of Pappus is valid.

It is known that from the axioms of the three-dimensional projective space the theorem of Desargues can be derived. In general, this system of axioms doesn't tell anything about the validity of the theorem of Pappus. If the field of coordinates is non-commutative then the Pappus theorem fails to be valid. Since commutativity is needed for building up the standard tools of a classical projective space, the theorem of Pappus has to be considered as an axiom which is equivalent to the commutativity of the field of coordinates.

The so-called *theorem of Gallucci* states that *if three mutually skew lines meet three other mutually skew lines, any transversal of the first triple meets any transversal of the second triple.*

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In this paper, we prove that this statement is a natural choice to consider it as an additional axiom for a synthetic definition of the three-dimensional projective space. First of all, it arises in its own right as a basic result concerning transversals of skew lines. Its equivalence to the Pappus theorem is known, but most proofs use either the concept of cross-ratios (see [9] or [2]) or the involution theorem of Desargues (see [3]). However, we found in the book of Bachmann a nice figure suggesting a purely synthetic proof for this equivalence (see [1, p. 256]). Coxeter claimed in [3] that the statement of Gallucci (explicitly stated first in its form in [5]) can be used as an axiom of the projective space.

To realize this idea, we present an elegant proof of the equivalence of the statement of Gallucci and the theorem of Pappus. Additionally, we give an immediate proof of the equivalence of the statement of Gallucci and the fundamental theorem of projectivities between ranges of points. (Also in Baker's book [2], this statement is proved in a synthetic way.) Finally, we prove that the fundamental theorem of central-axial collineations is equivalent to the above theorems, too. Our proof does not use any analytical tool and completes our synthetic approach.

2. Incidence of the space-elements

In the book [13] of Veblen and Young, we find a synthetic approach to define the alignments (incidences) in n-dimensional projective geometries, which we call here the *properties of incidences*. Only two undefined objects were used: the points and the lines. Any other elementary objects (like planes, spaces, 4-spaces, etc.) were defined. We suggest using a more didactic system of axioms for a faster building up of the *three-dimensional space of incidence*.

We call some points collinear (coplanar) if they are incident with the same line (the same plane). The incidence of points and lines may be considered in a so-called *plane of incidence* which satisfies the following three axioms:

Axiom P1. Two (different) points determine a unique line which is incident with both.

Axiom P2. Two (different) lines determine a unique point which is incident with both.

Axiom P3. There exist four points no three of which are collinear.

This concept can, e.g., be found in the books [11] and [8].

For the *three-space* we use the following set of axioms (see, e.g., [6]):

Axiom S1. Two (different) points determine a unique line which is incident with both.

Axiom S2. Two (different) planes determine a unique line which is incident with both. This is the line of intersection of the planes.

Axiom S3. Three non-collinear points determine a unique plane.

Axiom S4. Three planes which have no common line determine a unique point.

Axiom S5. If two points incident with a line are incident with a plane, then all points of this line are also incident with this plane.

Axiom S6. If two planes incident with a line are incident with a point, then all planes incident with this line are also incident with this point.

Axiom S7. In every plane, there exist four points, no three of which are collinear.

Axiom S8. There exists five points, no four of which are coplanar.

We note, that from these axioms the usual set of properties of projective incidences can be derived. A similar coherent set of axioms can be found, e.g., in the book of Kerékjártó [9], and the equivalence of the two systems was proved by the author in [6].

A plane in a space of incidence is also a plane of incidence. Furthermore, it can be shown that the duality of the projective space is valid in every space of incidence.

3. On the role of the theorem of Desargues

The plane of incidence is the forrunner of the concept of a classical projective plane. Theorems which state incidences among points and lines are called *configuration theorems*. One famous example is the theorem of Desargues (Figure 1).

Theorem 1 (Theorem of Desargues)**.** *The lines incident with three pairs of points* (A, A'), (B, B') *and* (C, C') *pass through a common point* S *if and only if the points* X, Y *and* Z, *respectively incident with the pairs of lines* (AB, $A'B'$), (BC, $B'C'$) *and* (AC, $A'C'$) *are incident with the same line* s.

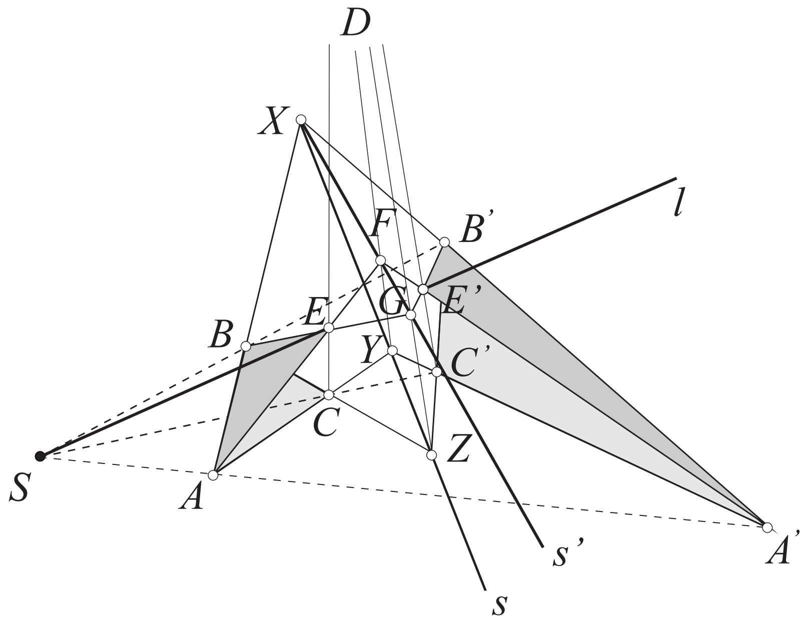

Figure 1: The theorem of Desargues

As a first question, we can ask whether, without any further information on the plane of incidence, we can prove or disprove the theorem of Desargues. The answer is known: in general, this theorem is not true in a plane of incidence. A nice counterexample has been given by Moulton in [10] (see also [6]). For more information, readers are referred to Weibel's survey [14].

4. On the transversals of lines in the space of incidence and the theorem of Gallucci

A line is called a *transversal* of two other lines if it intersects both of them. In a space of incidence, two lines are called *skew* if they are not located in a common plane. The lines which are transversals of a pair of skew lines (a, b) span two pencils of planes with the respective axes a and b. In the space, through each point P which doesn't lie either on a or b there pass two planes, one through a and the other through b. These planes intersect in a line t which goes through P. This is the *transversal of a and b from P*. If the space contains infinitely many points, every three pairwise skew lines a, b, and c have infinitely many common transversals which are also mutually skew.

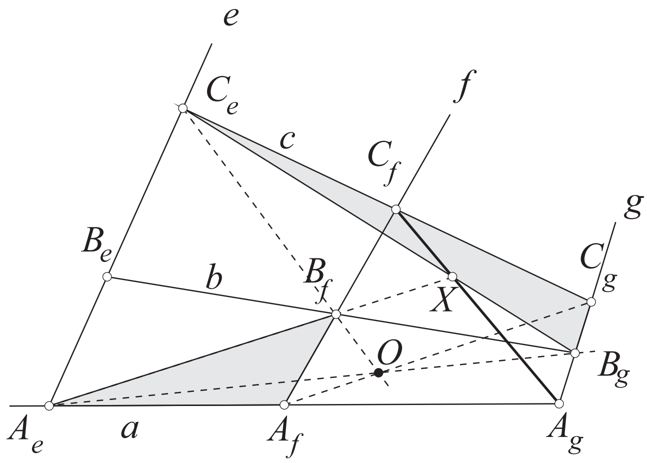

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Figure 2: The theorem of Desargues and the transversals of skew lines

The theorem of Desargues is also true for two non-coplanar triangles. Note that the corresponding point-line configuration strongly relates to the existence of three common transversals e, f, g of three pairwise skew lines a, b, c (consider Figure 2 and its notation):

Corresponding sides of the two triangles $(A_eA_fB_f)$ and $(B_gC_gC_e)$ intersect each other if and only if the planes $\alpha = (a, g)$ and $\beta := (c, f)$ exist and the lines A_eB_f and B_gC_e intersect each other at a point X. The existence of X is equivalent to the existence of the plane $\gamma = (e, b)$, where $e := (A_eC_e)$ and $b := (B_f, B_g)$ are located. (If either two of these planes coincide or all three are incident with a line, then the original two triangles are coplanar.) Since the triangles are non-coplanar, it follows that we have three lines of intersection which meet in a point O. These lines of intersection are (A_e, B_g), (A_f, C_g), and (B_f, C_e), respectively. Hence, the two triangles are perspective with respect to O. On the other hand, the existence of the points $a \cap g$, $c \cap f$, and $e \cap b$ is equivalent to their collinearity, because the respective pairs can meet each other only on the line of intersection between the two planes spanned by the respective triangles. As a conclusion, we can state:

Theorem 2 (Theorem of Desargues in space)**.** *In a space of incidence, two non-coplanar triangles are in perspective position with respect to a point if and only if the corresponding sides intersect each other.*

In Figure 2 we can immediately see that every non-coplanar Desargues configuration (containing the ten points and the same number of lines) can be associated to three pairwise skew lines a, b, c and their transversals e, f, g. Two from the first group of lines (in the figure we denote them by a and c) contain two skew edges of the triangles, respectively, and the third line ($b = B_g B_f$) is the common line of the remaining vertices of the triangle. Moreover, two lines (f and g) from the second group contain another two skew edges of the triangles, respectively, and the third line e is the common line of the remaining vertices of the triangles.

Hence, if we have two skew edges a, c and their transversals f, g, while the line b is a transversal of f and g and the line e is a transversal of a and c, then the associated lines of Figure 2 form a Desargues configuration if and only if the transversals b and e meet each other.

Since each Desargues configuration in the plane is a projection of a non-coplanar Desargues one, we can associate to each Desargues configuration a system of six lines, where the first three lines are transversal to the elements of the second group (and, obviously, vice versa).

5. The theorem of Gallucci and its equivalent forms

Consider three pairwise skew lines a, b, c and their three common transversals e, f and g (see Figure 3). The plane γ_e through the lines c and e intersects a and b in a corresponding pair of points (A_e, B_e). (Consequently, A_e, B_e and $C_e = c \cap e$ are collinear.) Conversely, if A_e is any point of the line a and we determine the plane (c, A_e), the line b intersects it in a point B_e determining the line $e := (A_e B_e)$ of the plane (c, A_e). This line e intersects c in a point C_e which is collinear to A_e and B_e.

For this situation, we can say that *the point A_e of a is projected to the point B_e of b through the line c.* With this method, we can also project A_f to B_f and A_g to B_g through c. If now d is another common transversal of the lines e, f, g, then we can project the points of a to the points of b through the line d, too. Since $(c, e) \cap (d, e) = e$, the corresponding pairs of points at the two projections (one of them through c and the other one through d) on the line e coincide. The same holds in the cases of the lines f and g.

Consider now the planes (c, A_h) and (d, A_h) with a common point A_h on a. *If the line b intersects these planes at the same point B_h,* then their intersection is the line $(A_h, B_h) =: h$ which intersects both of the lines c and d. In this case we get the following:

Theorem 3 (Theorem of Gallucci)**.** *If three mutually skew lines meet three other mutually skew lines, then any transversal of the first triple meets any transversal of the second triple.*

Assuming that we can speak about the concept of cross-ratios and the Steiner-Pappus theorem is valid in the space of incidence, the existence of B_h can be proved:

In fact, the cross-ratio of the point A_e, A_f, A_g, and A_h is equal to the cross-ratio of the corresponding planes through either c or d, implying that the cross-ratios $(B_e B_f B_g B_h(c))$ and $(B_e B_f B_g B_h(d))$ are equal. Since the cross-ratio determines the position of a point on a line (with respect to three fixed point of this line) uniquely, we obtain that $B_h(c) = B_h(d)$, and thus the theorem of Gallucci is proved. (Note that this proof works in the real projective space.)

Remark. We note that the statement of the theorem of Gallucci is self-dual with respect to the duality in the space of incidence. In fact, the dual of skew lines are skew lines and a transversal line of two skew lines a and b can be defined once more as the connecting line of two points and, on the dual way, as the intersection of two planes through the given lines. So, the theorem of Gallucci plays in space geometry a central role which is similar to that of the theorem of Desargues in plane geometry. We prove that the theorem of Gallucci implies all

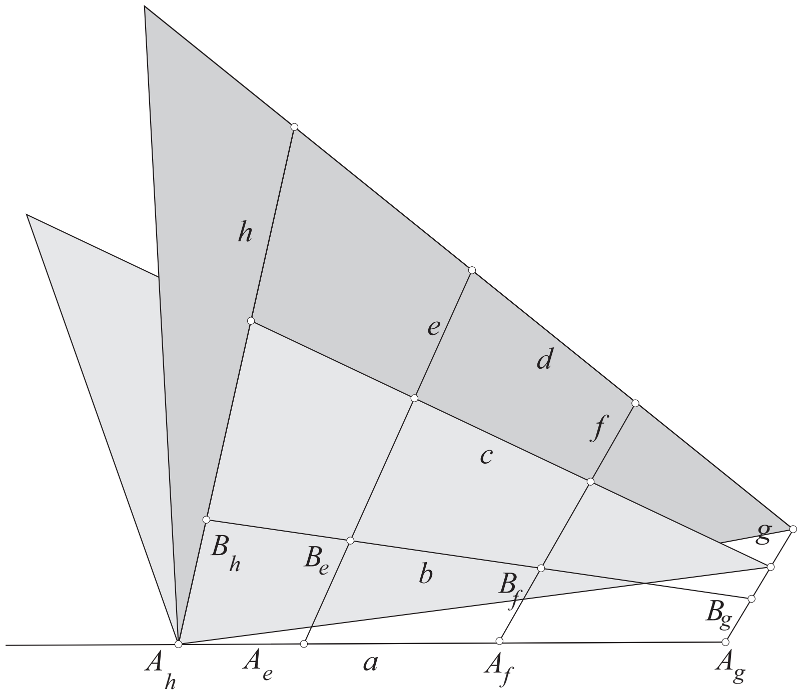

Figure 3: The theorem of Gallucci

important statements which are needed to build up of the framework of projective geometry in space.

The theorem of Pappus is an important statement of the real projective geometry. It is not valid in every planes of incidence, not even in the planes where the theorem of Desargues is true. Hessenberg proved in [7] that in a plane of incidence the Pappus theorem implies the Desargues theorem. This means that a so-called Pappian plane is always Desarguesian. The points of a Desarguesian plane have homogeneous coordinates taken from a skew field. In a Pappian plane, this field is commutative.

Theorem 4 (Theorem of Pappus)**.** *Assume that A, B, C are three points of a line and A', B', C' are three points of another line. Then the points $C'' = AB' \cap A'B$, $B'' = AC' \cap A'C$, and $A'' = BC' \cap B'C$ are collinear.*

It can be checked easily that the dual form of this theorem is equivalent to the original one.

The theorem of Pappus — contrary to the Desargues theorem — is not a consequence of the axioms of the space of incidence. To get a detailed image on its influence, we have to compare it with the theorem of Gallucci. On Figure 4, we visualized the connection between the two statements. The following theorem is known, and in both of the old books [2] and [9] we can find a synthetic proof.

Theorem 5. *In a space of incidence, the theorem of Gallucci is equivalent to the theorem of Pappus.*

Proof. First we prove that the theorem of Gallucci follows from the theorem of Pappus: Consider Figure 4. Assume first that the theorem of Pappus is valid in our planes and consider three pairwise skew lines a, b, and c with three transversals e, f, and g. Assume that d is a

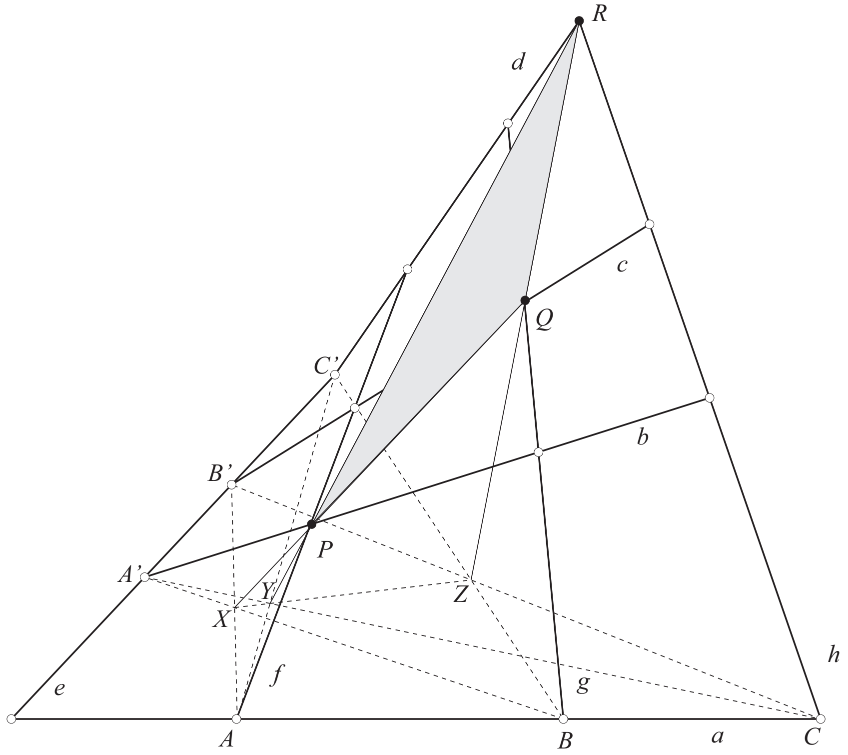

Figure 4: The theorem of Gallucci and the theorem of Pappus

transversal of e, f, g and h is a transversal of a, b, c. Then we have to prove that d intersects h in a point.

According to the notation in Figure 4, in the plane of the intersecting lines a and e we get a configuration of nine points $A, B, C,\ A', B', C'$, and $X = AB' \cap A'B$, $Y = AC' \cap A'C$, and $Z = BC' \cap B'C$. If the axiom of Pappus is valid, then the points X, Y, Z are collinear. Let $P = b \cap f$ and $Q = c \cap g$ and consider the plane π of the points X, Y, Z and P. Since the plane of b and g contains the points P, A', B, it contains the point X and also the line PX. Similarly, the plane of c and f contains the points P, A, B' and hence the point X and the line PX. From this follows that the points P, X, and Q are collinear.
Consider the intersection R of the coplanar lines YP and ZQ. Since point Z is on the line CB', and Q is on the line c, the line ZQ is on the plane spanned by the lines c and $B'C$, which is the same as the plane of c and h. Similarly, the line YP is on the plane of the points A', C, P, which is the plane of the lines b and h. Then, the intersection point R of the lines ZQ and YP has to lie on the intersection of the planes (c, h) and (b, h) which is the line h. Similarly, the line ZQ is in the plane (d, g), and the line YP is in the plane (d, f), showing that the point R is also in the line d. Hence, d and h intersect each other in a point, as the theorem of Gallucci states.

Conversely, we prove that the theorem of Gallucci implies the theorem of Pappus:
Consider a Pappian configuration of the nine points $A, B, C, A', B', C', X, Y, Z$ in the plane of the lines a and e (Figure 4). We would like to prove that the points X, Y, and Z are collinear. To this purpose, consider a point P of the space of incidence, which is not lying in the plane (a, e) and defines the lines $b := A'P$ and $f := AP$. Let Q be an arbitrary point of the line XP distinct from X and P, and define the lines $c := B'Q$ and $g := BQ$. Finally, let d be the unique transversal from C' to the skew lines f and g, and let h be the unique

transversal from C to the skew lines b and c.
If the theorem of Gallucci is true, then d and h meet in a point R. By the definition of the point Y, it lies in the plane (b,h) and it is also on the plane (f,d). Hence, the line YP is the line of intersection of these two planes. So, the common point R of d and h should lie on YP. Similarly, the points Z and Q lie in the planes (c,h) and (g,d), respectively; so Z, Q and R are collinear points, too. Hence, X, Y, and Z lie on the intersection of the planes (a,e) and (P,Q,R). This implies that they are collinear, as stated. □

The following theorem in our investigation is the fundamental theorem of projectivities. We prove that it is equivalent to Gallucci's theorem, too. This fact can be found in the books [13] and [2]. Baker's book [2] presents a synthetic proof.

Given two lines a and b and a point $P \notin a,b$ in a plane. If a bijective mapping between the points of two projective ranges[1] a and b is determined by the rays through P, it is called a *central perspectivity* (or, more precisely, the central perspectivity with centre P). A finite product of perspectivities is called a *projectivity.*

Theorem 6 (Fundamental theorem of projectivities)**.** *A projectivity between lines is determined by the images of three points of a projective range.*

Using cross-ratios, this theorem can be proved without any hardness. Hence, if the Pappus theorem is valid in the plane, then the fundamental theorem of projectivities is also true. But proving the other direction of the equivalence is a more complicated task. In this paper, we do not use the concept of cross-ratio and prove the required equivalence below.

Theorem 7. *The theorem of Gallucci is equivalent to the fundamental theorem of projectivities.*

Proof. Observe first that the theorem of Gallucci says that a perspectivity of a projective range of points to another projective range of points through a line is independent of the choice of the axis of perspectivity. More precisely, if a,b,c are three pairwise skew lines and $e,f,g,h,\dots$ are common transversals, then the correspondence $a\cap e \mapsto b\cap e$, $a\cap f \mapsto b\cap f$, $a\cap g \mapsto b\cap g$, $a\cap h \mapsto b\cap h$, and so one defines a mapping $\Phi(c)$ on the projective range of a to the projective range of b through the line c. This mapping is determined by the line c.
Let now d be any common transversal of the lines e,f,g distinct from a,b,c. Similarly, from the points of a to the points of b there is a similar mapping $\Phi(d)$ through d which agrees with $\Phi(c)$ in the three points $a\cap e$, $a\cap f$, and $a\cap g$. Let now h be a common transversal of a,b,c distinct from e,f,g. The theorem of Gallucci is true if and only if the lines d and h intersect each other in a point, implying that the equality $\Phi(c)(a\cap h) = \Phi(d)(a\cap h)$ holds for every point $a\cap h$ of a. Hence, any common transversal d of e,f,g can be the axis of the perspectivity under the investigation.

On the other hand, the perspectivity of the projective range of a to the projective range of b through the line c is the product of two central perspectivities. In fact, if we project from a point C of the line c the points of b to a plane α through the line a, we get a projective range of a line b' denoted by $(b\cap e)'$ $(b\cap f)'$, etc., respectively. These points are perspective images of the respective points $a\cap e$, $a\cap f$, etc. on the line a through the point of $\alpha\cap c = O(c)$. Hence, the mapping $\Phi(C)$ is the product of the perspectivity $a\cap e \mapsto (b\cap e)'$ through $O(c)$ and the perspectivity $(b\cap e)' \mapsto b\cap e$ through C. Similarly, the mapping $\Phi(d)$ is also the product of two perspectivities with centres $O(d)$ and D.

[1](A projective range of points is the set of points of a line.

Hence the two perspectivities through the respective lines c and d coincide if and only if the two products of perspectivities through the pairs of centres $O(c), C$ and $O(d), D$ coincide. By definition, these two products agree in the points $a \cap e$, $a \cap f$, and $a \cap g$ because c and d are transversals of the lines e, f, g. Consequently, the two perspectivities through the lines c and d coincide if the fundamental theorem of projectivities is valid in the space of incidence.

In order to prove that Gallucci's theorem implies the fundamental theorem of projectivities, we prove that any projectivity of the skew lines a and b can be considered as a perspectivity of the projective range of a to the projective range of b through the line c.

First observe, that if the lines a, a', and a'' meet in a common point, then the product of the two perspectivities $a \to a'$, $A \mapsto A'$, with center O and $a' \to a''$, $A' \mapsto A''$, with center O' can be simplified into one perspectivity $a \to a''$ with a center $O^\star$ which lies on the line OO'. To prove this, we have to use the Desargues theorem implying that the lines AA'', BB'', CC'', $\dots$ go through in the same point $O^\star$ of the line OO'.

As a second observation, we note that if the lines a, a', and a'' have no common point, but $a \cap a' \neq \emptyset$ and $a' \cap a'' \neq \emptyset$, then we can change the point range a' into any other point range b, which goes through the point $a \cap a'$ and does not contain the center O'. More precisely, we can change the second perspectivity $a' \overset{O'}{\to} a''$ to the composition $a' \overset{O'}{\to} b \overset{O'}{\to} a''$. Then, the investigated mapping $a \overset{O}{\to} a' \overset{O'}{\to} a''$ can be considered as the new product $a \overset{O}{\to} a' \overset{O'}{\to} b \overset{O'}{\to} a''$. By our first observation it can be simplified to the form $a \overset{O^\star}{\to} b \overset{O'}{\to} a''$.

Consider now a finite sequence of perspectivities $a^i \overset{O^i}{\to} a^{i+1}$ where i runs from 1 to $n-1$. Then, we can simplify the representation of the given projectivity as follows: We can assume that there are no three consecutive perspectivities with concurrent axes. If we have at least three perspectivities in the sequence, then we choose the line b of the previous observation as the line determined by the points $a^1 \cap a^2$ and $a^3 \cap a^4$. Line b avoids the centre O^2, and we can use the second observation to the first pair of the projectivities with this b. Then we get that the original product $a^1 \overset{O^1}{\to} a^2 \overset{O^2}{\to} a^3 \overset{O^3}{\to} a^4$ can be changed to the following one: $a^1 \overset{O^\star}{\to} b \overset{O^2}{\to} a^3 \overset{O^3}{\to} a^4$ where the lines b, a^3, and a^4 are concurrent. This means that, using the first observation, we can decrease the number of perspectivities.
From an inductive argument, this simplification leads to a product of two (suitable) perspectivities. If a^1 and a^n are skew, that the simplified chain of two perspectivities is of the form $a^1 \overset{O}{\to} b^{n-1} \overset{O^n}{\to} a^n$. Clearly, all lines A^1A^n, B^1B^n, C^1C^n, $\dots$ intersect the line OO^n. (Observe that, e.g., the line A^1X^{n-1} contains the centre O and the line A^nX^{n-1} contains the centre O^n, respectively, implying that A^1A^n and OO^n are coplanar.) Hence the projectivity is the perspectivity of a^1 to a^n through the line OO^n.
Now if we have two projectivities from $a = a_1$ to $b = a_n$ which send the point A^1 to A^n, the point B^1 to B^n, and the point C^1 to C^n, then they can be interpreted as two perspectivities from a to b through the respective lines c and d. These two maps agree at three points of the line a. If Gallucci's theorem is true, then the image of any further point of a is the same by the two mappings, which implies that the two projectivities coincide.

A last note remains: If the original lines have a common plane, then we can compose the respective projectivity by the same projectivity sending the second line to another one which is skew to the first one. Using again the above argument, we obtain that the original projectivities also coincide, and the fundamental theorem of projectivities is true in this case, too. $\square$

Our last-mentioned statement is the so-called fundamental theorem of central-axial collineations. A *collineation* of the projective plane to itself is a bijective mapping of its points with the property that the image of each line is again a line. The collineation is called *central* (*axial*) if there is a point (line) of the plane with the property, that the image of each line (point) through this point (line) is equal to itself. It can be proved, that every central collineation is an axial one, and vice versa. From the theorem of Desargues, it can be proved that a central-axial collineation (shortly c-a collineation) is uniquely determined by its centre, its axis, and a pair of points from which the second is the image of the first by this mapping. In general, the composition of c-a collineations is no longer a c-a collineation.

We consider now a finite product of c-a collineations, which we call a *projective mapping* of the plane to itself. Thus arises the problem whether four general points and their images determine a projective mapping of the plane uniquely. In the real projective plane, the answer is affirmative due to the fundamental theorem of c-a collineations.

Theorem 8 (Fundamental theorem of c-a collineations)**.** *In a plane of incidence, a projective mapping is uniquely defined by four general points (no three of them are collinear) and their respective images.*

Our goal to prove the following:

Theorem 9. *In a Desarguesian plane, the fundamental theorem of c-a collineations is equivalent to the fundamental theorem of projectivities.*

Proof. We prove first that the fundamental theorem of projectivities implies the fundamental theorem of c-a collineations. We show that if the pairs of points (A, A'), (B, B'), (C, C'), and (D, D') are in general position, then there is at least one projectivity ϕ of the plane with the property: $A' = \phi(A)$, $B' = \phi(B)$, $C' = \phi(C)$, and $D' = \phi(D)$. This projectivity of the plane is the product of at most four c-a collineations.

Assume that the line AB is not on the line $A'B'$. Then, the points $A, B, E := AB \cap CD$ and $A', B', E' := A'B' \cap C'D'$ form two triplets of two distinct lines, as we can see in Figure 5. We will define two c-a collineations whose product sends A to A', B to B', and E to E'. This product sends C to C''' and D to D''', respectively. Since $C'''D''' \cap C'D' = E'$, the third collineation ϕ_3 with center $O := C'C''' \cap D'D'''$ and axis $A'B'$ with a pair of points $C''' \mapsto C'$ fixes A', B', and sends D''' to D', implying the required result.

We have to give now the first two c-a collineations, respectively. Consider the line l of the points $B'' = AB' \cap A'B$ and $E'' = AE' \cap E'A$. We denote the intersection of l with the line AA' by A''. Note that the points A, B, E are in perspective position to the points A'', B'', E'' with respect to the point A'. Using the theorem of Desargues for the triplets A, B'', E and A'', B, E'', we obtain that the three lines AB, l, and $t' := (AB'' \cap A''B, BE'' \cap B''E)$ go through the same point. Similarly, the same is true for the lines AB, l, and $(AB'' \cap A''B, AE'' \cap A''E)$, thus implying that the third line in the two triplets is the same. Hence, the c-a collineation ϕ_1 defined by the center A', axis t', and corresponding pair of points A, A'' sends A, B, E to the points A'', B'', E'', respectively. Set $\phi_1(C) := C''$ and $\phi_1(D) := D''$.

Define the second c-a collineation ϕ_2 in an analogous way, originated from the points A'', B'', and E'', and getting the respective image points A', B', and C'. The center of ϕ_2 is A, the axis is $t := (A''B' \cap A'B'', B''E' \cap B'E'')$, and a pair of corresponding points is A'', A'. If we denote the respectives images of the points C'' and D'' under ϕ_2 by C''' and D''', we obtain the situation from which we can define ϕ_3.

If finally the line AB is equal to the line $A'B'$, we apply a suitable c-a collineation ϕ_0 which sends the line AB in another line A_0B_0, and for this situation we use the above arguments

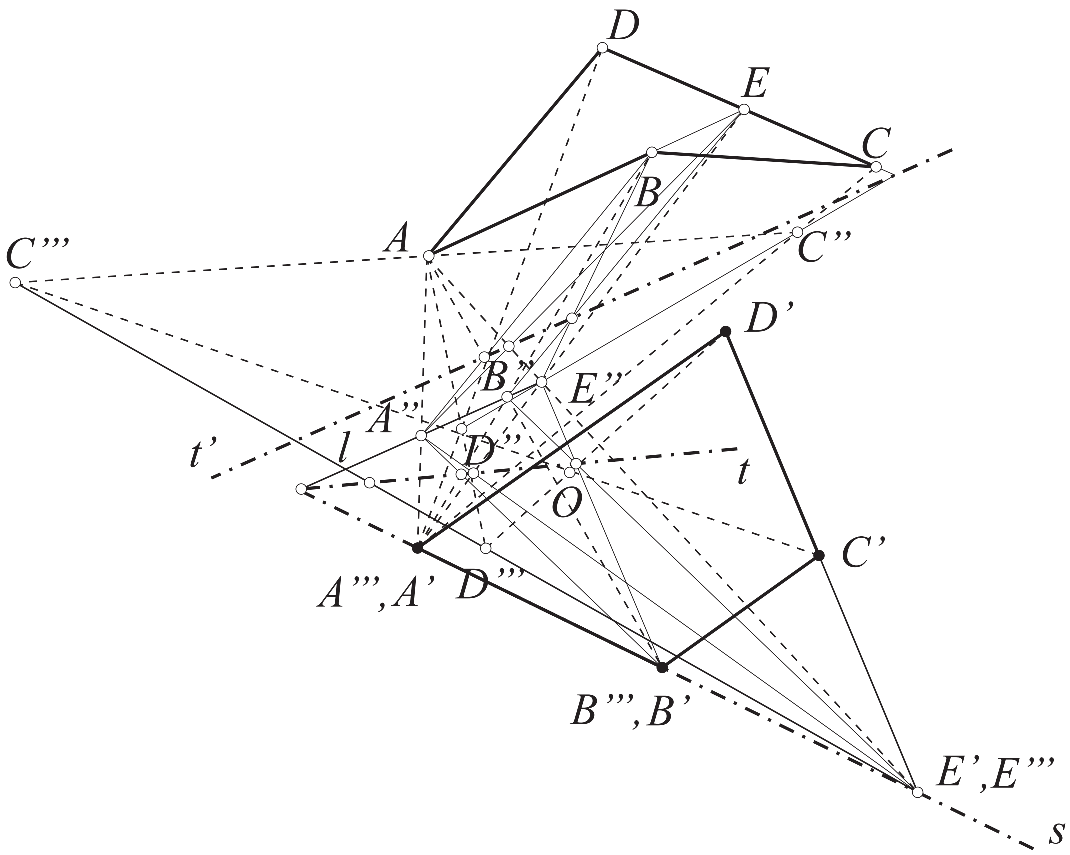

Figure 5: A composition of c-a collineations can send four given points $A \dots D$ two another four given points $A' \dots D'$

getting the three mappings ϕ_1, ϕ_2, and ϕ_3. Now the product of the four c-a collineations is the required mapping.

Assume now that there are two projectivities (say ϕ and ψ) which send the points A, B, C, D to the points A', B', C', D', respectively. Then the projectivity $\phi \circ \psi^{-1}$ gives a projectivity which fixes the points A, B, C, D and is distinct from the identity.
Since the point $E = AB \cap CD$ lies on two invariant lines, it is also fixed. Since the fundamental theorem of projectivities is true, all points of these two lines are fixed. Let P be any point of the plane, but outside of the lines AB and CD. Clearly, we have two lines through point P which intersects AB and CD implying that the point P is also a fixed point of the projectivity. Hence the mapping is the identity, contrary to our indirect hypothesis.

Conversely, we prove for the Desarguesian plane, that the fundamental theorem of c-a collineations implies the fundamental theorem of projectivities:
As we saw in the first part of this section, a perspectivity between two lines a and b can be considered as being induced by a c-a collineation $\phi_{a,b}$ on a, where the center of the perspectivity is the center of the c-a collineation. Hence, a finite product of perspectivities can be considered as being induced by a finite product of c-a collineations restricted to the given line a. If A and B are two points of a and C and D are two such points for which the four points are in general position, then there exists the point $E = AB \cap CD$, moreover A, B, and E are distinct. Consider now two projectivities ϕ and ψ on the projective range a to the projective range a' which send A to A', B to B', and E to E'. Then the corresponding products $\phi^\star$ and $\psi^\star$ of c-a collineations send C, D to C''', D''' and C, D to C'''', D'''', respectively. Define the c-a

collineation $\eta^\star$ with the axis a', the center $O = C'''C'''' \cap D'''D''''$, and for which the equalities $\eta(C''') = C''''$ and $\eta(D''') = D''''$ hold. (Of course, if the line $C'''D'''$ is distinct from the line $C''''D''''$, the center is the intersection of the lines $C'''C''''$ and $D'''D''''$; otherwise it is a point of this common line and can be determined easily, too.) We have $\eta^\star \circ \phi^\star = \psi^\star$, implying that the two induced projectivities on the line a agree, too. Since the line a' is point-wise fixed under $\eta^\star$, we proved that ϕ and ψ agree at all of a, too. Hence, the fundamental theorem of projectivities of point ranges is true in the plane, as stated. □

6. Conclusion

Due to the content of the previous sections, we suggest the following definition of the three-dimensional projective space:

Definition. A space of incidence is a projective space of dimension three if the theorem of Gallucci holds in it. More precisely, a system of points, lines and planes forms a projective space if the axioms **S1**–**S8** and **G** hold.

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