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Three-Dimensional Viviani Theorem on a Tetrahedron

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Abstract. A theorem of Viviani states that the sum of the distances from an inside point to the sides of a triangle is constant if, and only if the triangle is equilateral. It has a more general version that deals with a point anywhere in the plane. We will give a theorem similar to this general Viviani Theorem on a tetrahedron.

Key Words: three-dimensional Viviani Theorem, equifacial tetrahedron, isosceles tetrahedron.

MSC 2010: 51N20, 52B10

1. Introduction

A theorem of Vincenzo Viviani (1622–1703) states that the sum of the distances from an inside point to the sides of a triangle is constant if, and only if the triangle is equilateral. But it can be improved to deal with a point anywhere in the plane.

Notation 1. Let ABC be a triangle. Let P be an arbitrary point on the plane. Then let $\Delta(P;AB)$ be the distance between the point P and the line AB if P and C are on the same side of the line AB. And if P and C are on the opposite sides of the line AB, $\Delta(P;AB)$ is the negative of the distance between the point P and the line AB.

Viviani Theorem: *The triangle ABC is equilateral if, and only if*

$$\Delta(P;AB) + \Delta(P;BC) + \Delta(P;CA)$$

is constant for any point P in the plane.

Kawasaki et al. in [2] proved that the sum of the distances from an inside point to the sides of a regular tetrahedron is constant. We will prove a theorem similar to the above more general Viviani Theorem on a tetrahedron.

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2. The main result

We will prove our main theorem using two lemmas.

Definition 1. A tetrahedron $ABCD$ is said to be *equifacial* (or *isosceles*) if $AB = CD$, $AC = BD$, and $AD = BC$.

The next lemma gives us a characterization of an equifacial tetrahedron.

Lemma 1. *If $ABCD$ is an equifacial tetrahedron, then there is a rectangular box Ω that contains an equifacial tetrahedron $ABCD$ so that the six edges of the tetrahedron $ABCD$ are diagonals of six faces of Ω (see Figure 1). The rectangular box Ω is said to diagonally embed the tetrahedron $ABCD$.*
Conversely, the diagonally embedded tetrahedron in a rectangular box is equifacial.

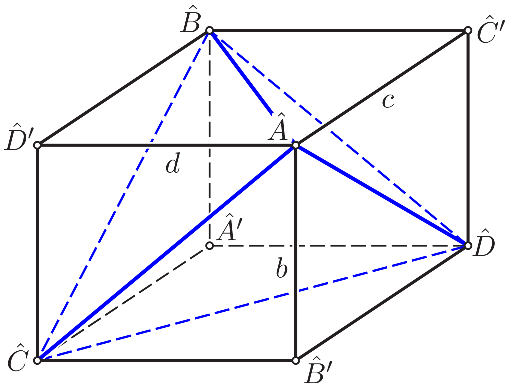

Figure 1: An equifacial tetrahedron diagonally embedded in a rectangular box.

Proof. Suppose $ABCD$ is an equifacial tetrahedron. Let

$$b = \sqrt{\frac{1}{2}\left[-(AB)^2 + (AC)^2 + (AD)^2\right]}, \quad c = \sqrt{\frac{1}{2}\left[(AB)^2 - (AC)^2 + (AD)^2\right]},$$

and

$$d = \sqrt{\frac{1}{2}\left[(AB)^2 + (AC)^2 - (AD)^2\right]}.$$

Let $\hat{A}\hat{A}'\hat{B}\hat{B}'\hat{C}\hat{C}'\hat{D}\hat{D}'$ be a rectangular box such that
a.) $(\hat{A}, \hat{A}')$, $(\hat{B}, \hat{B}')$, $(\hat{C}, \hat{C}')$, $(\hat{D}, \hat{D}')$ are pairwise diagonally opposite vertices, and
b.) $\hat{A}\hat{B}' = b$, $\hat{A}\hat{C}' = c$, $\hat{A}\hat{D}' = d$.
Then,

$$c^2 + d^2 = (\hat{A}\hat{B})^2, \quad b^2 + d^2 = (\hat{A}\hat{C})^2, \quad c^2 + b^2 = (\hat{A}\hat{B})^2$$

so that $\hat{A}\hat{A}'\hat{B}\hat{B}'\hat{C}\hat{C}'\hat{D}\hat{D}'$ is a $b \times c \times d$ box. Moreover, we have $AB = \hat{A}\hat{B}$, $AC = \hat{A}\hat{C}$, $AD = \hat{A}\hat{D}$. This shows that $ABCD$ and $\hat{A}\hat{B}\hat{C}\hat{D}$ are congruent equifacial tetrahedra. If $\hat{A}\hat{B}\hat{C}\hat{D}$ is not identical to $ABCD$, it is a mirror image of $ABCD$. In this case, the tetrahedron $\hat{A}\hat{B}\hat{C}\hat{D}$ is identical to $ABCD$.

We leave the verification of the converse to the readers. □

Notation 2. Let $ABCD$ be a tetrahedron. Let P be an arbitrary point in the space. The plane ABC divides the space into two regions. If P and D are on the same side of the plane ABC, let $\Delta(P;ABC)$ be the distance between the point P and the plane ABC. And if P and D are on the opposite side of the plane ABC, let $\Delta(P;ABC)$ be the negative of the distance between the point P and the plane ABC.

Lemma 2.

(1) *A tetrahedron $ABCD$ is equifacial if, and only if it can be represented by $A = (\alpha,\beta,\gamma)$, $B = (-\alpha,-\beta,\gamma)$, $C = (\alpha,-\beta,-\gamma)$, and $D = (-\alpha,\beta,-\gamma)$ for some three positive numbers α,β,γ.*

(2) *Let $P = (x,y,z)$ be an arbitrary point in the space. Suppose $ABCD$ is an equifacial tetrahedron. Using the notation in* (1)*, we have*

$$\Delta(P;ABC) = \frac{\alpha\beta\gamma - x\beta\gamma + \alpha y\gamma - \alpha\beta z}{\sqrt{(\beta\gamma)^2 + (\alpha\gamma)^2 + (\alpha\beta)^2}}, \quad \Delta(P;ABD) = \frac{\alpha\beta\gamma + x\beta\gamma - \alpha y\gamma - \alpha\beta z}{\sqrt{(\beta\gamma)^2 + (\alpha\gamma)^2 + (\alpha\beta)^2}},$$

$$\Delta(P;ACD) = \frac{\alpha\beta\gamma - x\beta\gamma - \alpha y\gamma + \alpha\beta z}{\sqrt{(\beta\gamma)^2 + (\alpha\gamma)^2 + (\alpha\beta)^2}}, \quad \Delta(P;BCD) = \frac{\alpha\beta\gamma + x\beta\gamma + \alpha y\gamma + \alpha\beta z}{\sqrt{(\beta\gamma)^2 + (\alpha\gamma)^2 + (\alpha\beta)^2}}.$$

Proof. (1): Let α,β,γ be positive numbers. Then the eight points $A = (\alpha,\beta,\gamma)$, $A' = (-\alpha,-\beta,-\gamma)$, $B = (-\alpha,-\beta,\gamma)$, $B' = (\alpha,\beta,-\gamma)$, $C = (\alpha,-\beta,-\gamma)$, $C' = (-\alpha,\beta,\gamma)$, and $D = (-\alpha,\beta,-\gamma)$, $D' = (\alpha,-\beta,\gamma)$ form vertices of a rectangular box. So Lemma 1 implies the part (1).

(2): The plane ABC has an equation $\alpha\beta\gamma - \beta\gamma x + \alpha\gamma y - \alpha\beta z = 0$. Since $\alpha\beta\gamma > 0$, the side of the plane ABC that contains the origin is given by the inequality $\alpha\beta\gamma - x\beta\gamma + \alpha y\gamma - \alpha\beta z > 0$. Since the origin is the centroid of the tetrahedron $ABCD$ and is inside of the tetrahedron $ABCD$, the origin and the point D are on the same side of the plane ABC. This shows that

$$\Delta(P;ABC) = \frac{\alpha\beta\gamma - x\beta\gamma + \alpha y\gamma - \alpha\beta z}{\sqrt{(\beta\gamma)^2 + (\alpha\gamma)^2 + (\alpha\beta)^2}}$$

since

$$\frac{|\alpha\beta\gamma - x\beta\gamma + \alpha y\gamma - \alpha\beta z|}{\sqrt{(\beta\gamma)^2 + (\alpha\gamma)^2 + (\alpha\beta)^2}}$$

is the distance from an arbitrary point $P = (x,y,z)$ to the plane ABC.

An equation of the plane ABD is given by $\alpha\beta\gamma + x\beta\gamma - \alpha y\gamma - \alpha\beta z = 0$, the plane ACD is $\alpha\beta\gamma - x\beta\gamma - \alpha y\gamma + \alpha\beta z = 0$, and the plane BCD is $\alpha\beta\gamma + x\beta\gamma + \alpha y\gamma + \alpha\beta z = 0$. So the other cases are left to the readers. □

Note. In Lemma 2(2), at most one of $\Delta(P;ABC)$, $\Delta(P;ABD)$, $\Delta(P;ACD)$, $\Delta(P;BCD)$ is a negative number.

Theorem 1 (Three-Dimensional Viviani Theorem).
The tetrahedron $ABCD$ is equifacial if, and only if

$$\Delta(P;ABC) + \Delta(P;ABD) + \Delta(P;ACD) + \Delta(P;BCD)$$

is constant for any point P in the space.

Proof. Suppose a tetrahedron $ABCD$ is equifacial. By Lemma 3, we assume $A = (\alpha, \beta, \gamma)$, $B = (-\alpha, -\beta, \gamma)$, $C = (\alpha, -\beta, -\gamma)$, and $D = (-\alpha, \beta, -\gamma)$ for some positive numbers α, β, γ. Let $P = (x, y, z)$ be an arbitrary point. Then we have

$$\begin{aligned}
&[\Delta(P;ABC) + \Delta(P;ABD) + \Delta(P;ACD) + \Delta(P;BCD)] \cdot \sqrt{(\beta\gamma)^2 + (\alpha\gamma)^2 + (\alpha\beta)^2} \\
&= (\alpha\beta\gamma - x\beta\gamma + \alpha y\gamma - \alpha\beta z) + (\alpha\beta\gamma + x\beta\gamma - \alpha y\gamma - \alpha\beta z) \\
&\quad + (\alpha\beta\gamma - x\beta\gamma - \alpha y\gamma + \alpha\beta z) + (\alpha\beta\gamma + x\beta\gamma + \alpha y\gamma + \alpha\beta z) \\
&= 4\alpha\beta\gamma.
\end{aligned}$$

Therefore, $\Delta(P;ABC) + \Delta(P;ABD) + \Delta(P;ACD) + \Delta(P;BCD)$ is a constant.

Conversely, suppose $\Delta(P;ABC) + \Delta(P;ABD) + \Delta(P;ACD) + \Delta(P;BCD)$ is constant for any P. Then by replacing P by A, B, C or D, we have

$$\Delta(D;ABC) = \Delta(C;ABD) = \Delta(B;ACD) = \Delta(A;BCD).$$

This shows that the four altitudes of the tetrahedron $ABCD$ from any four vertices are the same. Since the volume of a tetrahedron is $\frac{1}{2}$ base area · height , the areas of the four faces of the tetrahedron must be the same. Theorem 306 of [1] states that a tetrahedron is equifacial if, and only if the areas of the four faces are the same. This proves that the tetrahedron $ABCD$ is equifacial. □

Corollary 1.1. *The tetrahedron is equifacial if, and only if the sum of the distances from an inside point to the faces of a tetrahedron is constant.*

References

[1] N. Altshiller-Court: *Modern Pure Solid Geometry.* Macmillan Co., New York 1935.

[2] K. Kawasaki, Y. Yagi, K. Yanagawa: *On Viviani's Theorem in Three Dimension.* Math. Gaz. **89**, 283–287 (2005).

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