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On the Existence of Triangles with Given Lengths of Two Angle Bisectors and of the Cevian from the Third Angle Vertex

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Abstract. We prove the existence and uniqueness of a triangle with given lengths of two angle bisectors and the cevian from the third angle vertex (bisector, altitude and median). The results can be used in solving problems of computer graphics, architecture and other fields which include the construction of triangles with given elements.

Key Words: Triangle, angle bisector, geometry inequalities.

MSC 2010: 51M05, 51M16

1. Introduction

In this note we will put some relation between side a and two other sides of a triangle with given lengths of two angle bisectors l_b and l_c, and from this we will get as a corollary the famous result about the existence and uniqueness of a triangle with given internal angle bisectors lengths [3] (and also with given lengths of some other cevians).

In the triangle ABC, we denote $BC = a$, $AC = b$, $AB = c$ and the lengths of bisectors of angles B and C by l_b and l_c, respectively (see Figure 1).

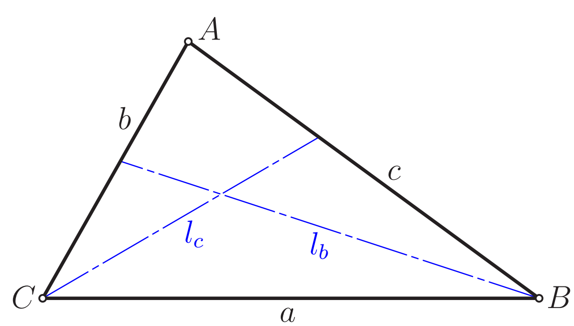

Figure 1: Find the triangle ABC with given lengths l_b, l_c and a.

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2. Main results

Lemma 1. *Given b, l_b, l_c, there is a unique triangle ABC with $AC = b$ and the lengths of bisectors of angles B and C are equal to l_b, l_c if and only if $\underline{b} < b$, where*

$$\underline{b} := \begin{cases} l_c\left(1 + \dfrac{l_c - \sqrt{l_b^2 - l_b l_c + l_c^2}}{2(l_b - l_c)}\right) & \text{if } l_b \neq l_c\,, \\ \dfrac{3}{4}\, l_b & \text{if } l_b = l_c\,. \end{cases}$$

Proof. In [5] it was proven that for given $b, l_b, l_c > 0$ there is such a unique triangle if and only if

$$l_c \le b, \tag{1}$$

or

$$b < l_c < 2b \tag{2}$$

and

$$l_b > \frac{4bl_c(l_c - b)}{(2b - l_c)(3l_c - 2b)}. \tag{3}$$

Then, if $l_c > b$, from (3) follows that

$$4b^2(l_b - l_c) + 4bl_c(l_c - 2l_b) + 3l_c^2 l_b < 0. \tag{4}$$

If $l_b = l_c$, we have $b > \frac{3}{4}\, l_b$.
For both cases $l_b < l_c$ and $l_b > l_c$, it is easy to check that from (2) and (3) we have

$$b > l_c\left(1 + \frac{l_c - \sqrt{l_b^2 - l_b l_c + l_c^2}}{2(l_b - l_c)}\right). \qquad \square$$

So, for each $b \in (\underline{b}, \infty)$ there is such a unique triangle ABC with bisectors l_b, l_c and therefore there is a unique value a for the length of the side BC. From [4] follows that

$$a \in \left(\frac{\sqrt{l_b^2 + l_c^2}}{2},\ \frac{l_b + l_c + \sqrt{l_b^2 - l_b l_c + l_c^2}}{2}\right) =: (\underline{a},\, \overline{a}).$$

Therefore, we can define a function $f\colon (\underline{b}, \infty) \to (\underline{a}, \overline{a})$ such that $f(b) = a$ holds if and only if there is a triangle ABC with given lengths of bisectors l_b, l_c, $BC = a$, and $AC = b$.

Theorem 1. *f is a continuous function.*

Proof. Let b_0 be any point of discontinuity of the function f with $b_0 \in (\underline{b}, \infty)$. Let $f(b_0) = a_0$. From the formulas in [4],

$$b = (a + c)\sqrt{1 - \frac{l_b^2}{ac}} \quad \text{and} \quad c = (a + b)\sqrt{1 - \frac{l_c^2}{ab}}$$

(both being real), we have $F(b, a) = 0$, where

$$F(b, a) = b - (a + c)\sqrt{1 - \frac{l_b^2}{ac}}$$

or

$$F(b,a) = b - \left(a + (a+b)\sqrt{1-\frac{l_c^2}{ab}}\right)\sqrt{1-\frac{l_b^2}{a(a+b)\sqrt{1-\frac{l_c^2}{ab}}}}\,.$$

The function $F(b,a)$ satisfies the conditions of the Implicit Function Theorem (see, for example, [2]). Actually, we have to check that F is a continuous function in a neighborhood of (b_0, a_0), that $F(b_0, a_0) = 0$, the partial derivative F_a exists and is continuous at (b_0, a_0), and $F_a(b_0, a_0) \neq 0$.

Let $R = [b_1, b_2] \times [a_1, a_2]$ with $b_1, b_2 \in (\underline{b}, \infty)$ and $a_1, a_2 \in (\underline{a}, \overline{a})$, where b_0 is an internal point of R. Then it is obvious that F is continuous in R and $F(b_0, a_0) = 0$.

$$\begin{aligned} F_a \;=\; &-\left(1+\sqrt{1-\frac{l_c^2}{ab}}+\frac{(a+b)l_c^2}{2a^2b\sqrt{1-\frac{l_c^2}{ab}}}\right)\sqrt{1-\frac{l_b^2}{a(a+b)\sqrt{1-\frac{l_c^2}{ab}}}} \\ &-\left(a+(a+b)\sqrt{1-\frac{l_c^2}{ab}}\right)\cdot l_b^2\cdot\frac{(2a+b)\sqrt{1-\frac{l_c^2}{ab}}+a(a+b)\frac{l_c^2}{2a^2b\sqrt{1-\frac{l_c^2}{ab}}}}{2\sqrt{1-\frac{l_b^2}{a(a+b)\sqrt{1-\frac{l_c^2}{ab}}}}} \neq 0\,. \end{aligned}$$

So F_a exists and is continuous at (b_0, a_0), and $F_a(a_0, b_0) \neq 0$.

Then, following the Implicit Function Theorem, there is a neighborhood $E \subseteq [b_1, b_2]$ of point b_0 that there exists a determined function $a = g(b)$ that satisfies $F(b, g(b)) = 0$, $a_0 = g(b_0)$, and $g(b)$ is continuous. For each $b \in E$, there is a unique value a that satisfies $F(b,a) = 0$. Then $g(b) \equiv f(b)$ in E and so f is continuous in E (and in b_0 in particular). □

For each $a \in (\underline{a}, \overline{a})$ there is a unique $b \in (\underline{b}, \infty)$ so that the triangle ABC with given l_b and l_c and with $BC = a$, $AC = b$ exists. Then we can define the inverse function of f, $f^{-1}\colon (\underline{a}, \overline{a}) \to (\underline{b}, \infty)$ with $f^{-1}(a) = b$. By the famous theorem about continuous functions, f^{-1} is a continuous monotonic function. It is easy to check that f^{-1} is a decreasing function. The analogous result may be received for the side c.

Corollary 1. *For given $l_a, l_b, l_c > 0$ there is a unique triangle ABC with internal angle bisectors that have lengths l_a, l_b, and l_c.*

Proof. Let $AL = l_a$ be the internal angle bisector in the triangle ABC. So for given $l_b, l_c > 0$,

$$l_a = \sqrt{bc\left(1-\frac{a^2}{(b+c)^2}\right)} =: l(a),$$

where $l(a)$ is a continuous monotonic decreasing function, $a \in (\underline{a}, \overline{a})$. Then,

$$l_{\underline{a}} = \sqrt{\underline{b}\,\underline{c}\left(1-\frac{\overline{a}^2}{(\underline{b}+\underline{c})^2}\right)} = 0$$

and $\lim_{a\to\underline{a}} l(a) = \infty$. So $l(a)\colon\ (\underline{a}, \overline{a}) \to (\infty, 0)$, and for given $l_a > 0$, there is a unique triangle ABC with internal angle bisectors that have the lengths l_a, l_b, l_c.

From this it follows that if two bisectors of a triangle are equal, then the triangle is isosceles (the Steiner-Lehmus Theorem [1, p. 9]). □

Corollary 2. *For a given $h_a > 0$, there is a unique triangle ABC with internal angle bisectors of lengths l_b, l_c and the altitude AH of length h_a.*

Proof. Let a $AH = h_a$ be the altitude in the triangle ABC. The area of the triangle ABC is equal to

$$\frac{1}{2}\, h_a\, a = \frac{1}{2}\, ab \sin \angle ACB = \frac{1}{2}(a+b) l_c \sin \frac{\angle ACB}{2}\,,$$

then

$$\cos \frac{\angle ACB}{2} = \frac{l_c(a+b)}{2ab} \quad \text{and} \quad \sin \frac{\angle ACB}{2} = \sqrt{1 - \frac{l_c^2(a+b)^2}{4a^2b^2}}\,.$$

So for given $l_b, l_c > 0$,

$$h_a = l_c \left(1 + \frac{b}{a}\right) \sqrt{1 - \frac{l_c^2(a+b)^2}{4a^2b^2}} =: h(a), \tag{5}$$

and $h(a)$ is a continous function for $a \in (\underline{a}, \overline{a})$.
Similarly,

$$h_a = l_b \left(1 + \frac{c}{a}\right) \sqrt{1 - \frac{l_b^2(a+c)^2}{4a^2c^2}}\,. \tag{6}$$

$$a = b \cos \angle ACB + c \cos \angle ABC = bF_1(a,b) + cF_2(a,c),$$

where

$$F_1(a,b) = \frac{l_c^2(a+b)^2}{2a^2b^2} - 1 \quad \text{and} \quad F_2(a,c) = \frac{l_b^2(a+c)^2}{2a^2c^2} - 1.$$

If $a_1 < a_2$, where $a_1, a_2 \in (\underline{a}, \overline{a})$ and

$$a_1 = b_1F_1(a_1,b_1) + c_1F_2(a_1,c_1), \quad a_2 = b_2F_1(a_2,b_2) + c_2F_2(a_2,c_2),$$

then $b_1 > b_2$, $c_1 > c_2$, and at least one of the inequalities

$$F_1(a_1,b_1) < F_1(a_2,b_2) \tag{7}$$

and

$$F_2(a_1,c_1) < F_2(a_2,c_2) \tag{8}$$

is true. Then from (5) and (6) it is easy to conclude that $h_a(a_1) > h_a(a_2)$. So $h(a)$ is continuous monotonic decreasing function. Then, $\underline{h}_a := h(\overline{a}) = 0$ and $\lim_{a\to\underline{a}} h(a) = \infty$. Thus, for given $h_a > 0$, there is a unique triangle ABC with internal angle bisectors of lengths l_b, l_c, and the altitude of length h_a. $\square$

Corollary 3. *CB and $BM : MC = k$ ($k > 0$ is constant). Then for given l_b, l_c and $t_a > \dfrac{|\underline{b}k - \underline{c}|}{k+1}$, there is a unique triangle ABC with internal angle bisectors' lengths l_b, l_c, and cevian $AM = t_a$.*

Proof. According to Figure 2 holds

$$\cos C = \frac{a^2+b^2-c^2}{2ab}, \quad CM = \frac{a}{k+1},$$

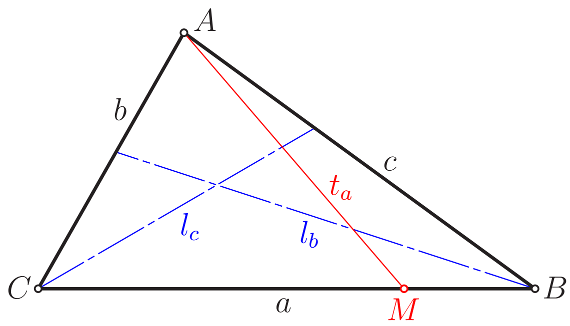

Figure 2: Illustration to Corollary 3.

and

$$AM^2 = b^2 + \left(\frac{a}{k+1}\right)^2 - \frac{a^2+b^2-c^2}{k+1} = \frac{b^2(k^2+k)+c^2(k+1)-a^2k}{(k+1)^2} = t(a).$$

Then, while a continuously decreases from $\overline{a}$ to $\underline{a}$, AM continuously increases from

$$\underline{AM} = \sqrt{\frac{\underline{b}^2(k^2+k)+\underline{c}^2(k+1)-\overline{a}^2k}{(k+1)^2}} = \frac{\sqrt{\underline{b}^2k^2-2\underline{b}\,\underline{c}\,k+\underline{c}^2}}{k+1} = \frac{|\underline{b}\,k-\underline{c}\,|}{k+1} \quad \text{to} \quad \infty.$$

Then, for given l_b, l_c and $t_a > \frac{|\underline{b}\,k-\underline{c}\,|}{k+1}$, there is a unique triangle ABC with internal angle bisector lengths l_b, l_c, and cevian $AM = t_a$. In particular for the median m_a ($k=1$) and $l_b \neq l_c$, we get the inequality

$$\underline{m}_a = \frac{1}{2}\left| l_c\left(1+\frac{l_c-\sqrt{l_b^2-l_bl_c+l_c^2}}{2(l_b-l_c)}\right) - l_b\left(1+\frac{l_b-\sqrt{l_b^2-l_bl_c+l_c^2}}{2(l_c-l_b)}\right)\right| < m_a < \infty,$$

and for $l_b = l_c$ (isosceles triangle) $0 < m_a < \infty$. □

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